Complete surfaces with zero curvatures
in conformally flat spaces

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Abstract. In this paper, we introduce a family of Riemannian manifolds $E^3_F$, which are Euclidean space $\mathbb{R}^3$ endowed with conformally flat metrics. We characterize rotational surfaces with constant Gaussian and extrinsic curvatures in $E^3_F$. We present a particular space that is isometric to $H^2 \times S^1$, and, using a special parametrization, we construct a family of complete rotational surfaces with zero Gaussian and extrinsic curvatures in $H^2 \times S^1$. We have built a special space that is a warped product $H^2 \times f \mathbb{R}$, which is a complete space foliated by complete surfaces of constant Gaussian curvature $-1$; this shows that the hyperbolic space $H^2$ is isometrically immersed into the space $H^2 \times f \mathbb{R}$, and this space is isometric to neither $H^3$ nor $H^2 \times \mathbb{R}$, showing that in the ambient space, $H^2 \times f \mathbb{R}$ Hilbert theorem does not hold.

1. Introduction

It is well known that only the cylinder and the plane are complete flat surfaces in Euclidean space $\mathbb{R}^3$. The study of flat surfaces is a fascinating topic in differential geometry. There are some recent papers that present results in this subject, for example, a study of flat surfaces with singularities in $\mathbb{R}^3$ can be found in [14]. Moreover, for flat surfaces in hyperbolic 3-space $H^3$, there is the well-known classification of complete flat surfaces obtained by Volkov–Vladimirova

Mathematics Subject Classification: 53C21, 53C42.
Key words and phrases: rotational flat surfaces, constant extrinsic curvature, conformally flat space.
The authors were partially supported by CAPES/PROCAD-NF.
and Sasaki [16]. In recent years, there have been several studies on complete flat surfaces with singularities in \( \mathbb{H}^3 \) (see [6] and the references therein).

There have been a number of studies on surfaces with constant Gaussian curvature [1], [3], [5], [13]. Contemporary extensive studies on product spaces suggest a renewed interest in these 3-manifolds. Recently, in [12], the authors classify complete rotational surfaces with constant Gaussian curvature both in \( \mathbb{H}^2 \times \mathbb{S}^1 \) and \( \mathbb{S}^2 \times \mathbb{S}^1 \). The study of surfaces with constant extrinsic curvature has been expanding, see, for example, the paper [10] for surfaces in product spaces.

In the present study, we consider the Euclidean space \( \mathbb{R}^3 \) endowed with a family of metrics \( \langle \cdot, \cdot \rangle_g = \frac{1}{F^2} \langle \cdot, \cdot \rangle \), which are conformal to Euclidean metric \( \langle \cdot, \cdot \rangle \). We take a variation of the conformal factor \( F \) to find interesting properties and get results for immersions of surfaces in these conformal spaces. In this case, \( \langle \cdot, \cdot \rangle_g \) is said to be a \textit{conformally flat metric}. Thus, we have a family of Riemannian manifolds, which we denote by \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g) \). Note that, if \( F \) is bounded, then \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g) \) is a complete manifold.

The study of surfaces immersed in spaces that are conformal to the Euclidean space is essential because they include very important spaces of constant curvature such as the sphere \( \mathbb{S}^3 \) and hyperbolic space \( \mathbb{H}^3 \).

The authors in [7] studied surfaces of rotation with constant extrinsic curvature in \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g) \), where \( F(x) = e^{-x_1^2-x_2^2-x_3^2}, x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). This particular metric appears as a solution to the Einstein equation obtained by Pina and Tenenblat [15], with a great potential for applications in physics (see [17] and the references therein). In [8], the authors studied rotational surfaces in \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g) \), where the conformal factor of the metric is rotationally symmetrical.

The first objective of the presented work is to study curvatures of surfaces in \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g) \), where \( \langle \cdot, \cdot \rangle_g \) is a conformally flat metric and its conformal factor \( 1/F^2 \) is invariant on a cylinder \( C_r \) with \( x_3 \)-axis and radius \( r \), for each \( r \), i.e., the value \( F(x), x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) depends only on \( r := x_1^2 + x_2^2 \), \( \forall x \in \mathbb{R}^3 \). So \( F(x_1, x_2, x_3) = F(x_1^2 + x_2^2) \), and \( F : \mathbb{R} \to \mathbb{R} \) is a differentiable function. The manifold \( \mathbb{R}^3 \) endowed with a metric as defined above will be denoted by \( \mathbb{E}^3_F \).

Helicoidal minimal surfaces and helicoidal surfaces with prescribed extrinsic curvature were studied in \( \mathbb{E}^3_F \), for a special conformal factor [2], [11]. As this space is invariant under the actions of the rotational group around \( x_3 \)-axis, it is logical to consider rotational surfaces around \( x_3 \)-axis, because they are invariant under the action of the same group.

We characterize a family of rotational surfaces with constant Gaussian curvature in \( \mathbb{E}^3_F \). We show that all the rotational cylinders with \( x_3 \)-axis are flat surfaces for all \( F \). We prove that all the planes perpendicular to \( x_3 \)-axis have
null extrinsic curvature for all $F$, and any rotational cylinder, with $x_3$-axis, has constant extrinsic curvature. Furthermore, we exhibit spaces, with $F(r) = e^{-r}$ and $F(r) = \sqrt{r} + 1$, for which all the cylinders with $x_3$-axis have constant positive and negative extrinsic curvature, respectively.

We present a special space $E^3_{\sqrt{r}}$, and prove that it is a complete manifold with non-positive sectional curvatures. We construct a family of rotational surfaces that are flat surfaces and have null extrinsic curvature in $E^3_{\sqrt{r}}$. We prove that the space $E^3_{\sqrt{r}}$ is isometric to $H^2 \times S^1$. Rotational surfaces in $H^2 \times S^1$, with zero Gaussian curvature, have also been studied in [12], where the authors considered another type of the parametrization. We show that the Catenoids are the only surfaces of rotation with zero extrinsic curvature when the conformal factor is $F(r) = \frac{1}{\sqrt{r}}$.

By Hilbert’s theorem, it is known that there does not exist a complete surface with constant negative Gaussian curvature in the Euclidean space $\mathbb{R}^3$. It is an interesting problem to construct 3-dimensional complete ambient space in which the hyperbolic space $H^2$ can be isometrically immersed. It is well known that there are two trivial ambient spaces, namely $H^3$ and $H^2 \times \mathbb{R}$, allowing such immersion of $H^2$. Here, we present the space $E^3_{\frac{1}{\sqrt{r}}}$, where $F(r) = \frac{1}{r}$, with $0 < r < 1$. We prove that $E^3_{\frac{1}{\sqrt{r}}}$ is the warped product $H^2 \times f \mathbb{R}$, where $f = \frac{1}{r}$, which is a complete space foliated by complete surfaces of constant Gaussian curvature $-1$, which show that the hyperbolic space $H^2$ is isometrically immersed into space $H^2 \times f \mathbb{R}$, and this space is isometric to neither $H^3$ nor $H^2 \times \mathbb{R}$, showing that in the ambient space $H^2 \times f \mathbb{R}$, Hilbert theorem does not hold.

We believe that the spaces $E^3_{\sqrt{r}}$ and $H^2 \times f \mathbb{R}$ as ambient spaces are intriguing to investigate surfaces with some special properties in the future.

2. Conformally flat 3-spaces

In this section, we consider the ambient space $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, which is a 3-dimensional real vector space $\mathbb{R}^3$ endowed with the canonical Euclidean metric, denoted by $\langle \cdot, \cdot \rangle$. Now, we consider a metric $\langle \cdot, \cdot \rangle_g$ that is conformal to the Euclidean metric $\langle \cdot, \cdot \rangle$. Similarly, we denote the Euclidean space $\mathbb{R}^3$ endowed with the metric $\langle \cdot, \cdot \rangle_g$ by $(\mathbb{R}^3, \langle \cdot, \cdot \rangle_g)$. Thus, the components of the metric $\langle \cdot, \cdot \rangle_g$ are given by

$$g_{ij}(x) = \frac{\delta_{ij}}{F^2(x)}, \quad x = (x_1, x_2, x_3), \quad 1 \leq i, j \leq 3,$$

(1)
where \( F : \mathbb{R}^3 \rightarrow \mathbb{R} \) is a positive differentiable function. We observe that if \( F \) is bounded, then \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\) is a complete Riemannian manifold.

Considering the Levi-Civita connection \( \nabla \) of \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\) and the canonical basis \( \{e_1, e_2, e_3\} \) of \( \mathbb{R}^3 \), we get
\[
\nabla_{e_i} e_j = \sum_{k=1}^{3} \Gamma^k_{ij} e_k \quad \text{and} \quad \nabla_{e_i} e_j = \nabla_{e_j} e_i.
\]

From equation (1), we have \( g_{ij} = 0 \) for \( i \neq j \). Therefore, the Christoffel symbols of this metric are given by
\[
\Gamma^k_{ij} = 0, \quad i \neq j \neq k \neq i, \quad \Gamma^j_{ii} = \frac{F_j}{F}, \quad \forall i \neq j,
\]
and
\[
\Gamma^i_{ij} = -\frac{F_j}{F}, \quad 1 \leq i,j \leq 3.
\]

The Riemannian manifold \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\) has sectional curvature given by
\[
K \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) (x) = \left[ \left( \frac{F_i}{F} \right)_i + \left( \frac{F_j}{F} \right)_j - \left( \frac{F_k}{F} \right)_k \right]^2 F^2,
\]
where \( 1 \leq i,k,j \leq 3 \) are distinct.

Let \( \beta(s) = (x_1(s), x_2(s), x_3(s)) \subset \mathbb{R}^3 \) be a curve parametrized by arc length. The curve \( \beta(s) \) is a geodesic in \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\) if and only if the components of \( \beta \) satisfy the following system of ordinary differential equations
\[
\frac{d^2 x_k}{ds^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds} = 0, \quad 1 \leq k \leq 3,
\]
where the Christoffel symbols are given by (3).

The following result (refer to [7, Theorem 1]) establishes some relations between the concepts defined in \((\mathbb{R}^3, \langle \cdot , \cdot \rangle)\) and \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\).

**Theorem 1** ([7]). Let \( X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a regular parametrized surface. Consider \( X(U) \) as a surface in \((\mathbb{R}^3, \langle \cdot , \cdot \rangle)\) with the Euclidean metric, let \( N \) be the normal Gauss mapping, \( \lambda_i \) the eigenvalues of \( N \), \( H \) and \( K \) the mean and Gaussian curvatures, respectively. Analogously, consider \( X(U) \) as a surface in \((\mathbb{R}^3, \langle \cdot , \cdot \rangle_g)\) with a metric conformal to the Euclidean metric, with the conformal factor \( F^{-2} \), let \( \tilde{\lambda}_i \) be the eigenvalues of \( \tilde{N} \), \( \tilde{H} \) and \( \tilde{K}_E \) the mean and the extrinsic curvatures, respectively. Then
\[
\tilde{\lambda}_i = F \lambda_i - \langle N, \text{grad } F \rangle, \quad \tilde{H} = F H - \langle N, \text{grad } F \rangle,
\]
\[
\tilde{K}_E = F^2 K - 2HF \langle N, \text{grad } F \rangle + \langle N, \text{grad } F \rangle^2,
\]
where \( F \) denotes the evaluation of \( F \) at \( X(u,v), (u,v) \in U \).
3. The conformally flat 3-spaces with cylindrical metrics

In this section, we study a conformally flat 3-space \((\mathbb{R}^3, \langle \cdot, \cdot \rangle_g)\), where the metric \(\langle \cdot, \cdot \rangle_g\) is any metric which is conformal to the Euclidean metric \(\langle \cdot, \cdot \rangle\) such that its conformal factor \(1/F^2\) is invariant on a cylinder \(C_r\) of radius \(r\), for each \(r\). So the real vector space \(\mathbb{R}^3\) endowed with such conformally flat metric will be denoted by \(E_3^F\), where \(F = F(r)\), with \(r = x_1^2 + x_2^2\), as mentioned in the Introduction.

In the next result, we establish the expressions for the principal curvatures \(\tilde{\lambda}_i\) in \(E_3^F\), as a direct consequence of Theorem 1.

**Lemma 1.** Let \(X: U \to \mathbb{R}^3\), with \(X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))\) be a surface in \(\mathbb{R}^3\), \(N\) the normal Gauss mapping of \(X\) in \(\mathbb{R}^3\), and \(\lambda_i\) the eigenvalues of \(N\). Then the eigenvalues \(\tilde{\lambda}_i\) of \(X\) in \(E_3^F\) satisfy

\[
\tilde{\lambda}_i = F\lambda_i - 2\dot{F}((x_1, x_2, 0), N), \quad i = 1, 2,
\]

where \(\dot{F}\) denotes the derivative of \(F\) with respect to the variable \(r\).

**Proposition 1.** The Riemannian manifold \(E_3^F\) has sectional curvature given by

\[
K \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)(x) = 4F\dot{F} + 4(x_1^2 + x_2^2)(-\dot{F}^2 + F\ddot{F}),
\]

\[
K \left( \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_3} \right)(x) = 2F\dot{F} + 4x_\alpha^2\dot{F} - 4(x_1^2 + x_2^2)\dot{F}^2,
\]

where \(1 \leq \alpha \leq 2\). Moreover, let \(\beta(s) = (x_1(s), x_2(s), x_3(s)) \subset E_3^F(c)\) be a curve parametrized by arc length. Then, \(\beta(s)\) is a geodesic if and only if

\[
\frac{d^2x_\alpha}{ds^2} + 2\frac{x_\alpha\dot{F}}{F} \sum_{p \neq \alpha} \left( \frac{dx_p}{ds} \right)^2 \left( \frac{dx_\alpha}{ds} \right)^2 - \frac{2x_\alpha\dot{F}}{F} \frac{dx_\alpha}{ds} \frac{dx_\alpha}{ds} = 0,
\]

\[
\frac{d^2x_3}{ds^2} - 4 \sum_{p \neq 3} \frac{x_p\dot{F}}{F} \frac{dx_p}{ds} \frac{dx_3}{ds} = 0,
\]

where \(1 \leq \tilde{\alpha}, \alpha \leq 2, \tilde{\alpha} \neq \alpha\).

**Proof.** Equation (7) follows by straightforward calculations using (4). Equation (8) follows by substitution of the Christoffel symbols (3) in equation (5), where

\[
\Gamma^k_{ij} = 0, \quad i \neq j \neq k \neq i, \quad \Gamma^i_{ij} = -f_j,
\]

\[
\Gamma^i_{ji} = f_j, \quad \Gamma^i_{ii} = -f_i \quad \text{and} \quad f_j = \frac{F_{x_j}}{F} = \frac{2x_j\dot{F}(r)}{F}.
\]

□
In the next result, we study geodesics in $\mathbb{E}^3_{F(r)}$.

**Theorem 2.** Let $\mathbb{E}^3_{F}$ be the ambient space. Then, the straight lines perpendicular to $x_3$-axis are geodesics in $\mathbb{E}^3_{F}$, $\forall F = F(r)$, where $r = x_1^2 + x_2^2$.

**Proof.** Let $\alpha(u) = (0, 0, v_3) + u(v_1, v_2, 0)$, an arbitrary straight line perpendicular to $x_3$-axis and its arc length $s(u) = \int_0^u ||\alpha'(u)||_g du$. Denote by $\beta(s) = \alpha \circ h(s)$, the reparametrization by the arc length, where $h = s^{-1}$ is the inverse function of $s$.

By using the system of ordinary differential equations of geodesics given by (8), we have that

$$
\frac{d^2 x_\alpha}{ds^2} + \frac{2x_\alpha \dot{F}}{F} \left( \sum_{p \neq \alpha} \left( \frac{dx_p}{ds} \right)^2 - \left( \frac{dx_\alpha}{ds} \right)^2 \right) - \frac{4x_\alpha \dot{F}}{F} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0,
$$

$$
\frac{d^2 x_3}{ds^2} - 4 \sum_{p \neq 3} \frac{x_p \dot{F}}{F} \frac{dx_p}{ds} \frac{dx_3}{ds} = 0,
$$

where $1 \leq \tilde{\alpha}, \alpha \leq 2$, $\tilde{\alpha} \neq \alpha$.

By using the curve components, where $\frac{d^2 x_3}{ds^2} = h''(s)v_k$, and observing that $h'(s) = F \circ \beta(s)$, the system is equivalent to

$$
h''v_\alpha + 2h'v_\alpha \frac{\dot{F}(\beta(s))}{F(\beta(s))} (-h'v_\alpha)^2 + \sum_{i \neq \alpha} (h'v_i)^2 - 4(v_\alpha)^2 \frac{h'\dot{F}(\beta(s))}{F(\beta(s))} v_\alpha = 0,
$$

where $1 \leq \tilde{\alpha}, \alpha \leq 2$, $\tilde{\alpha} \neq \alpha$. As the equation $h'' - 2hF \dot{F} = 0$ is verified, we can conclude that the first two equations are satisfied. The third equation is trivially satisfied, because the third component of the curve is $x_3 = v_3 = \text{constant}$. This concludes the proof of the theorem. $\square$

### 3.1. Flat rotational surfaces in $\mathbb{E}^3_{F}$

Because the rotations around $x_3$-axis are isometries of $\mathbb{E}^3_{F}$, up to the isometries, let us consider a family of the rotational surfaces around $x_3$-axis, which can be parametrized by

$$
X(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, v).
$$

We use the following result to characterize all the rotational surfaces $X$ with constant Gaussian curvature in $\mathbb{E}^3_{F}$. 

Lemma 2. Let \(X(u,v) = (\varphi(v) \cos u, \varphi(v) \sin u, v)\) be a rotational surface in \(E^3_F\). Then, the Gaussian curvature of \(X\) is given by

\[
K = -\frac{F^2}{a\varphi} \frac{\partial}{\partial v} \left( \frac{\varphi'(F - 2\varphi^2 \dot{F})}{aF^2} \right), 
\]

where \(\dot{F} = \frac{dF}{dr}\) and \(a^2 = 1 + \varphi'^2\).

Proof. The coefficients of the first fundamental form of \(X\), with respect to the metric that is conformal to the Euclidean metric, are given by

\[
\hat{E} = \frac{\varphi^2(v)}{F^2(\varphi^2(v))}, \quad \hat{G} = 1 + \left(\frac{\varphi'(v))^2}{F^2(\varphi^2(v))}\right) \quad \text{and} \quad \hat{F} = 0.
\]

We have that \(\hat{E}_v = \frac{F^2_2\varphi'^2 - \varphi^2_2F\dot{F}^22\varphi'}{F^4}\) and \(\hat{G}_u = 0\). Hence, we have that the Gauss equation gives us

\[
K = -\frac{1}{2\sqrt{EG}} \left\{ \frac{\partial}{\partial v} \left( \frac{\hat{E}_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left( \frac{\hat{G}_u}{\sqrt{EG}} \right) \right\}.
\]

Therefore, from the above expression we obtain equation (10). \(\square\)

In the next result, we show that all the cylinders with \(x_3\)-axis are flat surfaces for all \(F\) in \(E^3_F\). And, surprisingly, we get that all the rotational surfaces parametrized by \(X(u,v) = (\varphi(v) \cos u, \varphi(v) \sin u, v)\) are flat surfaces in the special ambient space \(E^3_{\sqrt{r}}\). We also establish a relation between the conformal factor \(F\) and the generatrix curve \(\alpha(v) = (\varphi(v), v, 0)\) of the rotational flat surfaces in \(E^3_{\sqrt{r}}\).

Theorem 3. Let \(X(u,v) = (\varphi(v) \cos u, \varphi(v) \sin u, v)\) be a rotational surface in \(E^3_F\). Then,

(a) if the conformal factor is \(F(r) = \sqrt{r}\), then all the rotational surface \(X\) are flat in \(E^3_{\sqrt{r}}\);

(b) \(X(u,v)\) is flat in \(E^3_F\) if and only if \(F(\varphi(v)) = e^{\int \frac{1}{2} \sqrt{\varphi'^2 - c_1 \varphi^2} dv}, c_1 \neq 0\);

(c) all the cylinders along \(x_3\)-axis are flat surfaces in \(E^3_F\) for all conformal factor \(F(r), r = x_1^2 + x_2^2\).

Proof. By using equation (10), \(X\) is flat if

\[
-\frac{F^2}{a\varphi} \frac{\partial}{\partial v} \left( \frac{\varphi'(F - 2\varphi^2 \dot{F})}{aF^2} \right) = 0,
\]
where $\dot{F} = \frac{dF}{dr}$ and $a^2 = 1 + \varphi^2$. Therefore

$$\frac{\varphi'(F - 2\varphi^2 \dot{F})}{aF} = c_1. \quad (11)$$

**Proof of (a) and (c).** If $c_1 = 0$, then either $\varphi'(v) = 0, \forall v$, or $F - 2\varphi^2 \dot{F} = 0$. From which we conclude, respectively, that $|\varphi(v)| = R = \text{(constant)} \forall v$ and all the cylinders are flat surfaces, or the conformal factor is $F(r) = \sqrt{r}$ and all the rotational surfaces, given as $X$, are flat surfaces in $E_3^\sqrt{r}$.

**Proof of (b).** If $c_1 \neq 0$, then equation (11) gives us $\varphi'(F - 2\varphi^2 \dot{F}) = c_1 \sqrt{1 + (\varphi')^2 F}$. Equivalently, $\frac{\dot{F}}{F} = \frac{1}{2\varphi^2} \sqrt{1 + (\varphi')^2 + 1}$. By integrating both sides, we get

$$\ln(F(\varphi^2)) = \int \frac{1}{\varphi} [\varphi' - c_1 \sqrt{1 + (\varphi')^2}] dv.$$ 

Therefore, from the above expression, we obtain the theorem. \qed

### 3.2. Rotational surfaces with constant extrinsic curvature in $E_3^F$. In this subsection, we study rotational surfaces with constant extrinsic curvature. We also introduce particular cases for $F$, and we obtain examples of complete conformally flat manifolds. We prove the following result.

**Lemma 3.** Let $X(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, v)$ be a rotational surface in $E_3^F$. Then, $X$ has extrinsic curvature given by

$$\tilde{K}_E = -\frac{1}{a^2 \varphi} (F - 2\varphi^2 \dot{F})(F\varphi'' + 2a^2 \dot{F}\varphi), \quad (12)$$

where $a^2 = 1 + \varphi^2$.

**Proof.** The coefficients of the first fundamental form of $X$, with respect to the Euclidean metric, are given by

$$E = \varphi^2(v), \quad G = 1 + \varphi^2(v) \quad \text{and} \quad F = 0.$$ 

The coefficients of the second fundamental form are given by

$$e = -\frac{\varphi(v)}{a}, \quad g = \frac{\varphi''(v)}{a} \quad \text{and} \quad f = 0.$$ 

By using equation (6), we have that

$$\tilde{\lambda}_i = F(r)\lambda_i - \frac{2\varphi \dot{F}}{a}.$$
where \( \lambda_1 = \frac{1}{a\phi} \) and \( \lambda_2 = \frac{2\phi''}{a^2} \). In this case, the extrinsic curvature is

\[
\tilde{K}_E = \tilde{\lambda}_1\tilde{\lambda}_2 = -\frac{1}{a\phi}(F - 2\phi^2\tilde{F})(F\phi'' + 2a^2\tilde{F}\phi).
\]

This concludes the proof of the lemma.

In the next result, we obtain a relation between the conformal factor \( F \) and the generatrix curve \( \gamma(v) = (\varphi(v),0,v) \) of rotational surface \( X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, v) \) in \( \mathbb{E}^3_F \), with zero extrinsic curvature. By using the obtained relation, we get a characterization of rotational surfaces \( X \) with zero extrinsic curvature in \( \mathbb{E}^3_F \).

**Theorem 4.** Let \( X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, v) \) be a rotational surface in \( \mathbb{E}^3_F \). Then,

(a) if \( F(r) = \sqrt{r} \), then all the rotational surfaces \( X \) have zero extrinsic curvature;

(b) if \( F(r) \neq \sqrt{r} \), \( X \) has zero extrinsic curvature in \( \mathbb{E}^3_F \) if and only if

\[
\varphi(v) = \pm \int \sqrt{\frac{1}{F(r)} - 1}dv + c_2, \text{ where } c_2 \text{ and } c_3 > 0 \text{ are constants};
\]

(c) the planes perpendicular to \( x_3 \)-axis have zero extrinsic curvature for all \( F \).

Proof of (a). Using equation \((12)\), we find that the extrinsic curvature vanishes everywhere, because for \( F(r) = \sqrt{r} \), we have that \( F - 2\phi^2\tilde{F} = 0 \).

Proof of (b). We can consider \( X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, v) \). Using equation \((12)\) for \( F(r) \neq \sqrt{r} \), \( \tilde{K}_E = 0 \) if and only if \( F\phi'' + 2a^2\tilde{F}\phi = 0 \). Hence \( 2a^2\phi\tilde{F} = -F\phi'' \), and then \( \int \frac{\phi}{F}\phi'\phi''dv = -\int \frac{\phi'}{\phi}\phi' dv \). Therefore, \( \ln(F) = -\frac{1}{2}\ln((\phi')^2) + \ln(c_3) \), for a positive constant \( c_3 \). Then \( F = \frac{c_3}{\sqrt{1+(\phi')^2}} \). We can get from this that \( \varphi = \pm \int \sqrt{\frac{c_3}{\phi^2} - 1}dv + c_2 \).

Proof of (c). The planes perpendicular to \( x_3 \)-axis have zero extrinsic curvature owing to equation \((6)\), for all \( F = F(x_1^2 + x_2^2) \). This concludes the proof of the theorem.

Remark. In item (b) above for \( F(r) = \frac{1}{\sqrt{r}} \), up to the planes perpendicular to \( x_3 \)-axis, rotational surface \( X \), with zero extrinsic curvature are the Catenoids with \( x_3 \)-axis. In fact, we can consider \( X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, v) \). Using equation \((12)\) for \( F(r) = \frac{1}{\sqrt{r}} \), \( \tilde{K}_E = 0 \) if and only if \( F\phi'' + 2a^2\tilde{F}\phi = 0 \), it is equivalent to equation \(-\varphi\phi'' + \varphi^2 + 1 = 0 \), whose general solution is \( \varphi(v) = c_1 \cosh(\frac{v}{c_1} + c_2) \). This solution generates the Catenoids.
In the next result, we obtain

**Theorem 5.** Let $C_R$ be the cylinder, of radius $R$ and $x_3$-axis, in $\mathbb{E}^3_F$.

(a) Then, for all $F$, $C_R$ has constant extrinsic curvature given by

$$\tilde{K}_E = 2\dot{F}(R^2)\left(-F(R^2) + 2R^2\dot{F}(R^2)\right).$$  \hspace{1cm} (13)

(b) If $F(r) = e^{-r}$, then $C_R$ has positive constant extrinsic curvature given by

$$\tilde{K}_E = \frac{2(1 + 2R^2)}{e^{2R^2}}.$$  \hspace{1cm} (14)

(c) If $F(r) = \sqrt{r} + 1$, then $C_R$ has negative constant extrinsic curvature given by

$$\tilde{K}_E = -\frac{1}{R}.$$  \hspace{1cm} (15)

Proof of (a). The cylinder of radius $R$ and $x_3$-axis, $C_R$, can be parametrized using $\varphi(v) = R$ in (9). By substituting $\varphi(v) = R$ in (12), we obtain (13).

Proof of (b). As $F(r) = e^{-r}$, we get that $\dot{F} = -F = -e^{-\varphi^2(v)}$ in (13), then we obtain (14).

Proof of (c). By a straightforward calculation, we can verify that substituting $F(r) = \sqrt{r} + 1$ and $\varphi(v) = R$ in equation (13), we obtain that $\tilde{K}_E = -\frac{1}{R}$.

This concludes the proof of the theorem. \hspace{1cm} $\square$

We observe that when $F(r) = e^{-r}$, because it is a bounded function, the space $\mathbb{E}_F^3$ is a complete manifold.

### 3.3. $\mathbb{E}_F^3$: a conformally flat space isometric to $\mathbb{H}^2 \times S^1$.

In this subsection, we study geodesics in $\mathbb{E}_F^3$, where $F(r) = \sqrt{r}$. We show that this special complete conformally flat space is isometric to $\mathbb{H}^2 \times S^1$ with non-positive sectional curvatures. We also show that all the rotational surfaces $X(u, v)$ have zero extrinsic curvature in $\mathbb{E}_F^3$.

In the next result, we study geodesics in $\mathbb{E}_F^3$.

**Theorem 6.** Consider the Riemannian manifold $\mathbb{E}_F^3$. Then,

(a) the circles centered at $(0, 0, x_3)$ and orthogonal to $x_3$-axis are geodesics in $\mathbb{E}_F^3$ if and only if $F(r) = \sqrt{r}$;

(b) if $F(r) = \sqrt{r}$, then the straight lines orthogonal to $x_3$-axis, which are geodesics in $\mathbb{E}_F^3$, are only the perpendicular ones.

Proof of (a). As the translation along $x_3$-axis are isometries, without loss of generality, we will show that only the circles centered at origin, in the plane $x_3 = 0$, are geodesics. Let $\beta(s)$ be a parametrization of such circles, then

$$\beta(s) = (R \cos(Fs/R), R \sin(Fs/R), 0), \quad F = F(R^2).$$
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In fact, the circle $\beta(s) = (R \cos(Fs/R), R \sin(Fs/R), 0)$ is parametrized by arc length, and the components $x_i(s)$ satisfy

$$
x'_1(s) = -F \sin(Fs/R), \quad x'_2(s) = F \cos(Fs/R),
$$
$$
x''_1(s) = -F^2/R \sin(Fs/R), \quad x''_2(s) = -F^2/R \cos(Fs/R),
$$

Then, by using the expressions of the Christoffel symbols from (3), we get that $\beta$ is geodesic if and only if the following system is satisfied:

$$
-F^2 \cos\left(\frac{Fs}{R}\right) \left[\frac{1}{R} - 2\frac{\dot{F}}{F}\right] = 0,
$$
$$
-F^2 \sin\left(\frac{Fs}{R}\right) \left[\frac{1}{R} - 2\frac{\dot{F}}{F}\right] = 0. \quad (15)
$$

This system (15) is satisfied if and only if $1 - 2\frac{\dot{F}}{F} = 0$, namely $F(r) = \sqrt{r}$.

**Proof of (b).** Suppose that a straight line $l$, orthogonal to $x_3$-axis, $l \cap Ox_3 = \emptyset$, is a geodesic. Then, there exists a circle centered at the $x_3$-axis that is tangent to $l$ at a point $p$. By item (a), this circle is a geodesic, this gives us a contradiction with respect to the uniqueness of the geodesic through the point $p$. This concludes the proof of the theorem. $\Box$

Now, we choose $F(r) = \sqrt{r}$, and we consider the space $\mathbb{E}^3_{\sqrt{r}} := (\mathbb{R}^3 \setminus \{x_3 \text{-axis}\}, \langle \cdot, \cdot \rangle_g)$ endowed with the conformal metric.

**Theorem 7.** Let $\mathbb{E}^3_{\sqrt{r}}$ be the ambient space. Then, $\mathbb{E}^3_{\sqrt{r}}$ is complete with non-positive sectional curvatures given by

$$
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 0 \quad \text{and} \quad K\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_3}\right) = -\frac{x_\alpha^2}{(x_1^2 + x_2^2)^2}, \quad 1 \leq \alpha \leq 2.
$$

**Proof.** The metric in the space $\mathbb{E}^3_{\sqrt{r}}$ is given by $g = \frac{1}{x_1^2 + x_2^2}(x_1^2 dx_1^2 + x_2^2 dx_2^2 + dx_3^2)$. If we consider the new parameters $(r, t)$, given by $x_1 = r \cos t, \ x_2 = r \sin t$, then the metric can be written as $g = \frac{1}{r^2}(dr^2 + dx_3^2) + dt^2$. Therefore, the ambient space is exactly $\mathbb{H}^2 \times S^1$; thus proving that the space $\mathbb{E}^3_{\sqrt{r}}$ is a complete manifold. Finally, as $F(r) = \sqrt{r}$ and by (7), the sectional curvatures satisfy

$$
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 0 \quad \text{and} \quad K\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_3}\right) = -\frac{x_\alpha^2}{(x_1^2 + x_2^2)^2}, \quad 1 \leq \alpha \leq 2.
$$

This concludes the proof of the theorem. $\Box$
Remark A. In Theorem 3, we show that all the rotational surfaces parametrized by $X(u,v) = (\varphi(v) \cos u, \varphi(v) \sin u,v)$ are flat surfaces in the special ambient space $E^3_{\sqrt{r}}$.

Remark B. In Theorem 4 we show that all the rotational surfaces parametrized by $X(u,v) = (\varphi(v) \cos u, \varphi(v) \sin u,v)$ have zero extrinsic curvature in the special ambient space $E^3_{\sqrt{r}}$.

We conclude this work by presenting the conformal space $E^3_F$, where the conformal factor is $F = F(r) = \frac{1}{1-r^2}$, $r < 1$, which is equivalent to the warped product $H^2 \times f \mathbb{R}$, where $f = \frac{1}{r}$. The warped product generalizes the product space $H^2 \times \mathbb{R}$. We have shown that $E^3_F$ is a complete space foliated by complete surfaces of constant Gaussian curvature $-1$.

We strongly believe that this complete space $E^3_F$ has a great potential for the purpose of constructing minimal immersions and constant mean curvature immersions.

Finally, we establish the following result:

**Theorem 8.** Let $H^2 \times f \mathbb{R}$ be the warped product space, where $f = \frac{1}{r}$. Then $H^2 \times f \mathbb{R}$ is

(a) a complete space foliated by complete surfaces with constant Gaussian curvature $-1$;

(b) isometric to neither $H^2 \times \mathbb{R}$ nor $H^3$.

**Proof of (a).** Let $(B,g_B)$ and $(A,g_A)$ be Riemannian manifolds, and $f$ be a positive differentiable function defined on $B$. The warped product $M = B \times f A$ is the product manifold equipped with the metric $\tilde{g} = g_B + f^2 g_A$, where $B$ is called the base, $A$ the fiber, and $f$ is called a torsion function on $B$. We know by [4] that $(M,\tilde{g})$ is a complete manifold if and only if $(B,g_B)$ and $(A,g_A)$ are complete manifolds. We prove that $E^3_{\frac{1}{1-r^2}}$ is the warped product $H^2 \times f \mathbb{R}$, where $f(r) = \frac{1}{r(r^2)} = \frac{2}{1-r^2}$, then $H^2 \times f \mathbb{R}$ is a complete space. As $H^2 \times \{p_0\}$, where $p_0 \in \mathbb{R}$, is isometric to the $H^2$, we have that $H^2 \times f \mathbb{R}$ is foliated by complete surfaces of constant Gaussian curvature $-1$.

**Proof of (b).** As the torsion function $f(r) = \frac{2}{1-r^2}$, $r < 1$, is not a constant function, we can conclude that $H^2 \times f \mathbb{R}$ is isometric to neither $H^2 \times \mathbb{R}$ nor $H^3$. □

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