The collapsibility of CAT(0) square 2-complexes

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Abstract. We give a sufficient condition for the collapsibility of finite square 2-complexes. We show that any finite, CAT(0) square 2-complex retracts to a point through CAT(0) subspaces.

1. Introduction

In this paper, we investigate metric conditions which guarantee the collapsibility of a finite, 2-dimensional square complex.

The metric curvature condition we have in mind is given by the CAT(0) inequality. A geodesic metric space is a CAT(0) space if geodesic triangles are thinner than comparison triangles in the Euclidean space (see [4], [5], [2], [24]). A 2-dimensional polyhedral space is a space of nonpositive curvature if and only if the link of each vertex does not contain a subspace isometric to a circle of length less than $2\pi$ (see [5, Chapter 4.2, p. 113]). Hence the standard piecewise Euclidean metric structure on a 2-dimensional simplicial complex is nonpositively curved if and only if the link of each vertex of the complex has girth at least 6. The girth of a graph is defined as the minimum number of edges in a circuit.

Combinatorially, one can express curvature using a condition, called $k$-systolicity ($k \geq 6$), which was introduced independently by CHEpoi [7] (under the name of bridged complexes), JANUSZKIEWICZ−ŚWIĄTKOWSKI [16] and HAGLUND [14].

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The theory of 7-systolic groups, that is groups acting geometrically on 7-systolic complexes, allowed to provide examples of highly dimensional Gromov hyperbolic groups (see [15], [16], [27]). However, for groups acting geometrically on CAT(-1) cubical complexes or on 7-systolic complexes, some very restrictive limitations are known. For example, 7-systolic groups are in a sense ‘asymptotically hereditarily aspherical’, i.e., asymptotically they cannot contain essential spheres. This yields in particular that such groups are not fundamental groups of negatively curved manifolds of dimension above two; see, e.g., [17], [25], [26], [31] and [30]. In [28], [8], [3] and [6], other conditions of this type are studied. They form a way of unifying CAT(0) cubical and systolic theories. Osajda introduced in [29] another local combinatorial condition called 8-location. He showed that under the additional hypothesis of local 5-largeness, this condition implies Gromov hyperbolicity of the universal cover. In [22], we study a version of 8-location, suggested in [29, Subsection 5.1]. This 8-location says that homotopically trivial loops of length at most 8 admit filling diagrams with one internal vertex. In [23], we introduce another combinatorial curvature condition, called the 5/9-condition, and we show that the complexes which fulfill it are also Gromov hyperbolic.

White showed that a finite, strongly convex 2-complex is collapsible (see [33]). Corson–Trace proved further that a finite, simply connected, 2-dimensional simplicial complex that has the 6-property collapses to a point (see [9]). In dimension above 2, systolic simplicial complexes are also collapsible (see [8], [20], [21]). Crowley showed in [11] that a finite simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidean metric that is nonpositively curved, and satisfies a technical condition, simplicially collapses to a point. She constructed a CAT(0) 2-complex by endowing the complex with the corresponding piecewise Euclidean metric and requiring that each interior vertex of the complex has degree at least 6. The naturally associated piecewise Euclidean metric on the 2-complex becomes then CAT(0). Crowley’s result was extended by Adiprasito–Benedetti to all dimensions (see [1]). Using discrete Morse theory (see [13], [12]), they proved that every simplicial complex that is CAT(0) with a metric for which all vertex stars are convex is collapsible.

It turns out that no combinatorial condition is necessary to prove that a finite, CAT(0) simplicial 2-complex is collapsible. A proof of this fact is given in [19, Chapter 3.1, p. 35]. The aim of the present paper is to extend this result on square 2-complexes. In [1, Corollary 3.2.9], it is shown that every CAT(0) cube complex is collapsible. We give an alternative proof of the same result only in the 2-dimensional case.
The main result of the paper states that a finite, CAT(0) square 2-complex retracts to a point through subspaces which are, at each step of the retraction, CAT(0) spaces. When finding the new geodesic segments in the subspace obtained by performing an elementary collapse on a finite, CAT(0) square 2-complex, we reduce the problem to the simplicial case. The proof for the fact that the subcomplex obtained by performing an elementary collapse on a finite, CAT(0) square 2-complex remains non-positively curved, on the other hand, does not reduce to the simplicial case. Instead, we argue on a square 2-complex, which is the novelty of the paper.

In the infinite case, the equivalent notion of collapsibility is called arborescent structure. In [9], it is shown that any locally finite, simply connected simplicial 2-complex with the 6-property is a monotone union of a sequence of collapsible subcomplexes. We show that a similar result holds for locally finite CAT(0) square 2-complexes. This is a consequence of the fact that finite, CAT(0) square 2-complexes are collapsible.

2. Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Definition 2.1. Let \((X, d)\) be a metric space. If \(x, m, y\) are three points in \(X\) such that \(d(x, m) + d(m, y) = d(x, y)\), then we say that \(m\) lies between \(x\) and \(y\). We call \(m\) the midpoint of \(x\) and \(y\) if \(d(x, m) = d(m, y) = \frac{1}{2}d(x, y)\).

Definition 2.2. Let \((X, d)\) be a metric space. \(X\) is a convex metric space if for any two points \(x, y\) in \(X\), there exists at least one midpoint \(m\). \(X\) is a strongly convex metric space if for any two points \(x, y\) in \(X\), there exists exactly one midpoint \(m\).

Definition 2.3. Let \((X, d)\) be a metric space, and let \(c : [a, b] \to X\) be a path in \(X\). The length \(l(c)\) of \(c\) is defined by:

\[
l(c) = \sup_{a = t_0 \leq t_1 \leq \cdots \leq t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),
\]

where the supremum is taken over all possible partitions with \(a = t_0 \leq t_1 \leq \cdots \leq t_n = b\).
Definition 2.4. Let \((X, d)\) be a metric space, and let \(x, y\) be two distinct points in \(X\). A segment \(c : [a, b] \to X\) in \(X\) connecting \(x\) to \(y\) is a path which has, among all paths joining \(x\) to \(y\) in \(X\), the shortest length.

Theorem 2.5. Let \((X, d)\) be a metric space. Let \(x\) and \(y\) be two distinct points in \(X\).

1. A subset \(S\) of \(X\) containing \(x\) and \(y\) is a segment joining \(x\) to \(y\) if there exists a closed real line interval \([a, b]\) and an isometry \(c : [a, b] \to X\) such that \(c(a) = x\) and \(c(b) = y\).

2. A path \(c : [a, b] \to X\) joining \(x\) to \(y\) is a segment if and only if \(d(c(t), c(t')) = |t - t'|\) for all \(t, t' \in [a, b]\).

For the proof, see [2, Chapter II.2, p. 76].

Theorem 2.6. Let \((X, d)\) be a complete metric space. There exists a segment (there exists a unique segment) in \(X\) between any two distinct points \(x, y\) in \(X\) if and only if \(X\) is a convex metric space (strongly convex metric space).

For the proof, see [24].

Definition 2.7. Let \((X, d)\) be a metric space. A geodesic path joining \(x\) to \(y\) is a path \(c : [a, b] \to X\) such that \(c(a) = x\), \(c(b) = y\) and \(d(c(t), c(t')) = |t - t'|\) for all \(t, t' \in [a, b]\). The image \(\alpha\) of \(c\) is called a geodesic segment with endpoints \(x\) and \(y\).

A geodesic metric space \((X, d)\) is a metric space in which every pair of points can be joined by a geodesic segment. We denote any geodesic segment from a point \(x\) to a point \(y\) in \(X\), by \([x, y]\). We emphasize that any such geodesic segment is not determined by its endpoints. Thus, without further assumptions on \(X\), there may be many geodesic segments joining \(x\) to \(y\).

A geodesic triangle in \(X\) consists of three points \(p, q, r \in X\), called vertices, and a choice of three geodesic segments \([p, q], [q, r], [r, p]\) joining them, called sides. Such a geodesic triangle is denoted by \(\Delta(p, q, r)\). If a point \(x\) in \(X\) lies in the union of \([p, q], [q, r]\) and \([r, p]\), then we write \(x \in \Delta\). A triangle \(\Delta = \Delta(\bar{p}, \bar{q}, \bar{r})\) in \(\mathbb{R}^2\) is called a comparison triangle for \(\Delta = \Delta(p, q, r)\) if \(d(p, q) = d_{\mathbb{R}^2}(\bar{p}, \bar{q})\), \(d(q, r) = d_{\mathbb{R}^2}(\bar{q}, \bar{r})\) and \(d(r, p) = d_{\mathbb{R}^2}(\bar{r}, \bar{p})\). A point \(\bar{p} \in [\bar{q}, \bar{r}]\) is called a comparison point for \(x \in [q, r]\) if \(d(q, x) = d_{\mathbb{R}^2}(\bar{q}, \bar{p})\). The interior angle of \(\Delta = \Delta(\bar{p}, \bar{q}, \bar{r})\) at \(\bar{p}\) is called the comparison angle between \(q\) and \(r\) at \(p\), and it is denoted by \(\angle_{\bar{p}}(q, r)\) (the comparison angle is well-defined provided \(q\) and \(r\) are both distinct from \(p\)).

Definition 2.8. Let \((X, d)\) be a metric space, and let \(c : [0, a] \to X\) and \(c' : [0, a'] \to X\) be two geodesic paths with \(c(0) = c'(0)\). Given \(t \in (0, a]\) and
$t' \in (0, a']$, we consider the comparison triangle $\overline{\triangle}(c(0), c(t), c'(t'))$ in $\mathbb{R}^2$ and the comparison angle $\angle_{c(0)}(c(t), c'(t'))$. The Alexandrov angle between the geodesic paths $c$ and $c'$ is the number $\angle(c, c') \in [0, \pi]$ defined by:

$$\angle(c, c') := \limsup_{t, t' \to 0} \angle_{c(0)}(c(t), c'(t')) = \limsup_{\varepsilon \to 0} \angle_{c(0)}(c(t), c'(t')).$$

The Alexandrov angle between two geodesic segments which have a common endpoint is defined to be the Alexandrov angle between the unique geodesics which issue from this point and whose images are the given segments. Alexandrov angles in $\mathbb{R}^2$ are equal to the usual Euclidean angles.

**Remark 2.9.** The Alexandrov angle between the geodesic paths $c : [0, a] \to X$ and $c' : [0, a'] \to X$ in a metric space $(X, d)$ depends only on the germs of these paths at $0$. If $c'' : [0, a''] \to X$ is any geodesic path for which there exists $\varepsilon > 0$ such that $c''|_{[0, \varepsilon]} = c'|_{[0, \varepsilon]}$, then the Alexandrov angle between $c$ and $c''$ is the same as that between $c$ and $c'$.

**Definition 2.10.** Let $(X, d)$ be a convex metric space. Let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0) = p$ which have no other common points in the neighborhood of $p$. The geodesic paths $c$ and $c'$ divide a sufficiently small neighborhood of $p$ into two sectors $U$ and $V$. We consider in $U$ the geodesic paths $c_1, c_2, \ldots, c_n$, numbered according to their position relative to $c$ and $c'$. We denote by $\alpha_0, \alpha_1, \ldots, \alpha_n$ the Alexandrov angles between $c$ and $c_1$ and $c_2, \ldots, c_n$ and $c'$, respectively. The upper limit of the sum $\alpha_0 + \alpha_1 + \cdots + \alpha_n$ for any geodesic paths $c_i$ in $U$, $1 \leq i \leq n$, is called the **Alexandrov angle of the sector** $U$.

**Definition 2.11.** Let $(X, d)$ be a convex metric space, and let $p$ be a point in $X$. Let $U_1, \ldots, U_n$ be sectors around $p$ which form a full neighborhood of $p$. We call the sum of the Alexandrov angles of the sectors $U_1, \ldots, U_n$ in $X$ the **full angle around the point** $p$ in $X$.

**Definition 2.12.** Let $(X, d)$ be a convex metric space. Let $\triangle(p, q, r)$ be a geodesic triangle in $X$. Let $\alpha, \beta, \gamma$ be the Alexandrov angles between the sides of $\triangle$. The **curvature of the geodesic triangle** $\triangle$ is defined by $\omega(\triangle) = \alpha + \beta + \gamma - \pi$.

**Definition 2.13.** Let $(X, d)$ be a convex metric space. Let $p$ be a point of $X$. Let $\theta$ be the full angle around the point $p$. The **curvature at the point** $p$ is defined by $\omega(p) = 2\pi - \theta$.

**Theorem 2.14.** Let $(X, d)$ be a convex metric space, and let $\triangle(p, q, r)$ be a geodesic triangle in $X$ whose curvature equals zero. Then $\triangle(p, q, r)$ is isometric to its comparison triangle $\overline{\triangle}(\overline{p}, \overline{q}, \overline{r})$ in $\mathbb{R}^2$. 
For the proof, we refer to [2, Chapter V.6, p. 218].

We define next CAT(0) spaces and present some of their basic properties.

Definition 2.15. Let \((X, d)\) be a metric space. Let \(\triangle(p, q, r)\) be a geodesic triangle in \(X\). Let \(\triangle(p, q, r) \subset \mathbb{R}^2\) be a comparison triangle for \(\triangle\). The metric \(d\) is CAT(0) if for all \(x, y \in \triangle\) and all comparison points \(x, y \in \triangle\), the CAT(0) inequality holds: \(d(x, y) \leq d_{\mathbb{R}^2}(x, y)\). A metric space \(X\) is called a CAT(0) space if it is a geodesic space all of whose geodesic triangles satisfy the CAT(0) inequality.

A metric space \(X\) is said to be of curvature \(\leq 0\) (or non-positively curved) if it is locally a CAT(0) space, i.e., for every \(x \in X\) there exists \(r_x > 0\) such that the ball \(B(x, r_x)\), endowed with the induced metric, is a CAT(0) space.

Theorem 2.16. Let \(X\) be a CAT(0) space.

1. The balls in \(X\) are convex (i.e., any two points in such a ball are joined by a unique geodesic segment and this segment is contained in the ball) and contractible.
2. (Approximate midpoints are close to midpoints.) For every \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon) > 0\) such that if \(m\) is the midpoint of a geodesic segment \([x, y] \subset X\) and if \(\max\{d(x, m'), d(y, m')\} \leq \frac{1}{2}d(x, y) + \delta\), then \(d(m, m') < \varepsilon\).

For the proof, we refer to [4, Chapter II.1, p. 160].

In CAT(0) spaces, angles exist in the following strong sense.

Theorem 2.17. Let \(X\) be a CAT(0) space, and let \(c : [0, a] \to X\) and \(c' : [0, a'] \to X\) be two geodesic paths issuing from the same point \(c(0) = c'(0)\). Given \(t \in [0, a]\) and \(t' \in [0, a']\), let \(\triangle(c(t), c(0), c'(t'))\) be a comparison triangle in \(\mathbb{R}^2\) for \(\triangle(c(t), c(0), c'(t'))\). The comparison angle \(\angle_{c(0)}(c(t), c'(t'))\) is a non-decreasing function of both \(t, t' \geq 0\), and the Alexandrov angle \(\angle(c, c')\) is equal to \(\lim_{t, t' \to 0} \angle_{c(0)}(c(t), c'(t')) = \lim_{t \to 0} \angle_{c(0)}(c(t), c'(t))\). Hence

\[
\angle(c, c') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(c(t), c'(t))
\]

For the proof, see [4, Chapter II.3, p. 184].

Let \(p, x, y\) be points of a metric space \(X\) such that \(p \neq x\) and \(p \neq y\). If there are unique geodesic segments \([p, x]\) and \([p, y]\), then we write \(\angle_p(x, y)\) to denote the Alexandrov angle between these segments.

Theorem 2.18. Let \(X\) be a metric space. The following conditions are equivalent:

1. \(X\) is a CAT(0) space;
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(2) for every geodesic triangle $\triangle(p, q, r)$ in $X$ and for every point $x \in [q, r]$, the following inequality is satisfied by the comparison point $\bar{x} \in [\overline{q}, \overline{r}] \subset \overline{\triangle}(p, q, r) \subset \mathbb{R}^2$: $d(p, x) \leq d(p, \bar{x})$;

(3) the Alexandrov angle between the sides of any geodesic triangle in $X$ with distinct vertices is not greater than the angle between the corresponding sides of its comparison triangle in $\mathbb{R}^2$.

For the proof, see [4, Chapter II.1, p. 161].

**Theorem 2.19.** Any CAT(0) space is contractible; in particular, it is simply connected.

For the proof, we refer to [4, Chapter II.1, p. 161].

**Theorem 2.20.** Let $(X, d)$ be a CAT(0) space. Then the distance function $d : X \times X \to \mathbb{R}$ is convex and strongly convex.

For the proof, see [4, Chapter II.2, p. 176 and Chapter II.1, p. 160].

**Theorem 2.21.** Let $X$ be a complete connected metric space. If $X$ is simply connected and of curvature $\leq 0$, then $X$ is a CAT(0) space.

For the proof, we refer to [4, Chapter II.4, p. 194].

Alexandrov’s Lemma, given below, will be referred to frequently when showing the main result of the paper.

**Lemma 2.22.** Consider four distinct points $A, B, B', C$ in the Euclidean plane. Suppose that $B$ and $B'$ lie on opposite sides of the line through $A$ and $C$. Consider the geodesic triangles $\triangle = \triangle(A, B, C)$ and $\triangle' = \triangle(A, B', C)$. Let $\alpha, \beta, \gamma$ ($\alpha', \beta', \gamma'$) be the angles of $\triangle$ ($\triangle'$) at the vertices $A, B, C$ ($A, B', C$).

Let $\overline{\triangle}$ be a triangle in $\mathbb{R}^2$ with vertices $\overline{A}, \overline{B}, \overline{B'}$ such that $d(\overline{A}, \overline{B}) = d(A, B)$, $d(\overline{A}, \overline{B'}) = d(A, B')$ and $d(\overline{B}, \overline{B'}) = d(C, B')$. Let $\overline{C}$ be the point of $[\overline{B}, \overline{B'}]$ with $d(\overline{B}, \overline{C}) = d(B, C)$. Let $\overline{\alpha}, \overline{\beta}, \overline{\beta'}$ be the angles of $\overline{\triangle}$ at the vertices $\overline{A}, \overline{B}, \overline{B'}$.

If $\gamma + \gamma' \geq \pi$, then $d(B, C) + d(B', C) \leq d(B, A) + d(B', A)$. Also $\overline{\alpha} \geq \alpha + \alpha', \overline{\beta} \geq \beta, \overline{\beta'} \geq \beta'$ and $d(\overline{A}, \overline{C}) \geq d(A, C)$.

If $\gamma + \gamma' \leq \pi$, then $d(B, C) + d(B', C) \geq d(B, A) + d(B', A)$. Also, $\overline{\alpha} \leq \alpha + \alpha', \overline{\beta} \leq \beta, \overline{\beta'} \leq \beta'$ and $d(\overline{A}, \overline{C}) \leq d(A, C)$.

Either equality is equivalent to the others, and occurs if and only if $\gamma + \gamma' = \pi$.

For the proof, see [4, Chapter I.2, p. 25].
Definition 2.23. The unit $n$-cube $I^n$ is the $n$-fold product $[0, 1]^n$; it is isometric to a cube in the Euclidean $n$-space with edges of length one. By convention, $I^0$ is a point. A cubical complex $K$ is the quotient of a disjoint union of cubes $X = \bigcup_{\lambda} I^n_\lambda$ by an equivalence relation $\sim$. The restrictions $p_{\lambda} : I^n_\lambda \to K$ of the natural projection $p : X \to K = X|_\sim$ are required to satisfy:

(1) for every $\lambda \in \Lambda$, the map $p_{\lambda}$ is injective;
(2) if $p_{\lambda}(I^n_\lambda) \cap p_{\lambda'}(I^n_{\lambda'}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset I^n_\lambda$ onto a face $T_{\lambda'} \subset I^n_{\lambda'}$ such that $p_{\lambda}(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

In other words, $K$ is a cubical complex if and only if each of its cells $C_\lambda$ is isometric to a cube $I^n_\lambda$, each of the maps $p_{\lambda}$ is injective, and the intersection of any two cells in $K$ is empty or a single face. We call a 2-dimensional cubical complex a square complex.

Definition 2.24. Let $K$ be a square complex. Let $\sigma$ be a 2-cell of $K$ with vertices at the points $a, b, c, d$. The curvature of $\sigma$ is equal to $\omega(\sigma) = [\angle(a, b, d) + \angle(b, a, d) + \angle(c, b, d) + \angle(b, c, d) - 2\pi]$.

Definition 2.25. Let $K$ be a square 2-complex, and let $\alpha$ be an $i$-cell of $K$, $0 \leq i \leq 2$. If $\beta$ is a $k$-dimensional face of $\alpha$, $k < i$ but not of any other cell in $K$, then we say there is an elementary collapse from $K$ to $K \setminus \{\alpha, \beta\}$. If $K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n = L$ are square complexes such that there is an elementary collapse from $K_{j-1}$ to $K_j$, $1 \leq j \leq n$, then we say that $K$ collapses to $L$.

Definition 2.26. A locally finite space is a topological space in which every point has a finite neighborhood.

Definition 2.27. Let $K$ be a locally finite square complex. We say $K$ has an arborescent structure if it is a monotone union $\bigcup_{n=1}^{\infty} L_n$ of a sequence of collapsible subcomplexes $L_n$.

Definition 2.28. Let $K$ be an $n$-dimensional square complex. A $k$-dimensional subcomplex $K'$ of $K$ is called a spine of $K$ if $K$ collapses to $K'$ for any $k < n$.

Definition 2.29. A point $a$ of an $n$-dimensional square complex $K$ is an interior point of $K$ if it is contained in a subspace $U$ of $K$ which is homeomorphic to an $n$-ball $B^n$ of finite radius. Otherwise we call $a$ an exterior point of $K$.

Definition 2.30. Let $K$ be a square complex, and let $e$ be an edge of $K$. We denote by $i(e)$, the initial vertex of $e$, by $t(e)$, the terminus of $e$. A finite sequence $e_0 e_1 \ldots e_n$ of edges in $K$ such that $t(e_i) = i(e_{i+1})$ for all $0 \leq i \leq n - 1$, is called an edge-path in $K$. 
Definition 2.31. A subcomplex $L$ of a square complex $K$ is called full (in $K$) if any cell of $K$ spanned by a set of vertices in $L$ is a cell of $L$.

Definition 2.32. We call a square complex $K$ flag if any set of vertices is included in a face of $K$ whenever each pair of its vertices is contained in a face of $K$.

3. Collapsing a CAT(0) square 2-complex

This section provides a metric characterization of collapsible square 2-complexes. We show that finite, CAT(0) square 2-complexes are collapsible. Namely, they retract to a point through CAT(0) subspaces. Similar results are obtained in [19] on finite CAT(0) simplicial 2-complexes.

We start investigating the collapsibility of finite, CAT(0) square 2-complexes by characterizing the curvature at the interior points of such complex. Besides, in the following proposition we show that finite, CAT(0) square 2-complexes have a 2-cell with a free 1-dimensional face.

Proposition 3.1. Let $K$ be a finite square 2-complex. If $|K|$ admits a CAT(0) metric $d$, then:

1. $|K|$ has curvature $\leq 0$ at any of its interior points;
2. $K$ has a 2-cell with a free 1-dimensional face.

Proof. (1) Let $\tau = [c, h]$ be a 1-cell of $K$ that is the face of at least two 2-cells in $K$, $\sigma_1$ and $\sigma_2$. Let $a, b, c, h$ be the vertices of the 2-cell $\sigma_1$. Let $c, h, e, f$ be the vertices of the 2-cell $\sigma_2$. Let $g \in \tau$.

Let $\triangle(b', h', g')$ be a comparison triangle for the geodesic triangle $\triangle(b, h, g)$. Let $\triangle(b', c', g')$ be a comparison triangle for the geodesic triangle $\triangle(b, c, g)$. 

Figure 1. Comparison triangles
We place the comparison triangles \( \triangle(b', h', g') \) and \( \triangle(b', c', g') \) in different half-planes with respect to the line \( b'g' \) in \( \mathbb{R}^2 \). Let \( \triangle(h'', f'', g'') \) be a comparison triangle for the geodesic triangle \( \triangle(h, f, g) \). Let \( \triangle(c'', f'', g'') \) be a comparison triangle for the geodesic triangle \( \triangle(c, f, g) \). We place the comparison triangles \( \triangle(h'', f'', g'') \) and \( \triangle(c'', f'', g'') \) in different half-planes with respect to the line \( f''g'' \).

Because \( d_{\mathbb{R}^2}(b', g') = d(b, g) \) and \( d_{\mathbb{R}^2}(f'', g'') = d(f, g) \), Alexandrov’s Lemma implies that \( \angle_{g'}(h', b') + \angle_{g'}(b', c') = \pi \) and \( \angle_{g''}(h'', f'') + \angle_{g''}(f'', c'') = \pi \). So \( \angle_g(h, b) + \angle_g(b, c) = \pi \) and \( \angle_g(h, f) + \angle_g(f, c) = \pi \). Hence, because the 1-cell \( \tau \) is contained in at least two 2-cells of \( K \), the full angle around the point \( g \) in \( |K| \) equals at least \( 2\pi \).

We denote by \( \theta \) the full angle around the point \( g \) in \( |K| \). Because \( |K| \) has a convex metric, the curvature at the interior point \( g \) of \( |K| \) is equal to \( \omega(g) = 2\pi - \theta \leq 0 \). \( |K| \) has therefore curvature \( \leq 0 \) at any of its interior points.

(2) Let \( \tau = [c, b] \) be a 1-cell of \( K \) such that there exist at least two 2-cells \( \sigma_1 \) and \( \sigma_2 \) in \( K \) with \( \tau < \sigma_1 \) and \( \tau < \sigma_2 \). Let \( a, b, c, h \) be the vertices of the 2-cell \( \sigma_1 \), and let \( c, h, e, f \) be the vertices of the 2-cell \( \sigma_2 \). Let \( g \in \tau \).

![Figure 2. Comparison triangles](image)

Let \( \triangle(a', h', g') \) be a comparison triangle for the geodesic triangle \( \triangle(a, h, g) \) in \( |K| \). Let \( \triangle(a', c', g') \) be a comparison triangle for the geodesic triangle \( \triangle(a, c, g) \) in \( |K| \). We place the comparison triangles \( \triangle(a', h', g') \) and \( \triangle(a', c', g') \) in different half-planes with respect to the line \( a'g' \).

Because any geodesic triangles in \( |K| \) satisfies the CAT(0) inequality and \( g \in [h, c] \),

\[
\pi = \angle_g(h, c) \leq \angle_g(h, a) + \angle_g(a, c) \leq \angle_{g'}(h', a') + \angle_{g'}(a', c').
\]

So \( \angle_g(h', a') + \angle_g(a', c') \geq \pi \). According to Alexandrov’s Lemma, we have \( d_{\mathbb{R}^2}(a', g') \leq d(a, g) \). But \( \triangle(a', h', g') \) is a comparison triangle for the geodesic triangle \( \triangle(a, h, g) \), and hence \( d_{\mathbb{R}^2}(a', g') = d(a, g) \). Because one equality in
Alexandrov’s Lemma implies the others, the following equalities hold
\[ \angle g'(h', a') + \angle g'(a', c') = \pi, \quad \angle h(a, g) = \angle h'(a', g') \quad \text{and} \quad \angle a(h, g) + \angle a(g, c) = \angle a(h', c'). \]
Note that \( \angle a(h', c') + \angle h'(a', g') + \angle c'(a', g') = \pi. \) So the sum of the angles between the sides of the geodesic triangle \( \triangle(a, h, c) \) equals \( \pi. \) Therefore, since \( |K| \) has a convex metric, the curvature of the geodesic triangle \( \triangle(a, h, c) \) equals \( \omega(\triangle(a, h, c)) = \pi - \pi = 0. \)

It similarly follows that the geodesic triangles \( \triangle(a, b, c), \triangle(b, h, c), \) and \( \triangle(a, b, h) \) in \( |K| \) have curvature zero. So these triangles are isometric to their comparison triangles and are therefore flat. Note that the geodesic triangles \( \triangle(a, h, c), \triangle(b, h, c), \triangle(a, b, c) \) and \( \triangle(a, b, h) \) overlap and cover the 2-cell \( \sigma_1. \)
So the curvature of \( \sigma_1 \) is equal to \( \omega(\sigma_1) = [\angle a(h, c) + \angle h(a, c) + \angle c(a, h)] + [\angle a(b, c) + \angle h(a, c) + \angle c(a, b)] - 2\pi = \pi + \pi - 2\pi = 0. \) Thus \( \sigma_1 \) is flat.

Assume that \( K \) has no 2-cell with a free 1-dimensional face. Hence each 1-cell of \( K \) is a face of at least two 2-cells whose 1-cells are further faces of at least two 2-cells and so on. \( K \) is contractible because it is a CAT(0) space. Therefore, since \( K \) is finite and the 2-cells of \( K \) are flat, this implies a contradiction. So \( K \) has a 2-cell with a free 1-dimensional face.

Related to the result above, we note the following.

**Remark 3.2.** There are contractible finite 2-complexes which are not collapsible. The dunce hat space, for instance, is a contractible surface without boundary, so no triangulation of it can have any free faces (see [34]). But the dunce hat is the flag triangulation of a contractible 2-complex which does not admit a CAT(0) metric (see [10]).

We show further that the subcomplex \( K' \) obtained by performing an elementary collapse on a finite, CAT(0) square 2-complex \( K \), remains non-positively curved. We treat only the general case when \( K' \) is obtained by pushing in an entire 2-cell with a free 1-dimensional face, by starting at its free face. We emphasize that the same result holds for any deformation retract of a finite, CAT(0) square 2-complex \( K \) obtained by pushing in any geodesic triangle \( \delta \) in \( |K| \) that belongs to the 2-cell of \( K \) which has a free 1-dimensional face. Namely, one side of \( \delta \) is included in the free 1-dimensional face of this 2-cell.

For the remainder of the paper, we fix the following notation. Let \( K \) be a finite, CAT(0) square 2-complex such that it has a 2-cell \( \sigma \) which has a free 1-dimensional face \( e. \) Let \( d \) be the CAT(0) metric \( |K| \) is endowed with. We show that the subcomplex \( K' = K \setminus \{e, \sigma\} \) is non-positively curved.

Let \( a, b, c, h \) be the vertices of the 2-cell \( \sigma. \) Let \( e = [b, c] \) be its free 1-dimensional face. We denote by \( r := \max\{d(a, b), d(a, c), d(a, h)\}. \) We consider
in $|K|$ a neighborhood of $a$ homeomorphic to a closed ball of radius $r$, $U = \{x \in |K| \mid d(a, x) \leq r\}$. $U$ endowed with the induced metric, is a CAT(0) space. Because $U$ is complete and it has a strongly convex metric, any two points in $U$ are joined by a unique geodesic segment which is contained in $U$. So any geodesic triangle with vertices at any three points in $U$, belongs to $U$ and it satisfies the CAT(0) inequality.

We consider in $|K'|$ a neighborhood of $a$ homeomorphic to a closed ball of radius $r$, $U' = \{x \in |K'| \mid d'(a, x) \leq r\}$, endowed with the induced metric $d'$. We notice that $U' = U \setminus \{e, \sigma\}$. We find next the geodesic segments in $U'$ with respect to $d'$. Let $p, q$ be two distinct points in $U$ that do not belong to $\sigma$ such that the geodesic segment $[p, q]$ intersects the interior of $\sigma$. One of the points $p$ and $q$ may lie on one of the edges of $\sigma$ but not both. There are two cases to consider, which are given below.

**Case A.** The segment $[p, q]$ intersects $[a, b]$ in $m$, and $[a, h]$ in $n$. The segment $[p, q]$ does not intersect $[c, h]$ and $[b, c]$. The point $p$ may belong to $[a, b]$ and the point $q$ may belong to $[a, h]$, but not simultaneously.

![Figure 3. Case A](image)

**Case B.** The segment $[p, q]$ intersects $[a, b]$ in $m$, and $[c, h]$ in $n$. The segment $[p, q]$ does not intersect $[a, h]$ and $[b, c]$. The point $p$ may belong to $[a, b]$ and the point $q$ may belong to $[a, h]$, but not simultaneously.

The aim of Lemmas 3.3–3.5 below is to find some geodesic segments in the subcomplex obtained by performing an elementary collapse on a finite, CAT(0) square 2-complex.

**Lemma 3.3.** Let $c : [0, 1] \to U$ be a path in $U$ joining $p$ to $q$ that does not intersect $\sigma$. The points $p$ and $q$ are chosen as in Case A. Then there exists a point $s_0$ on $c$ such that the geodesic segments $[p, s_0]$ and $[q, s_0]$ do not intersect $\sigma$ and such that the following inequality holds:

$$d'(p, a) + d'(a, q) < d'(p, s_0) + d'(s_0, q).$$
Proof. We turn the square 2-complex $K$ into a simplicial 2-complex $L$. So $|K| = |L|$. Let $L$ be a finite simplicial 2-complex such that all $i$-simplices of $K$ are also $i$-simplices of $L$, $0 \leq i \leq 1$. Besides $L$ contains, for each 2-cell $\sigma$ of $K$, the 1-simplex $[a, h]$ and the 2-simplices $\triangle(a, b, h)$, $\triangle(a, c, h)$. Note that $L$ is a CAT(0) space. The lemma now follows due to [19, Chapter 3.1.4, p. 39].

This completes the proof. □

Lemma 3.4. Let the segment $[p, q]$ be as in Case A. Then the geodesic segment $[p, q]$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, a]$ and $[a, q]$.

Proof. We denote by $c : [0, 1] \to U'$ the path obtained by concatenating the segments $[p, a]$ and $[a, q]$. Among all paths joining $p$ to $q$ in $U'$ that pass through $a$, the path $c$ has the shortest length.

Suppose that there exists a path $c_0 : [0, 1] \to U'$ connecting $p$ to $q$ in $U'$ that does not pass through $a$ and whose length is less or equal to the length of the path $c$. Because the path $c_0$ does not intersect $\sigma$, there exists, according to Lemma 3.3, a point $s_0$ on $c_0$ such that the geodesic segments $[p, s_0]$ and $[s_0, q]$ in $U$ do not intersect $\sigma$. The geodesic segments $[p, s_0]$ and $[s_0, q]$ in $U$ belong
therefore to $U'$. So
\[ d'(p, s_0) + d'(s_0, q) \leq l(c_0) \leq l(c) = d'(p, a) + d'(a, q) \]
which implies, by Lemma 3.3, a contradiction. Any path in $U'$ joining $p$ to $q$ and that does not pass through $a$ is therefore longer than $c$.

Altogether, it follows that the geodesic segment joining $p$ to $q$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, a]$ and $[a, q]$. □

**Lemma 3.5.** Let the segment $[p, q]$ be as in Case B. Then the geodesic segment $[p, q]$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, a]$, $[a, h]$ and $[h, q]$.

**Proof.** Let $f$ be the midpoint of the geodesic segment $[a, h]$. Because $U$ is a CAT(0) space, the midpoint $f$ exists and it is unique. Let $[p, f] \cap [a, h] = \{m'\}$, and let $[q, f] \cap [c, d] = \{n'\}$. The previous lemma implies that the geodesic segment $[p, f]$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, a]$ and $[a, f]$. It also implies that the geodesic segment $[q, f]$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[q, h]$ and $[h, f]$. The geodesic segment $[p, q]$ in $U'$ with respect to $d'$ is therefore the union of the geodesic segments $[p, a]$, $[a, f]$, $[f, h]$ and $[h, q]$. So the geodesic segment $[p, q]$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, a]$, $[a, h]$ and $[h, q]$. □

In Lemmas 3.6–3.10 below, we study whether certain geodesic triangles in $U'$ fulfill the CAT(0) inequality. This will be useful when showing that $U'$ is non-positively curved.

**Lemma 3.6.** Let the segment $[p, q]$ be as in Case A. Let $r$ be a point in $U$ such that the geodesic segments $[r, p]$ and $[r, q]$ do not intersect $\sigma$. Also the quadrilaterals $\text{ramp}$ and $\text{ranq}$ are convex. Then, the geodesic triangle $\triangle(p, r, q)$ in $U'$ satisfies the CAT(0) inequality.
Proof. By Lemma 3.4, $d'(p, q) = d'(p, a) + d'(a, q)$.

Let $\triangle(p', a', r')$ be a comparison triangle for $\triangle(p, a, r)$ in $U'$. Let $\triangle(r', a', q')$ be a comparison triangle for $\triangle(r, a, q)$ in $U'$. We place the comparison triangles $\triangle(p', a', r')$ and $\triangle(r', a', q')$ in different half-planes with respect to the line $a'r'$. The CAT(0) inequality implies that $\angle_p(r, a) \leq \angle_{p'}(r', a')$, $\angle_r(p, a) \leq \angle_{r'}(p', a')$, $\angle_r(a, q) \leq \angle_{r'}(a', q')$ and $\angle_q(r, a) \leq \angle_{q'}(r', a')$.

Let $\triangle(p'', q'', r'')$ be a comparison triangle for $\triangle(p, q, r)$ in $U'$. Note that either $\angle_{r'}(p', r') + \angle_{r'}(r', q') \geq \pi$ or $\angle_{r'}(p', a') + \angle_{r'}(a', q') \geq \pi$. Assume w.l.o.g.
that \( \angle_a(p', r') + \angle_a(r', q') \geq \pi \). Alexandrov's Lemma implies that \( \angle_{a'}(r', a') \leq \angle_{a''}(r'', q'') \), \( \angle_{r'}(p', a') + \angle_{r'}(a', q') \leq \angle_{r''}(p'', q'') \) and \( \angle_{q'}(r', a') \leq \angle_{q''}(r'', q'') \).

In conclusion, \( \angle_{p'}(r', a') \leq \angle_{p''}(r'', q'') \), \( \angle_{r'}(p', a') + \angle_{r'}(a', q') \leq \angle_{r''}(p'', q'') \), \( \angle_{q'}(r', a') \leq \angle_{q''}(r'', q'') \) and \( \angle_{q'}(r', a') \leq \angle_{q''}(r'', q'') \). So the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality.

\[ \square \]

**Lemma 3.7.** Let the segment \([p, q]\) be as in Case A. Let \( r \) be a point in \( U \) such that the geodesic segments \([r, p]\) and \([r, q]\) do not intersect \( \sigma \). The point \( r \) is chosen such that the quadrilateral \( \text{ramp} \) is convex and the quadrilateral \( \text{ranq} \) is concave. Then, the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) satisfies the CAT(0) inequality.

**Proof.** By Lemma 3.4, \( d'(p, n) = d'(p, a) + d'(a, n) \) and \( d'(r, n) = d'(r, a) + d'(a, n) \).

\[ \text{Figure 10. Case A, the quadrilateral \text{ramp} is convex and the quadrilateral \text{ranq} is concave.} \]

\[ \text{Figure 11} \]

Let \( \triangle(p'', a'', r'') \) be a comparison triangle for \( \triangle(p, a, r) \) in \( U' \). The CAT(0) inequality implies that \( \angle_p(a, r) \leq \angle_{p''}(a'', r'') \) and \( \angle_r(p, a) \leq \angle_{r''}(p'', a'') \).

Let \( \triangle(p', n', r') \) be a comparison triangle for \( \triangle(p, n, r) \) in \( U' \). We consider a point \( a' \) in the interior of the geodesic triangle \( \triangle(p', n', r') \) such that \( d_{\mathbb{R}^2}(p', a') = d(p, a) \) and \( d_{\mathbb{R}^2}(r', a') = d(r, a) \). We can choose the point \( a' \) in this manner, because in \( U' \) we have \( a \in [p, n] \) and \( a \in [r, n] \). Thus \( d(p, a) < d(p, n) \) and
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Thus, \( d(r, a) < d(r, n) \). Thus \( \angle_{p'}(a', r') < \angle_{p'}(n', r') \) and \( \angle_{r'}(a', p') < \angle_{r'}(n', p') \). Since the geodesic triangles \( \triangle(p', a', r') \) and \( \triangle(p'', a'', r'') \) are congruent to each other, \( \angle_{p'}(a', r') \equiv \angle_{p''}(a', r'') \) and \( \angle_{r'}(a', p') \equiv \angle_{r''}(a', p'') \).

In conclusion, for the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) we have \( \angle_p(n, r) < \angle_p'(n', r') \) and \( \angle_r(n, p) < \angle_r'(n', p') \). Because the Alexandrov angle between the geodesic segments \( [p, n] \) and \( [r, n] \) equals zero, we have \( \angle_p(n, p) = 0 < \angle_p'(n', p') \).

Hence the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) satisfies the CAT(0) inequality.

Lemma 3.8. Let the segment \( [p, q] \) be as in Case B. Let \( r \) be a point in \( U \) such that the geodesic segments \( [r, p] \) and \( [r, q] \) do not intersect \( \sigma \). Also \( r \) is chosen such that either the quadrilaterals \( ramp \) and \( rhnq \) are concave or the quadrilateral \( rhpq \) is concave. Then, the geodesic triangles \( \triangle(p, r, q) \) and \( \triangle(p, n, r) \) in \( U' \) satisfy the CAT(0) inequality.

Proof. By Lemma 3.5, \( d'(p, q) = d'(p, a) + d'(a, h) + d'(h, q) \) and \( d'(p, n) = d'(p, a) + d'(a, h) + d'(h, n) \). By Lemma 3.4, \( d'(r, n) = d'(r, h) + d'(h, n) \).

![Figure 12. Case B, the quadrilaterals ramp and rhnq are concave.](image12)

![Figure 13. Case B, the quadrilateral ramp is convex and the quadrilateral rhnq is concave.](image13)
Let $\triangle(p', a', r')$ be a comparison triangle for $\triangle(p, a, r)$ in $U$. Let $\triangle(r', a', h')$ be a comparison triangle for $\triangle(r, a, h)$ in $U$. We place the comparison triangles $\triangle(p', a', r')$ and $\triangle(r', a', h')$ in different half planes with respect to the line $r'a'$. The CAT(0) inequality implies that $\angle_{p}(a) \leq \angle_{p'}(r', a')$, $\angle_{r}(p, a) \leq \angle_{r'}(p', a')$ and $\angle_{r}(a, h) \leq \angle_{r'}(a', h')$.

Let $\triangle(p'', r'', h'')$ be a comparison triangle for $\triangle(p, r, h)$ in $U$. Let $\triangle(q'', r'', h'')$ be a comparison triangle for $\triangle(q, r, h)$ in $U$. We place the comparison triangles $\triangle(p'', r'', h'')$ and $\triangle(q'', r'', h'')$ in different half planes with respect to the line $r''h''$. Note that either $\angle_{a'}(p', r') + \angle_{a'}(r', h') \geq \pi$ or $\angle_{a'}(p', a') + \angle_{a'}(a', h') \geq \pi$. Assume w.l.o.g. $\angle_{a'}(p', r') + \angle_{a'}(r', h') \geq \pi$. Alexandrov’s Lemma implies that $\angle_{p'}(a', r') \leq \angle_{p'}(h'', r'')$ and $\angle_{r'}(p', a') + \angle_{r'}(a', h') \leq \angle_{r'}(p'', h'')$. Hence

$$\angle_{r}(p, a) + \angle_{r}(a, h) \leq \angle_{r'}(p', a') + \angle_{r'}(a', h') \leq \angle_{r'}(p'', h'').$$

The CAT(0) inequality implies that $\angle_{r}(h, q) \leq \angle_{r'}(h'', q'')$, $\angle_{q}(r, h) \leq \angle_{q'}(r'', h'')$ and

$$\angle_{r}(p, q) \leq \angle_{r}(p, h) + \angle_{r}(h, q) \leq \angle_{r'}(p'', h'') + \angle_{r'}(h'', q'') = \angle_{r'}(p'', q'').$$

Let $\triangle(p'', r'', q'')$ be a comparison triangle for $\triangle(p, r, q)$ in $U$. Note that either $\angle_{h''}(p'', r'') + \angle_{h''}(r'', q'') \geq \pi$ or $\angle_{r''}(p'', h'') + \angle_{r''}(h'', q'') \geq \pi$. Assume w.l.o.g. $\angle_{h''}(p'', r'') + \angle_{h''}(r'', q'') \geq \pi$. Alexandrov’s Lemma implies that $\angle_{r''}(p'', h'') + \angle_{r''}(h'', q'') \leq \angle_{r''}(p'', q'')$, $\angle_{p''}(r'', h'') \leq \angle_{p''}(r'', q'')$ and $\angle_{q''}(r'', h'') \leq \angle_{q''}(r'', p'')$. Hence

$$\angle_{p}(r, q) = \angle_{p}(r, a) \leq \angle_{p'}(r', a') \leq \angle_{p'}(r'', a'') \leq \angle_{p''}(r'', q''),$$

$$\angle_{q}(r, h) \leq \angle_{q'}(r', h') \leq \angle_{q''}(r'', p''),$$

$$\angle_{r}(p, a) + \angle_{r}(a, h) \leq \angle_{r'}(p', a') + \angle_{r'}(a', h') \leq \angle_{r'}(p'', h'').$$
\[ \angle_r(p, q) \leq \angle_r(p, a) + \angle_r(a, h) + \angle_r(h, q) \leq \angle_{r''}(p'', h'') + \angle_{r''}(h'', q'') \leq \angle_{r''}(p'', q''). \]

So the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality.

We show further that the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) satisfies the CAT(0) inequality.

Let \( \triangle(p'', a'', r'') \) be a comparison triangle for \( \triangle(p, a, r) \) in \( U' \). The CAT(0) inequality implies that \( \angle_p(a, r) \leq \angle_{p''}(a'', r'') \) and \( \angle_r(p, a) \leq \angle_{r''}(p'', a'') \).

Let \( \triangle(p', n', r') \) be a comparison triangle for \( \triangle(p, n, r) \) in \( U' \). We consider a point \( a' \) in the interior of the geodesic triangle \( \triangle(p', n', r') \) such that \( d_{\mathbb{R}^2}(p', a') = d(p, a) \) and \( d_{\mathbb{R}^2}(r', a') = d(r, a) \). Thus \( \angle_{p'}(a', r') \leq \angle_{p''}(n', r') \) and \( \angle_{r'}(a', p') \leq \angle_{r''}(n', p') \). Because the geodesic triangles \( \triangle(p', a', r') \) and \( \triangle(p'', a'', r'') \) are congruent to each other, we have \( \angle_{p'}(a', r') = \angle_{p''}(a'', r'') \) and \( \angle_{r'}(a', p') = \angle_{r''}(a'', p'') \).

In conclusion, for the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) we have \( \angle_p(n, r) < \angle_{p''}(n', r') \) and \( \angle_r(n, p) < \angle_{r''}(n', p') \). Because the Alexandrov angle between the geodesic segments \([p, n]\) and \([r, n]\) equals zero, we have \( \angle_n(r, p) = 0 < \angle_{n'}(p', r') \).

Hence the geodesic triangle \( \triangle(p, n, r) \) in \( U' \) satisfies the CAT(0) inequality. \( \square \)

**Lemma 3.9.** Let the segment \([p, q]\) be as in Case B. Let \( r \) be a point in \( U \) such that the segment \([r, q]\) does not intersect \( \sigma \) and the segment \([p, r]\) intersects \( \sigma \) \( ([p, r] \cap [a, b] \neq \emptyset, [p, r] \cap [c, h] \neq \emptyset) \). Then, the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality.

**Proof.** By Lemma 3.5, \( d'(p, q) = d'(p, a) + d'(a, h) + d'(h, q) \) and \( d'(p, r) = d'(p, a) + d'(a, h) + d'(h, r) \).

Let \( \triangle(r'', h'', q''') \) be a comparison triangle for \( \triangle(r, h, q) \) in \( U \). Because \( U \) is a CAT(0) space, we have \( \angle_r(h, q) \leq \angle_{r''}(h'', q'') \) and \( \angle_q(h, r) \leq \angle_{q''}(h'', q'') \).

Let \( \triangle(p', q', r') \) be a comparison triangle for \( \triangle(p, q, r) \) in \( U \). We consider a point \( h' \) in the interior of the geodesic triangle \( \triangle(p', q', r') \) such that \( d_{\mathbb{R}^2}(r', h') = \)
Figure 16. Case B

Figure 17

d(r, h) and \( d_{\mathbb{R}^2}(q', h') = d(q, h) \). Thus \( \angle_r(h', q') \leq \angle_r(p', q') \) and \( \angle_q(h', r') \leq \angle_q(p', r') \). Because the geodesic triangles \( \triangle(r', h', q') \) and \( \triangle(r'', h'', q'') \) are congruent to each other, we have \( \angle_r(h', q') \equiv \angle_r(h'', q'') \) and \( \angle_q(h', r') \equiv \angle_q(h'', r'') \).

In conclusion, in \( U' \) we have \( \angle_r(p, q) \leq \angle_r(p', q') \) and \( \angle_q(p, r) \leq \angle_q(p', r') \). Thus the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality.

Lemma 3.10. Let the segment \([p, q]\) be as in Case B. Let \( r \) be a point in \( U \) such that the segments \([p, r]\) and \([r, q]\) intersect \( \sigma \) \( (p, r) \cap [a, b] \neq \emptyset, [p, r] \cap [a, h] \neq \emptyset, [r, q] \cap [a, h] \neq \emptyset, [r, q] \cap [c, h] \neq \emptyset) \). Then, the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality.

Proof. By Lemma 3.4, \( d'(p, r) = d'(p, a) + d'(a, r) \) and \( d'(r, q) = d'(r, h) + d'(h, q) \). Lemma 3.5 implies that \( d'(p, q) = d'(p, a) + d'(a, h) + d'(h, q) \).

Since \( d'(a, h) < d'(a, r) + d'(r, h) \), we have that \( d'(p, q) < d'(p, r) + d'(r, q) \). So in \( U' \) the geodesic triangle \( \triangle(p, q, r) \) is well defined.

Let \( \triangle(p', q', r') \) be a comparison triangle for \( \triangle(p, q, r) \) in \( U' \). Let \( \triangle(r'', h'', q'') \) be a comparison triangle for \( \triangle(r, a, h) \) in \( U \). Let \( a' \in [p', r'] \) be a comparison point
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for \( a \in [p, r] \). Let \( h' \in [r', q'] \) be a comparison point for \( h \in [r, q] \). Note that in \( U' \) we have \( 0 = \angle_p(r, q) < \angle_{p'}(r', q') \) and \( 0 = \angle_q(r, p) < \angle_{q'}(r', p') \).

By the CAT(0) inequality, we have that \( d(a, h) \leq d_{R^2}(a', h') \). Hence, since \( d_{R^2}(a'', h'') = d(a, h) \), we have \( d_{R^2}(a'', h'') \leq d_{R^2}(a', h') \). Therefore, because \( d_{R^2}(a', r') = d_{R^2}(a'', r'') \) and \( d_{R^2}(h', r') = d_{R^2}(h'', r'') \), we have \( \angle_{r'}(a'', h'') \leq \angle_{r'}(a', h') \).

The CAT(0) inequality implies in \( U' \) we have that

\[
\angle_r(p, q) = \angle_r(a, h) \leq \angle_{r'}(a'', h'') \leq \angle_{r'}(a', h') = \angle_{r'}(p', q').
\]

So the geodesic triangle \( \triangle(p, r, q) \) in \( U' \) satisfies the CAT(0) inequality. \( \square \)

**Proposition 3.11.** Every point in \( |K'| \) has a neighborhood that is a CAT(0) space.

**Proof.** Let \( u, v, w \) be three distinct points in \( U \) chosen such that they do not belong to \( \sigma \), and such that the geodesic segments \([u, v], [u, w]\) and \([v, w]\) in \( U \) do not intersect \( \sigma \). Note that the geodesic triangle \( \triangle(u, v, w) \) in \( U' \) fulfills the CAT(0) inequality. Hence, due to Lemmas 3.6–3.10, we may conclude that any geodesic triangle in \( U' \) fulfills the CAT(0) inequality. So \( U' \) is a CAT(0) space.
Let \( y \) be a point in \( |K| \) that does not belong to \( \sigma \). Let \( U_y \) be a neighborhood of \( y \) homeomorphic to a closed ball of radius \( r_y \), \( U_y = \{ x \in |K| \mid d(y, x) \leq r_y \} \). The radius \( r_y \) is chosen small enough such that \( U_y \) does not intersect \( \sigma \). For any \( y \) in \( |K'| \) that does not lie on \([a, b], [a, c] \) or \([a, h] \), we consider a neighborhood \( U'_y \) that coincides with \( U_y \). \( U'_y \) is hence a CAT(0) space.

So every point in \( |K'| \) has a neighborhood which is a CAT(0) space. □

We are now in the position to show the main result of the paper: any finite, CAT(0) square 2-complex retracts to a point through subspaces which remain, at each step of the retraction, CAT(0) spaces.

**Theorem 3.12.** Let \( K \) be a finite, CAT(0) square 2-complex. Then \( K \) collapses to a point through CAT(0) subspaces \( |K'| \).

**Proof.** Proposition 3.1 implies that \( K \) has a 2-cell with a free 1-dimensional face. We fix a point \( p \) in the interior of a 2-cell of \( K \). We define the map \( R : |K| \times [0, 1] \rightarrow |K| \) which associates for any \( x \in |K| \) and for any \( t \in [0, 1] \), to \((x, t)\) the point of a distance \( t \cdot d(p, x) \) from \( x \) along the geodesic segment \([p, x]\). Because \( |K| \) has a strongly convex metric, the map \( R \) is a continuous retraction of \( |K| \) to \( p \). \( R(|K| \times [0, 1]) \) is therefore contractible, and then it is simply connected. Let \( a, b, c \) be any three distinct points in \( R(|K| \times [0, 1]) \) such that the unique geodesic segment \([b, c] \) belongs to a 1-cell that is the face of a single 2-cell \( \sigma \) in the complex. Also the points \( a, b, c \) are chosen such that the geodesic triangle \( \delta = \triangle(a, b, c) \) is contained in \( \sigma \). For each such \( \delta = \triangle(a, b, c) \), we deformation retract \( R(|K| \times [0, 1]) \) by pushing in \( \delta \) starting at \([b, c]\). We obtain each time a subspace \( |K'| = R(|K| \times [0, 1]) \) which remains simply connected and, by Proposition 3.11, non-positively curved. So \( |K'| \) is a CAT(0) space implying that any two points in \( |K'| \) are joined by a unique geodesic segment in \( |K'| \). If at a certain step we delete the point \( p \), we fix another point \( p \) in the interior of a 2-cell of \( K' \), define the map \( R \) as before, and retract the space through CAT(0) subspaces further. Since \( K \) is finite, we reach, after a finite number of steps, a 1-dimensional spine \( L \). Since \( |L| \) is also a CAT(0) space, it is contractible. Taking into account that a contractible 1-complex is collapsible, the result follows. □

As a consequence of the above result, we have the following.

**Corollary 3.13.** Let \( K \) be a locally finite, CAT(0) square 2-complex. Then \( K \) has an arborescent structure.

**Proof.** We fix a vertex \( v \) of \( K \). For each integer \( n \), let \( B_n \) be the full subcomplex of \( K \) generated by the vertices that can be joined to \( v \) by an edge-path
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of length at most \( n \). Note that for each \( n \), \( B_n \) is a ball in a CAT(0) space. It is therefore contractible and, in particular, it is simply connected.

Furthermore, because \( K \) is a CAT(0) space, it is locally a CAT(0) space. For each \( n > 0 \), \( B_n \) is therefore itself locally a CAT(0) space. Since each \( B_n, n > 0 \) is simply connected and locally a CAT(0) space, it is a CAT(0) space. Hence for each \( n \), \( B_n \) is a finite square 2-complex that is a CAT(0) space. So, according to Theorem 3.12, each such \( B_n \) is collapsible. Therefore \( K \) is the monotone union \( \bigcup_{n=1}^{\infty} B_n \) of a sequence of collapsible subcomplexes. This ensures that \( K \) has an arborescent structure.

We give below a few applications of the collapsibility of square 2-complexes. Due to the fact that median 2-complexes are CAT(0) square 2-complexes (see [7, Theorem 6.1]), the following holds.

**Corollary 3.14.** Median 2-complexes are collapsible.

The collapsibility of CAT(0) square 2-complexes implies that the first derived subdivision of such complexes is also collapsible ([32, Theorem 2.10]). This happens because if a complex is non-evasive (see [32]), then it is collapsible ([18, Proposition 1]). Hence the following result holds.

**Corollary 3.15.** CAT(0) square 2-complexes admit collapsible triangulations.

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**References**

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