A remark on an algorithm for testing Pisot polynomials

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Abstract. We improve an algorithm, due to R. J. Duffin, to test minimal polynomials of Pisot numbers.

1. Introduction

A Schur polynomial is a non-constant polynomial with complex coefficients whose roots are of modulus less than one. In [3], Duffin proposed the following rule, based on the classical Schur–Cohn algorithm (named after [2], [5]), for testing Schur polynomials.

Proposition 1 ([3]). Let $F(x) = c_0 + c_1x + \cdots + cx^d \in \mathbb{C}[x]$, where $d = \deg(F) \geq 1$, and let $F^*(x) := x^dF(1/x) = c_0x^d + c_1x^{d-1} + \cdots + c_d$. Then, $F$ is a Schur polynomial if and only if $|c_d| > |c_0|$, and

$$\frac{\overline{c_d}F(x) - c_0F^*(x)}{x}$$

is also a Schur polynomial, when it is not constant.

Setting $F(0) := F$, and inductively

$$F_{i+1}(x) := \frac{\overline{m_i}F_i(x) - n_iF^*_i(x)}{x},$$

where $m_i$ and $n_i$ are, respectively, the leading and constant coefficients of $F_i$ ($m_0 = c_d$ and $n_0 = c_0$), we see that

$F$ is a Schur polynomial $\Leftrightarrow |m_i| > |n_i|$, $\forall i \in \{0, 1, \ldots, d - 1\}$.

Mathematics Subject Classification: 11R06, 11R09.

Key words and phrases: Pisot polynomials, Schur polynomials, Pisot numbers.

I would like to thank the referees for reading these pages very carefully and for having indicated some related references.
A monic polynomial $P \in \mathbb{Z}[x]$, satisfying $P(0) \neq 0$, is said to be a Pisot polynomial if $P$ has a unique root, say $\theta$, with modulus greater than one, and no root with modulus one. In this case, $\theta \in \mathbb{R}$, $P$ is irreducible (over $\mathbb{Q}$), and $|\theta|$ is a Pisot number, that is, a real algebraic integer with modulus greater than one whose other conjugates are of modulus less than one. Clearly, if $\deg(P) = 1$ (resp. if $\deg(P) = 2$), then $P$ is a Pisot polynomial if and only if $|P(0)| \neq 1$ (resp. $P(-1)P(1) < 0$). In [4], Duffin formulated an algorithm, for testing Pisot polynomials, with degree at least 3. This algorithm is very simple when the corresponding constant coefficients have modulus greater than 1.

**Proposition 2 ([4]).** Let $P(x) = k_0 + k_1 x + \cdots + k_{d-1} x^{d-1} + x^d \in \mathbb{Z}[x]$, where $d \geq 3$ and $|k_0| \geq 2$. Then, $P$ is a Pisot polynomial if and only if $k_0 P - P^\ast$ is a Schur polynomial (of degree $d - 1$).

Using Propositions 2 and 1, we may easily check whether a monic polynomial $P \in \mathbb{Z}[x]$ is a Pisot polynomial, when $|P(0)| \geq 2$ and $\deg(P) \geq 3$; this algorithm involves at most $\deg(P) - 1$ rational steps (only integer arithmetic is used).

Clearly, Proposition 2 is not true when $|P(0)| = 1$, and in this case Duffin [4] found another test based on the application of similar computations to a certain multiple of $P$, with degree $\deg(P) + 1$. The aim of the present note is to give a related simple test independent of the value of $P(0)$ and involving at most $\deg(P) - 3$ rational steps (see Example 1 below, where the number of steps is zero). Although the application of this algorithm, presented in the next section, to some family of polynomials shows that it is better than the corresponding one in [4], we are still far from the hope to obtain a test which is independent of the degree of the Pisot polynomial and involves only a finite number of steps.

2. A variation on a test of Duffin

In a similar way as in the propositions above, and using Rouche’s theorem and Hurwitz’s theorem, we easily obtain the following assertion.

**Proposition 3.** Let $P(x) = k_0 + k_1 x + \cdots + k_{d-1} x^{d-1} + x^d \in \mathbb{Z}[x]$, where $d \geq 3$. Then,

$$
F(x) := \frac{(k_0 x^2 + (k_1 - k_0 k_{d-1}) x - k_0) P(x) - (x^2 - k_0^2) P^\ast(x)}{x^2} \in \mathbb{Z}[x],
$$

$\deg(F) \leq d - 2$, and $P$ is a Pisot polynomial if and only if $k_0 \neq 0$,

$$
|k_1 - k_0 k_{d-1}| > |k_0^2 - 1|,
$$

and $F$ is a Schur polynomial (of degree $d - 2$).
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PROOF. A short calculation shows that the first (resp. the last) two coefficients of the polynomial

\[ N(x) := (k_0 x^2 + (k_1 - k_0 k_{d-1}) x - k_0) P(x) - (x^2 - k_0^2) P^s(x) \]

are zeros, and so \( F(x) = N(x)/x^2 \in \mathbb{Z}[x] \), with \( \deg(F) \leq d - 2 \). Also, for every complex number \( z \) with modulus one, we have

\[ |k_0 z^2 + (k_1 - k_0 k_{d-1}) z - k_0| > |z^2 - k_0^2| \Leftrightarrow |k_1 - k_0 k_{d-1}| > |k_0^2 - 1|. \quad (2) \]

Now, suppose that \( P \) is a Pisot polynomial. Then, \( k_0 \neq 0 \) and \( P(x) - k_0 P^s(x) \) is not identically zero, as \( d > 2 \). To prove inequality (1), when \( |k_0| = 1 \), consider the family of polynomials \( R_\lambda(x) = P(x) - k_0 \lambda P^s(x) \), where \( \lambda \) is a real number greater than one. Clearly, \( \lim_{\lambda \to 1+} R_\lambda(x) = P(x) - k_0 P^s(x) \), the first (resp. the second) coefficient of the polynomial \( P(x) - k_0 P^s(x) \) is 0 (resp. is \( k_1 - k_0 k_{d-1} \)), and Rouché’s theorem gives that \( R_\lambda(x) \) has exactly one root with modulus less than one, since \( \lambda > 1 \). It follows by Hurwitz’s theorem that \( k_1 - k_0 k_{d-1} \neq 0 \), yielding (1). For the case \( |k_0| > 1 \), consider the polynomial \( k_0 P(x) - P^s(x) \). Also, Rouché’s theorem implies that \( k_0 P(x) - P^s(x) \) has \((d-1)\) roots with modulus less than one, and as \( k_0 P(x) - P^s(x) = (k_0 k_{d-1} - k_1) x^{d-1} + \cdots + (k_0^2 - 1) \), we deduce that \( k_0 P(x) - P^s(x) \) is a Schur polynomial, and so (1) holds.

It follows, by (2) and again Rouché’s theorem, that \( N \) has \( d \) roots with modulus less than one (if we evaluate the polynomial \( k_0 x^2 + (k_1 - k_0 k_{d-1}) x - k_0 \) at \pm 1, we immediately see that it has a root in \((-1,1)\) and a root with modulus greater than one), and so \( F \) has exactly \((d-2)\) roots with modulus less than one. Therefore, \( \deg(F) = d - 2 \) and \( F \) is a Schur polynomial. In a similar manner, we show the “if” part of the equivalence in Proposition 3, as the condition \( k_0 \neq 0 \) implies that \( P \) is a Pisot polynomial when \( P \) has \( d - 1 \) roots with modulus less than one.

Example 1. Consider the polynomial \( P(x) = x^d - x^{d-1} - x^{d-2} + \varepsilon x^2 - \varepsilon, \) where \( \varepsilon \in \{-1,1\} \) and \( d \geq 3 \). With the notation above, we have \( k_1 - k_0 k_{d-1} = -\varepsilon \neq 0 \) and \( F(x) = \varepsilon x^{d-2} \); thus \( F \) is a Schur polynomial, and so, by Proposition 3, \( P \) is a Pisot polynomial (it is well known, see, for instance [1], that \( P \) is the minimal polynomial of a Pisot number \( \theta_d \) and \( \lim_{d \to \infty} \theta_d = (1 + \sqrt{5})/2 \)). Let us now apply the corresponding algorithm of [4] to the same polynomial \( P \). Following the related rule in [4], the multiple of \( P \) is \( M(x) := (x + 2) P(x) = x^{d+1} + \cdots + (-2 \varepsilon) \). Then, we compute \( Q(x) := 2 \varepsilon M(x) + M^*(x) = \varepsilon x^d + \cdots + (-3) \) (step one) and \( R(x) := 3Q(x) + \varepsilon Q^*(x) = -8 - 7x - 8 \varepsilon x^{d-2} - 13 \varepsilon x^{d-1} \) (step two); the test says that \( P \) is a Pisot polynomial if and only if \( R \) is a Schur polynomial. Hence,
we need more steps to prove that $P$ is a Pisot polynomial, since it is not trivial that $R$ is a Schur polynomial.

Example 2. Let $P(x) = x^6 - 2x^5 + x^4 - x^2 + x - 1$. Then, $|k_1 - k_0 k_5| = 1 \neq 0$, $F(x) = 2x^4 - 2x^3 + 2x - 1$, $F_1(x) = 3x^3 - 2x^2 + 2$, $F_2(x) = 5x^2 - 6x + 4$ and $F_3(x) = 9x - 6$. Therefore, for each $F_i$, where $i \in \{0, 1, 2, 3\}$, the leading coefficient is greater than the absolute value of the constant coefficient, and so, by Proposition 1, $F$ is a Schur polynomial. It follows, by Proposition 3, that $P$ is the minimal polynomial of a Pisot number, as $P(1) < 0$.

Example 3. To show that $P(x) = x^4 - 3x^3 - x^2 + x + 1$ is a Pisot polynomial, Duffin [4] determined a multiple of $P$ of degree 5, and after three steps he obtained a quadratic Schur polynomial, leading to the desired conclusion. This result is trivial, using Proposition 3, since $|k_1 - k_0 k_3| = 4 \neq 0$, $F(x) = -4(3x^2 - x - 1)$, $F(-1) = -12$, $F(0) = 4$ and $F(1) = -4$.

References