On clp-paracompact spaces

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Abstract. The clp-paracompact spaces were defined and studied by A. Sondore. These spaces are those such that each clopen cover of them has a locally finite clopen refinement. Then, these spaces are related to ultraparacompact and to clp-compact spaces. In this paper, we obtain a theorem showing that every clp-paracompact Hausdorff space is the image of a clp-paracompact zero-dimensional Hausdorff space for a clopen continuous map with clp-compact fibers.

1. Introduction

SONDORE and ŠOSTAK [5] defined clp-compact spaces and related these spaces with compact, connected and zero-dimensional spaces. They also studied some properties of clp-compact spaces (e.g., products, countability). Afterwards, SONDORE [6] defined clp-paracompact spaces, and studied some properties of them (for example, their preservation for preimages of clopen maps with clp-compact fibers).

On the other hand, there exist various early results [1]–[4] on the existence of zero-dimensional paracompact spaces and perfect maps onto certain spaces (metric, paracompact, etc.). These results allow us to reduce the study of these spaces to those of them that are zero-dimensional.

In this paper, we prove that each clp-paracompact Hausdorff space is the image of a clp-paracompact zero-dimensional Hausdorff space for a clopen continuous map with clp-compact fiber.

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First, we list some previous definitions:

Definition 1 ([5]). A topological space is clp-compact if each clopen cover contains a finite subcover.

Definition 2 ([6]). A topological space is clp-paracompact if each clopen cover has a locally finite clopen refinement.

Definition 3. A map \( f \) between two spaces \( X \) and \( Y \) is clopen if the image for \( f \) of each clopen set of \( X \) is a clopen set of \( Y \).

2. Main results

Theorem 1. Let \( X \) be a clp-paracompact Hausdorff space. Then there exists a clp-paracompact Hausdorff topological space \( X^* \) with \( \dim X^* = 0 \) and a clopen continuos map \( f \) from \( X^* \) onto \( X \) such that \( f^{-1}(x) \) is clp-compact for every point \( x \in X \).

Proof. Let \( F_\lambda = \{ F_\alpha \mid \alpha \in A_\lambda \} \) be a locally finite clopen covering of \( X \), and \( \{ F_\lambda \mid \lambda \in \Lambda \} \) be the family of all locally finite clopen coverings of \( X \). Let

\[
X^* = \left\{ \alpha = (\alpha_\lambda)_{\lambda \in \Lambda} \mid \prod_{\lambda \in \Lambda} A_\lambda \text{ discrete topological spaces such that } \cap_{\lambda \in \Lambda} F_{\alpha_\lambda} \neq \emptyset \right\}.
\]

If \( \bigcap_{\lambda \in \Lambda} F_{\alpha_\lambda} \neq \emptyset \), then it is a point.

Let \( f : X^* \to X \) be the map \( f(\alpha) = \bigcap_{\lambda \in \Lambda} F_{\pi_{\alpha}(\alpha)} \), where \( \pi_{\alpha} \) is the projection from \( \prod_{\lambda \in \Lambda} A_\lambda \) onto \( A_\lambda \). Clearly, \( f \) is continuous and onto.

We prove that \( f \) is clopen: Let \( B \) be a non-empty clopen subset of \( X^* \). We have that \( f(B) \) is closed by [2, Th. 2]. For each \( x \in f(B) \), there exits \( \alpha \in B \) such that \( f(\alpha) = x \) and a base member \( \bigcap_{\lambda \in F} \pi^{-1}_{\lambda}(\alpha_\lambda) \subset B \) (with \( F \) finite and contained in \( \Lambda \)) such that \( \alpha \in \bigcap_{\lambda \in F} \pi^{-1}_{\lambda}(\alpha_\lambda) \).

Then \( x \in f \left( \bigcap_{\lambda \in F} \pi^{-1}_{\lambda}(\alpha_\lambda) \right) \subset f(B) \), thus \( x \in f \left( \bigcap_{\lambda \in F} \pi^{-1}_{\lambda}(\alpha_\lambda) \right) \subset \bigcap_{\lambda \in F} F_{\alpha_\lambda} \subset f(B) \). Since \( \bigcap_{\lambda \in F} F_{\alpha_\lambda} \) is an open set, we have that \( f(B) \) is open in \( X \).
Moreover, \( f^{-1}(x) = \prod_{\lambda \in \Lambda} B_\lambda \), with \( B_\lambda \) finite (for every \( \lambda \in \Lambda \)), then \( f^{-1}(x) \) is compact (and clp-compact).

Finally, we will show that \( X^* \) is a clp-paracompact Hausdorff space with \( \dim X^* = 0 \). Let \( \mathcal{U} \) be an arbitrary clopen covering of \( X^* \), then \( \mathcal{U} \) can be refined by a family \( \mathcal{B} \) of clopen sets. Since \( f^{-1}(x) \) is compact for any \( x \in X \), there exists a finite subfamily \( \mathcal{V}_{x,1}, \ldots, \mathcal{V}_{x,n(x)} \in \mathcal{B} \) such that \( f^{-1}(x) \subset \bigcup_{i=1}^{n(x)} \mathcal{V}_{x,i} = \mathcal{W}_x \), which is a clopen set also. If \( D(x) = X \setminus f(X^* \setminus \mathcal{W}_x) \), it is also clopen and there exists a \( \lambda_0 \in \Lambda \) such that \( \mathcal{F}_{\lambda_0} \) refines \( \{D(x) \mid x \in X\} \). Since

(a) \( \{\pi_{\lambda_0}^{-1}(\alpha) \mid \alpha \in A_{\lambda_0}\} \) refines \( \{f^{-1}(D(x)) \mid x \in X\} \);  
(b) the order of \( \{\pi_{\lambda_0}^{-1}(\alpha) \mid \alpha \in A_{\lambda_0}\} \) is 1,

we have (by transfinite induction on \( x \in X \)) that there is a clopen covering \( \{U_x/x \in X\} \) of order 1 with \( U_x \subset \mathcal{W}_x \), for every \( x \in X \). Let \( \mathcal{V} = \left\{ U_x \cap \left( \mathcal{V}_{x,i} \setminus \bigcup_{j \leq i} \mathcal{V}_{x,j} \right) \mid i = 2, \ldots, n(x), x \in X \right\} \).

Then \( \mathcal{V} \) is a clopen covering of \( X^* \), of order 1, which refines \( \mathcal{U} \). Thus \( X^* \) is a clp-paracompact Hausdorff space with \( \dim X^* = 0 \).

**Proposition 1.** Let \( (X,T) \) be a topological space. Then \( (X,T) \) is clp-paracompact if and only if each clopen covering of \( (X,T) \) has a \( \sigma \)-locally finite clopen refinement.

**Proof.** For each clopen covering \( \mathcal{U} \) of \( (X,T) \), there exists a clopen refinement \( \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \) such that \( \mathcal{V}_n \) is locally finite (for each \( n \in \mathbb{N} \)). If we call \( \mathcal{V}_n = \bigcup_{V \in \mathcal{V}_n} V \) and \( \mathcal{W}_n = \mathcal{V}_n \setminus \bigcup_{m \in \mathbb{N}} \mathcal{V}_m \), then \( \mathcal{A} = \{\mathcal{W}_n \cap V \mid n \in \mathbb{N}, V \in \mathcal{V}_n\} \) is a locally finite clopen refinement of \( \mathcal{U} \). \( \square \)

**References**


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