On Nemytskii operator in the space of set-valued functions of bounded $p$-variation in the sense of Riesz

By N. MERENTES (Caracas) and S. RIVAS (Caracas)

Abstract. We consider the Nemytskii operator, i.e. the composition operator defined by $(Nu)(t) = H(t, u(t))$, where $H$ is a given set-valued function. It is shown that if the operator $N$ maps the space of set-valued functions of bounded $p$-variation in the sense of Riesz into the space of set-valued functions of bounded $q$-variation in the sense of Riesz, there is $1 \leq q \leq p < \infty$, and if it is globally Lipschitzian, then it has to be of the form $(Nu)(t) = A(t)u(t) + B(t)$, where $A(t)$ are linear continuous set-valued and $B$ is a set-valued function of bounded $q$-variation in the sense of Riesz. This generalizes results of G. Zawadzka [8], A. Smajdor and W. Smajdor [7], N. Merentes and K. Nikodem [3].

Introduction

In [7] A. Smajdor and W. Smajdor proved that every Nemytskii operator $N$, i.e. $(Nu)(t) = H(t, u(t))$ mapping the space $\text{Lip}([a, b], cc(Y))$ into itself and globally Lipschitzian has to be of the form

$$(Nu)(t) = A(t)u(t) + B(t), \quad u \in \text{Lip}([a, b], cc(Y)), \quad t \in [a, b],$$

where $A(t)$ are linear continuous set-valued functions and $B$ is a set-valued function belonging to the space $\text{Lip}([a, b], cc(Y))$. For the first time a theorem of such a type for single-valued functions was proved by J. Matkowski [1] in the space of Lipschitz functions. Similar characterizations of the Nemytskii operator have been also obtained by G. Zawadzka (see [8]) in the space of set-valued functions of bounded variation in the classical Jordan sense. For single-valued functions it was proved by

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J. Matkowski and J. Miś [2]. Recently, N. Merentes and K. Nikodem (see [3]) proved an analogous theorem in the space of set-valued functions of bounded $p$-variation in the sense of Riesz. The aim of this paper is to prove an analogous result in the case when the Nemytskii operator $N$ maps the space of set-valued functions of bounded $p$-variation in the sense of Riesz into the space of set-valued functions of bounded $q$-variation in the sense of Riesz, where $1 \leq q \leq p < \infty$ and $N$ is globally Lipschitzian. The particular cases $p = q$ have been already considered by N. Merentes and K. Nikodem (see [3]), but the present case of possibly different spaces requires a different proof technique, and this extension may turn out to be useful in some applications.

1. Preliminary results

Let $(X, \| \cdot \|)$ be a normed space and $p \geq 1$ be a fixed number. Given a function $u : [a, b] \to X$ and a partition $\pi : a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$, we define:

$$\sigma_p(u; \pi) := \sum_{i=1}^{n} \frac{\|u(t_i) - u(t_{i-1})\|^p}{|t_i - t_{i-1}|^{p-1}}.$$ 

The number:

$$V_p(u, [a, b]; X) := \sup_{\pi} \sigma_p(u, \pi),$$ 

where the supremum is taken over all partitions $\pi$ of $[a, b]$, is called the $p$-variation of $u$ in $[a, b]$. A function $u$ is said to be of bounded $p$-variation if $V_p(u, [a, b]; X) < \infty$. Denote by $RV_p([a, b], X)$ the space of all functions $u : [a, b] \to X$ of bounded $p$-variation equipped with the norm

$$\|u\|_p := \|u(a)\| + (V_p(u, [a, b]; X))^\frac{1}{p}.$$ 

Clearly, for $p = 1$ the space $RV_1([a, b], X)$ coincides with the classical space $BV([a, b], X)$ of functions of bounded variation. In the particular case when $X = \mathbb{R}$ and $1 < p < \infty$, then we have the space $RV_p[a, b]$ of functions of bounded Riesz $p$-variation, and the following characterization is well-known:

**Lemma 1** (see [5]). $u \in RV_p([a, b], \mathbb{R})$ if and only if $u$ is absolutely continuous on $[a, b]$ and its derivative $u' \in L^p([a, b]; \mathbb{R})$. In that case we also have the equality

$$V_p(u, [a, b]; \mathbb{R}) = \int_a^b |u'(t)|^p dt.$$
Let $cc(X)$ be the family of all non-empty convex compact subsets of $X$ and $D$ be the Hausdorff metric in $cc(X)$, i.e.

$$D(A, B) := \inf \{ t > 0 : A \subseteq B + tS, \ B \subseteq A + tS \},$$

where $S = \{ y \in X : \| y \| \leq 1 \}$.

We say that a set-valued function $F : [a, b] \rightarrow cc(X)$ has bounded $p$-variation ($1 \leq p < \infty$) if

$$W_p(F, [a, b]; cc(X)) := \sup_{\pi} \sum_{i=1}^{n} \left( \frac{D(F(t_i), F(t_{i-1}))^p}{|t_i - t_{i-1}|^{p-1}} \right) < \infty,$$

where the supremum is taken over all partitions $\pi$ of $[a, b]$.

Denote by $RW_p([a, b]; cc(X))$ the space of all set-valued functions $F : [a, b] \rightarrow cc(X)$ of bounded $p$-variation equipped with the metric

$$D_p(F_1, F_2) := \left( D(F_1(a), F_2(a)) + \left( \sup_{\pi} \sum_{i=1}^{n} \left( \frac{D(F_1(t_i) + F_2(t_{i-1}), F_1(t_{i-1}) + F_2(t_i))^p}{|t_i - t_{i-1}|^{p-1}} \right) \right)^{\frac{1}{p}} \right).$$

Clearly, for $p = 1$ the space $RW_1([a, b]; cc(X))$ coincides with the space $BV([a, b]; cc(X))$ of set-valued functions of bounded variation.

Now, let $(X, \| \cdot \|), (Y, \| \cdot \|)$ be two normed spaces and $K$ be a convex cone in $X$. Given a set-valued function $H : [a, b] \times K \rightarrow cc(Y)$ we consider the Nemytskii operator $N$ generated by $H$, that is the composition operator defined by:

$$(Nu)(t) := H(t, u(t)), \quad u : [a, b] \rightarrow K, \quad t \in [a, b].$$

We denote by $L(K; cc(Y))$ the space of all set-valued function $A : K \rightarrow cc(Y)$ additive and positively homogeneous. We say that $A$ is linear if $A \in L(K; cc(Y))$.

In the proof of the main results of this paper we will use some facts which we list here as lemmas.

**Lemma 2** (see [6], Lemma 3). Let $(X, \| \cdot \|)$ be a normed space and let $A, B, C$ be subsets of $X$. If $A, B$ are convex compact and $C$ is non-empty and bounded, then

$$D(A + C, B + C) = D(A, B).$$
Lemma 3 (see [4], Th. 5.6). Let \((X, \| \cdot \|), (Y, \| \cdot \|)\) be normed spaces and \(K\) be a convex cone in \(X\). A set-valued function \(F : K \to \text{cc}(Y)\) satisfies the Jensen equation

\[
F\left(\frac{x+y}{2}\right) = \frac{1}{2}(F(x) + F(y)), \quad x, y \in K,
\]

if and only if there exists an additive set-valued function \(A : K \to \text{cc}(Y)\) and a set \(B \in \text{cc}(Y)\) such that

\[
F(x) = A(x) + B, \quad x \in K.
\]

Lemma 4. If \(F \in RW_p([a, b], \text{cc}(Y))\) with \(p > 1\), then \(F\) is continuous. In the case \(p = 1\), we have \(F^-(\cdot, x) \in BW([a, b], \text{cc}(Y))\) for all \(x \in K\), where

\[
F^-(t, x) := \begin{cases} \lim_{s \uparrow t} F(s, x), & t \in (a, b), \ x \in K, \\ F(a, x), & t = a, \ x \in K. \end{cases}
\]

PROOF. For \(1 < p < \infty\), this follows immediately from the inequality

\[
D(F(t), F(t_0)) = \left(\frac{(D(F(t), F(t_0)))^p|t - t_0|^{p-1}}{|t - t_0|^{p-1}}\right)^{\frac{1}{p}}
\leq W_p(F, [a, b]; \text{cc}(Y))|t - t_0|^{1-\frac{1}{p}}.
\]

For the case \(p = 1\), see [8].

2. Main results

In this section we shall present a characterization of functions \(H : [a, b] \times K \to \text{cc}(Y)\) for which the Nemytskii operator \(N\) generated by \(H\) maps the space \(RV_p([a, b], K)\) into the space \(RW_q([a, b], \text{cc}(Y))\), where \(1 < q < p\), and it is globally Lipschitzian. On the other hand if \(1 < p < q\), then the Nemytskii operator \(N\) is constant.

Theorem 1. Let \((X, \| \cdot \|), (Y, \| \cdot \|)\) be normed spaces and \(K\) be a convex cone in \(X\) and \(1 < q < p\). If the Nemytskii operator \(N\) generated by a set-valued function \(H : [a, b] \times K \to \text{cc}(Y)\) maps the space \(RV_p([a, b], K)\) into the space \(RW_q([a, b], \text{cc}(Y))\) and if it is globally Lipschitzian, then the set-valued function \(H\) satisfies the following conditions:

a) For all \(t \in [a, b]\) there exists \(M(t)\), such that

\[
D(H(t, x), H(t, y)) \leq M(t)\|x - y\| \quad (x, y \in X)
\]
b) \( H(t, x) = A(t)x + B(t) \) (\( t \in [a, b], \ x \in K \)),
where \( A : [a, b] \to L(K, cc(Y)) \) and \( B \in RW_q([a, b], cc(Y)) \).

**Proof.** The Nemytskii operator \( N \) is globally Lipschitzian, then there exists a constant \( M \), such that
\[
D_q(Nu_1, Nu_2) \leq M\|u_1 - u_2\|_p \quad (u_1, u_2 \in RV_p([a, b], K)).
\]

Let \( t \in (a, b) \). Using the definition of the operator \( N \) and of the metric \( D_q \) we have
\[
D_q(H(t, u_1(t))) + H(a, u_2(a)), H(a, u_1(a)) + H(t, u_2(t)) \leq 
M|t - a|^{1 - \frac{1}{q}}\|u_1 - u_2\|_p, \quad (u_1, u_2 \in RV_p([a, b], K)).
\]

Define the function \( \alpha : [a, b] \to [0, 1] \) by:
\[
\alpha(\tau) := \begin{cases} 
\frac{\tau - a}{t - a}, & a \leq \tau \leq t, \\
1, & t \leq \tau \leq b.
\end{cases}
\]

The function \( \alpha \in RV_p[a, b] \) and
\[
V_p(\alpha; [a, b]) = \frac{1}{|t - a|^{p - 1}}.
\]

Let us fix \( x, y \in K \) and define the functions \( u_i : [a, b] \to K \) (\( i = 1, 2 \)) by:
\[
(3) \quad u_1(\tau) := x, \quad \tau \in [a, b], \quad u_2(\tau) := \alpha(\tau)(y - x) + x, \quad \tau \in [a, b].
\]

The functions \( u_i \in RV_p([a, b], K) \) (\( i = 1, 2 \)) and
\[
\|u_1 - u_2\|_p = (V_p(\alpha; [a, b]))^{\frac{1}{q}} \|x - y\| = \frac{|x - y|}{|t - a|^{1 - \frac{1}{p}}},
\]

Hence, substituting in inequality (2) the particular functions \( u_i \) (\( i = 1, 2 \)) defined by (3), we obtain
\[
(4) \quad D(H(t, x) + H(a, x), H(t, y)) \leq M\left|\frac{t - a}{|t - a|^{1 - \frac{1}{q}}}ight| \|x - y\|,
\]
for all \( t \in [a, b], \ x, y \in K \).

By Lemma 2 and the inequality (4) we have
\[
D(H(t, x), H(t, y)) \leq M\left|\frac{t - a}{|t - a|^{1 - \frac{1}{p}}}ight| \|x - y\|,
\]
for all \( t \in (a, b), x, y \in K \).
Now, let \( t = a \). Define the function \( \beta : [a, b] \to [0, 1] \) by
\[
\beta(\tau) := \frac{\tau - a}{b - a}, \quad (\tau \in [a, b]).
\]
The function \( \beta \in RV_p[a, b] \) and
\[
\beta(\tau) = \frac{1}{|b - a|^{p - 1}}.
\]

Let us fix \( x, y \in K \) and define the functions \( u_i : [a, b] \to K \) \((i = 1, 2)\) by
\[
(5) \quad u_1(\tau) := x \quad (\tau \in [a, b]), \quad u_2(\tau) := \beta(\tau)(x - y) + y, \quad (\tau \in [a, b]).
\]
The functions \( u_i \in RV_p([a, b], K) \) \((i = 1, 2)\) and
\[
\|u_1 - u_2\|_p = \left(1 + (V_p(\beta; [a, b]))^{\frac{1}{p}}\right)\|x - y\| = \left(1 + \frac{1}{|b - a|^{1 - \frac{1}{p}}}\right)\|x - y\|.
\]

Hence, substituting in the inequality (2), the particular functions \( u_i \) \((i = 1, 2)\) defined by (5), we obtain
\[
D(H(b, x) + H(a, y), H(a, x) + H(b, x)) \leq \frac{1}{|b - a|^{1 - \frac{1}{q}} \left(1 + \frac{1}{|b - a|^{1 - \frac{1}{p}}}\right)}\|x - y\|.
\]

By Lemma 2 and the above inequality, we have
\[
D(H(a, y), H(a, x)) \leq M|b - a|^{1 - \frac{1}{q}} \left(1 + \frac{1}{|b - a|^{1 - \frac{1}{p}}}\right)\|x - y\|.
\]

Define the function \( M : [a, b] \to \mathbb{R} \) by
\[
M(t) := \begin{cases} 
M\frac{|t - a|^{1 - \frac{1}{q}}}{|t - a|^{1 - \frac{1}{p}}}, & a < t \leq b, \\
M|b - a|^{1 - \frac{1}{q}} \left(1 + \frac{1}{|b - a|^{1 - \frac{1}{p}}}\right), & t = a.
\end{cases}
\]
Hence
\[
D(H(t, x), H(t, y)) \leq M(t)\|x - y\| \quad (x, y \in X, \ t \in [a, b]),
\]
and, consequently, for every \( t \in [a, b] \) the function \( H(t, \cdot) : K \to cc(Y) \) is continuous.
Next we shall prove that $H$ satisfies equality b).

Let us fix $t, t_0 \in [a, b]$ such that $t_0 < t$. Since the Nemytskii operator $N$ is globally Lipschitzian, there exists a constant $M$, such that

$$D(H(t, u_1(t)) + H(t_0, u_2(t_0)), H(t, u_1(t_0)) + H(t, u_2(t))) \leq \leq M\|u_1 - u_2\|_p |t - t_0|^{1 - \frac{1}{q}}. \tag{6}$$

Define the function $\gamma : [a, b] \to [0, 1]$ by

$$\gamma(\tau) := \begin{cases} \frac{\tau - a}{t_0 - a}, & a \leq \tau \leq t_0, \\ -\frac{\tau - t}{t - t_0}, & t_0 \leq \tau \leq t, \\ 0, & t \leq \tau \leq b. \end{cases}$$

The function $\gamma \in RV_p[a, b]$. Let us fix $x, y \in K$ and define the functions $u_i : [a, b] \to K$ by

$$u_1(\tau) := \frac{\gamma(\tau)}{2} x + \left(1 - \frac{\gamma(\tau)}{2}\right) y, \quad (\tau \in [a, b]) \tag{7}$$

$$u_2(\tau) := \frac{1 + \gamma(\tau)}{2} x + \frac{1 - \gamma(\tau)}{2} y, \quad (\tau \in [a, b]).$$

The functions $u_i \in RV_p([a, b], K)$ ($i = 1, 2$) and

$$\|u_1 - u_2\|_p = \|x - y\|_p = \frac{1}{2}. \tag{8}$$

Hence, substituting in the inequality (6) the particular functions $u_i$ ($i = 1, 2$) defined by (7), we obtain

$$D \left( H(t_0, x) + H(t, y), H \left( t_0, \frac{x + y}{2} \right) + H \left( t, \frac{x + y}{2} \right) \right) \leq \leq \frac{M}{2} |t - t_0|^{1 - \frac{1}{q}} \|x - y\|. \tag{8}$$

Since $N$ maps $RV_p([a, b], K)$ into $RW_q([a, b], cc(Y))$ ($1 < q < p$), then $H(\cdot, z)$ is continuous for all $z \in K$. Hence, letting $t_0 \uparrow t$ in the inequality (8), we get

$$D \left( H(t, x) + H(t, y), H \left( t, \frac{x + y}{2} \right) + H \left( t, \frac{x + y}{2} \right) \right) = 0,$$

for all $t \in [a, b]$ and $x, y \in K$. 

Thus for all $t \in [a, b]$, $x, y \in K$, we have
\[
H \left( t, \frac{x + y}{2} \right) + H \left( t, \frac{x + y}{2} \right) = H(t, x) + H(t, y).
\]

Since that values of $H$ are convex, we have
\[(9)\]
\[
H \left( t, \frac{x + y}{2} \right) = \frac{1}{2}(H(t, x) + H(t, y)),
\]
for all $t \in [a, b]$, $x, y \in K$. Thus for all $t \in [a, b]$, the set-valued function $H(t, \cdot) : K \to cc(Y)$ satisfies the Jensen equation (9). Now by the Lemma 3, there exists an additive set-valued function $A(t) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that
\[
H(t, x) = A(t)(x) + B(t), \quad (x \in K, \ t \in [a, b]).
\]

Substituting $H(t, x) = A(t)(x) + B(t)$ into inequality (1), we obtain, for all $t \in [a, b]$ that there exists $M(t)$, such that
\[
D(A(t)(x), A(t)(y)) \leq M(t)\|x - y\| \quad (x, y \in K),
\]
consequently, the set-valued function $A(t) : K \to cc(Y)$ is continuous, and $A(t)(\cdot) \in L(K, cc(Y))$.

$A(t)(\cdot)$ is additive and 0 $\in K$, then $A(t) = \{0\}$, thus $H(\cdot, 0) = B(\cdot)$.

The Nemytskii operator $N$ maps the space $RV_p([a, b], K)$ into the space $RW_q([a, b], cc(Y))$, then $H(\cdot, 0) = B(\cdot) \in RW_q([a, b], K)$. Consequently the set-valued function $H$ has to be of the form
\[
H(t, x) = A(t)(x) + B(t),
\]
for all $t \in [a, b]$, $x \in K$, where $A(t) \in L(K, cc(Y))$ and $B \in RW_q([a, b], cc(Y))$.

**Theorem 2.** Let $(X, \| \cdot \|)$, $(Y, \| \cdot \|)$ be normed spaces, $K$ a convex cone in $X$ and $1 < p < q$. If the Nemytskii operator $N$ generated by a set-valued function $H : [a, b] \times K \to cc(Y)$ maps the space $RV_p([a, b], K)$ into the space $RW_q([a, b], cc(Y))$ and if it is globally Lipschitzian, then the set-valued function $H$ satisfies the following condition
\[
H(t, x) = H(t, 0) \quad (t \in [a, b], \ x \in K);
\]
i.e. the Nemytskii operator is constant.

**Proof.** Since the Nemytskii operator $N$ is globally Lipschitzian between $RV_p([a, b], K)$ and the space $RW_q([a, b], cc(Y))$, $1 < p < q$, then there exists a constant $M$, such that
\[
D_q(Nu_1, Nu_2) \leq M\|u_1 - u_2\|_p \quad (u_1, u_2 \in RV_p([a, b], K)).
\]
Let us fix \( t, t_0 \in [a, b] \) such that \( t_0 < t \). Using the definitions of the operator \( N \) and of the metric \( D_q \), we have

\begin{equation}
D(H(t, u_1(t)) + H(t_0, u_2(t_0)), H(t_0, u_1(t_0)) + H(t, u_2(t)) \leq M|t - t_0|^{1 - \frac{q}{p}}\|u_1 - u_2\|_p, \quad (u_1, u_2) \in RV_p([a, b], K).
\end{equation}

Define the function \( \alpha : [a, b] \to [0, 1] \) by

\[
\alpha(\tau) := \begin{cases} 
1, & a \leq \tau \leq t_0, \\
-\frac{\tau - t}{t - t_0}, & t_0 \leq \tau \leq t, \\
0, & t \leq \tau \leq b.
\end{cases}
\]

The function \( \alpha \in RV_p[a, b] \) and

\[
V_p(\alpha; [a, b]) = \frac{1}{|t - t_0|^{p-1}}.
\]

Let us fix \( x \in K \) and define the functions \( u_i : [a, b] \to K \) \((i = 1, 2)\) by

\begin{equation}
(11)\quad u_1(\tau) := x \quad \tau \in [a, b], \quad u_2(\tau) := \alpha(\tau)x \quad \tau \in [a, b].
\end{equation}

The functions \( u_i \in RV_p([a, b], K) \) \((i = 1, 2)\) and

\[
\|u_1 - u_2\|_p = \frac{\|x\|}{|t - t_0|^{1 - \frac{1}{p}}}.
\]

Hence, substituting in the inequality \(10\) the particular functions \( u_i \) \((i = 1, 2)\) defined by \((11)\), we obtain

\[
D(H(t, x) + H(t_0, x), H(t_0, x) + H(t, 0)) \leq M|t - t_0|^{1 - \frac{1}{q}}\|x\|.
\]

By Lemma 2 and the above inequality, we get

\[
D(H(t, x), H(t, 0)) \leq M\frac{|t - t_0|^{1 - \frac{1}{q}}}{|t - t_0|^{1 - \frac{1}{p}}}\|x\|.
\]

Since \( q > p \). Letting \( t_0 \uparrow t \) in the above inequality, we have \( D(H(t, x), H(t, 0)) = 0 \), thus for all \( t \in [a, b] \) and for all \( x \in K \), we get

\[
H(t, x) = H(t, 0).
\]
Theorem 3. Let \((X, \| \cdot \|), (Y, \| \cdot \|)\) be normed spaces, \(K\) be a convex cone in \(X\) and \(1 < p < \infty\). If the Nemytskii operator \(N\) generated by a set-valued function \(H : [a, b] \times X \rightarrow cc(Y)\) maps the space \(RV_p ([a, b], K)\) into the space \(BW ([a, b], cc(Y))\) and if it is globally Lipschitzian, then the left regularization \(H^* : [a, b] \times K \rightarrow cc(Y)\) of the function \(H\) defined by

\[
H^*(t, x) := \begin{cases} 
H^-(t, x), & t \in (a, b), \ x \in K, \\
\lim_{s \downarrow a} H(s, x), & t = a, \ x \in K,
\end{cases}
\]

satisfies the following conditions:

a) for all \(t \in [a, b]\) there exists \(M(t)\), such that

\[
D_1(H^*(t, x), H^*(t, y)) \leq M(t)\|x - y\| \quad (x, y \in X)
\]

b) \(H^*(t, x) = A(t)x + B(t) \quad (t \in [a, b], \ x \in K)\), where \(A(t)\) is linear continuous set-valued function, and \(B \in BW ([a, b], cc(Y))\).

Proof. Take \(t \in [a, b]\), and define the function \(\alpha : [a, b] \rightarrow [0, 1]\) by:

\[
\alpha(t) := \begin{cases} 
1, & a \leq \tau \leq t, \\
\frac{\tau - b}{t - b}, & t \leq \tau \leq b.
\end{cases}
\]

The function \(\alpha \in RV_p [a, b]\) and

\[
V_p(\alpha, [a, b]) = \frac{1}{|b - t|^{p-1}}.
\]

Let us fix \(x, y \in K\) and define the functions \(u_i : [a, b] \rightarrow K\) \((i = 1, 2)\) by

\[
u_1(\tau) := x \quad \tau \in [a, b], \quad u_2(\tau) := \alpha(\tau)(y - x) + x, \quad \tau \in [a, b].
\]

The functions \(u_i \in RV_p ([a, b], K)\) \((i = 1, 2)\) and

\[
\|u_1 - u_2\|_p = (V_p(\alpha; [a, b]))^{\frac{1}{p}}\|x - y\| = \left(1 + \frac{1}{|b - t|^{1-\frac{1}{p}}}\right)\|x - y\|.
\]

Since the Nemytskii operator \(N\) is globally Lipschitzian between \(RV_p ([a, b], K)\) and \(BW ([a, b], cc(Y))\), then there exists a constant \(M\), such that

\[
D(H(b, u_1(b)) + H(t, u_2(t)), H(t, u_1(t)) + H(b, u_2(b)) \leq M\|u_1 - u_2\|_p.
\]

By Lemma 2, substituting the particular functions \(u_i (i = 1, 2)\) defined by (12) in the above inequality, we obtain

\[
D(H(t, x), H(t, y)) \leq M(t)\|x - y\| \quad (x, y \in K, \ t \in [a, b]),
\]

\[
(13)
\]
where

\[ M(t) := M \left[ 1 + \frac{1}{|b - t|^{1 - \frac{1}{p}}} \right]. \]

In the case where \( t = b \), by a similar reasoning as above, we obtain that there exists a constant \( M(b) \), such that

\[
D(H(b, x), H(b, y)) \leq M(b)\|x - y\| \quad (x, y \in K).
\]

Hence, passing to the limit in the inequality (13) by the inequality (14) and the definition of \( H^* \) we have for all \( t \in [a, b] \) that there exists \( M(t) \), such that

\[
D(H^*(t, x), H^*(t, y)) \leq M(t)\|x - y\| \quad (x, y \in K).
\]

Now we shall proof that \( H^* \) satisfies the following equality

\[
H^*(t, x) = A(t)x + B(t) \quad (t \in [a, b], x \in K),
\]

where \( A(t) \) is linear continuous set-valued functions, and \( B \in BW([a, b], cc(Y)) \).

Let us fix \( t, t_0 \in [a, b] \), \( n \in \mathbb{N} \) such that \( t_0 < t \). Define the partition \( \pi_n \) of the interval \([t_0, t]\) by \( \pi_n : a < t_0 < t_1 < \cdots < t_{2n-1} < t_2n = t \), where

\[ t_i - t_{i-1} = \frac{t - t_0}{2n}, \quad i = 1, 2, \ldots, 2n. \]

The Nemytskii operator \( N \) is globally Lipschitzian between \( RV_p([a, b], K) \) and \( BW([a, b], cc(Y)) \), then there exists a constant \( M \), such that

\[
\sum_{i=1}^{n} D(H(t_{2i}, u_1(t_{2i})), H(t_{2i-1}, u_2(t_{2i-1}))) + H(t_{2i-1}, u_1(t_{2i-1})), H(t_{2i}, u_2(t_{2i}))) \leq M\|u_1 - u_2\|
\]

\[ (u_1, u_2 \in BV_p([a, b], K)). \]

Define the function \( \alpha : [a, b] \to [0, 1] \) in the following way:

\[
\alpha(\tau) := \begin{cases} 
0, & a \leq \tau \leq t_0, \\
\frac{\tau - t_{i-1}}{t_i - t_{i-1}}, & t_{i-1} \leq \tau \leq t_i, \quad i = 1, 3, \ldots, 2n - 1, \\
\frac{\tau - t_i}{t_i - t_{i-1}}, & t_{i-1} \leq \tau \leq t_i, \quad i = 2, 4, \ldots, 2n, \\
0, & t \leq \tau \leq b.
\end{cases}
\]
The function $\alpha \in RV_p([a,b])$ and
$$V_p(\alpha; [a,b]) = \frac{2^n p^n}{|t-t_0|^{p-1}}.$$Let us fix $x, y \in K$ and define the functions $u_i : [a, b] \to K$ by:

\begin{align*}
u_1(\tau) &:= \frac{\alpha(\tau)}{2} x + \left(1 - \frac{\alpha(\tau)}{2} y\right), \quad (\tau \in [a, b]) \\
u_2(\tau) &:= \frac{1 + \alpha(\tau)}{2} x + \frac{1 - \alpha(\tau)}{2} y, \quad (\tau \in [a, b]).
\end{align*}

The functions $u_i \in RV_p([a,b], K)$ ($i = 1, 2$) and
$$\|u_1 - u_2\|_p = \frac{\|x - y\|}{2}.$$Substituting in the inequality (15) the particular functions $u_i$ ($i = 1, 2$) defined in (16), we obtain

\begin{align*}
\sum_{i=1}^n D\left(H(t_{2i-1}, x) + H(t_{2i}, y), H\left(t_{2i-1}, \frac{x+y}{2}\right) + H\left(t_{2i}, \frac{x+y}{2}\right)\right) &\leq \\
&\leq M \frac{n}{2} \|x - y\|, (t \in [a, b], x, y \in K).
\end{align*}

for all $x, y \in K$.

The Nemytskii operator $N$ maps the space $RV_p([a,b], K)$ into the space $BW([a,b], cc(Y))$, then for all $z \in K$, the function $H(\cdot, z) \in BW([a,b], cc(Y))$. Letting $t_0 \uparrow t$ in the inequality (17), we get

$$D\left(H^*(t, x) + H^*(t, y), H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right)\right) \leq M \frac{n}{2n} \|x - y\|.$$Passing to the limit when $n \to \infty$, we get

$$H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right) + H^*(t, y) + H^*(t, x), \quad (t \in [a, b], x, y \in K).$$

$H^*(t, x)$ is a convex set, then

$$H^*\left(t, \frac{x+y}{2}\right) = \frac{1}{2} (H^*(t, x) + H^*(t, y)) \quad (t \in [a, b], x, y \in K).$$

Thus for every $t \in [a, b]$, the set-valued function $H^*(t, \cdot)$ satisfies the Jensen equation. By Lemma 3 and by the property a) previously established, we get that for all $t \in [a, b]$ there exist an additive set-valued
function $A(t) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$H^*(t, x) = A(t)x + B(t) \quad (t \in [a, b], \ x \in K).$$

By the same reasoning as in the proof of Theorem 1, we obtain that $A(t)(\cdot) \in L(K, cc(Y))$ and $B \in BW([a, b]cc(Y))$.

References


