Tangent prolongation of $C^r$-differentiable loops

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Abstract. The aim of our paper is to generalize the tangent prolongation of Lie groups to non-associative multiplications and to examine how the weak associative and weak inverse properties are transferred to the multiplication defined on the tangent bundle. We obtain that the tangent prolongation of a $C^r$-differentiable loop ($r \geq 1$) is a $C^{r-1}$-differentiable loop that has the classical weak inverse and weak associative properties of the initial loop.

1. Introduction

The tangent space of a differentiable manifold $M$ at $\xi \in M$ is $T_{\xi}(M)$ and the tangent bundle is $T(M)$. The tangent map of a differentiable map $\varphi : M \to M$ at $\xi \in M$ will be denoted by $d_\xi \varphi : T_{\xi}(M) \to T_{\varphi(\xi)}(M)$. For a Lie group $G$ with Lie algebra $\mathfrak{g}$ any element $(\xi, X) \in T(G)$ can be identified with the element $(\xi, d_\xi \lambda_\xi^{-1} X)$ of the Cartesian product $G \times \mathfrak{g}$, where $\lambda_\xi : G \to G$ denotes the left translation on $G$. It is well-known that the Lie group structure of $G$ has a natural prolongation to tangent bundle $T(G)$ such that the corresponding multiplication

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on $G \times \mathfrak{g}$ is expressed by

$$(\xi, x) \cdot (\eta, y) = \left( \xi \eta, d_{\xi \eta} \lambda_{\xi \eta}^{-1} \frac{d}{dt} \big|_{t=0} (\xi \exp tx \cdot \eta \exp ty) \right) = (\xi \eta, d_{\xi}(\lambda_{\eta}^{-1} \rho_{\eta})x + y),$$

(1)

where $\xi, \eta \in G$ and $x, y \in \mathfrak{g}$. Clearly, the tangent prolongation structure on $G \times \mathfrak{g}$ is a semi-direct product $G \ltimes \mathfrak{g}^+$, where $\mathfrak{g}^+$ is the additive group of $\mathfrak{g}$, cf. [12], [5] and [8].

The aim of our paper is to generalize the tangent prolongation (1) to non-associative multiplications and to show that the weak associative and weak inverse properties are preserved by the tangent prolongation. We prove that the tangent prolongation of a $C^r$-differentiable loop ($r \geq 1$) is a $C^{r-1}$-differentiable loop that has the same classical weak inverse and weak associative properties as the initial loop. It is worth noting that, unlike the associative case, the multiplications of differentiable loops are not necessarily analytic (cf. the loop constructions in [6, Part 2]). Our research is a reflection on the subject raised by J. Grabowski in his conference lecture *Tangent and cotangent loopoids* [2].

We examine in Section 3 the two-sided inverse, left and right inverse, monoassociative, left and right alternative, flexible, left and right Bol properties of general abelian linear extensions. The loops in this class of extensions (cf. [11], [1]) are natural abstract generalizations of the tangent prolongation of differentiable loops. Section 4 is devoted to the discussion of the differentiability properties of tangent prolongations. In Section 5, we investigate such abelian linear extensions which are determined by a homomorphism of the inner mapping group into the linear group acting on the tangent space at the identity element. We apply our results to the tangent prolongation of $C^r$-differentiable loops, where the above homomorphism is determined by the tangent map of inner mappings. At the end, in Section 6, we obtain our main result on the tangent prolongation of $C^r$-differentiable loops and on their classical weak inverse and weak associative properties.

2. Preliminaries

There are two usual formal definitions of quasigroups: as a set with a binary multiplication, or the other, as a set with three binary operations: multiplication, left division and right division. We will study, among others, the differentiability properties of each operation; therefore, it is worth using the definition with three operations.
A quasigroup \((Q, \cdot, \setminus, /)\) is a set equipped with three binary operations: multiplication \(\cdot\), left division \(\setminus\) and right division \(/\) satisfying the identities

\[
y = x \cdot (x \setminus y), \quad y = x \setminus (x \cdot y), \quad y = (y/x) \cdot x, \quad y = (y \cdot x)/x, \quad x, y \in Q.
\]

A loop \((L, \cdot, \setminus, /)\) is a quasigroup with an identity element \(e \in L\), \((e \cdot x = x \cdot e = x\) for each \(x \in L\)). The left translations \(\lambda_x : y \mapsto x \cdot y\), and the right translations \(\rho_x : y \mapsto y \cdot x\) are bijective maps and \(x \setminus y = \lambda_x^{-1} y\), \(x/y = \rho_y^{-1} x\), \(x, y \in L\). The opposite loop of \((L, \cdot, \setminus, /)\) is the loop \((L, *, \setminus, /)*\) defined by the operations \(x * y = y \cdot x\), \(x \setminus y = y/x\), \(y */ x = x \setminus y\) on the same underlying set.

The automorphism group of a loop \(L\) is denoted by \(\text{Aut}(L)\). The group \(\text{Mlt}(L)\) generated by all left and right translations of the loop \(L\) is called the multiplication group of \(L\). The inner mapping group \(\text{Inn}(L) = \{\varphi \in \text{Mlt}(L) ; \varphi(e) = e\}\) of \(L\) is the subgroup of \(\text{Mlt}(L)\) consisting of all bijections in \(\text{Mlt}(L)\) fixing the identity element \(e\).

We will reduce the use of parentheses on the loop \(L\) by the following convention: juxtaposition will denote multiplication, the division operations are less binding than juxtaposition, and the multiplication is less binding than the divisions. For instance, the expression \(xy/u \cdot v/w\) is a short form of \((x \cdot y)/u \cdot (v/w)\).

We say that a loop \(L\) has a weak associative property if it satisfies one of the following properties:

\begin{align*}
\text{monoassociative : } & x \cdot x^2 = x^2 \cdot x \quad \text{for all } x \in L, \\
\text{left alternative : } & x \cdot xy = x^2 \cdot y \quad \text{for all } x, y \in L, \\
\text{right alternative : } & yx \cdot x = y \cdot x^2 \quad \text{for all } x, y \in L, \\
\text{flexible : } & x \cdot yx = xy \cdot x \quad \text{for all } x, y \in L, \\
\text{left Bol loop : } & (x \cdot yx)z = x(y \cdot xz) \quad \text{for all } x, y, z \in L, \\
\text{right Bol loop : } & z(xy \cdot x) = (zx \cdot y)x \quad \text{for all } x, y, z \in L.
\end{align*}

The element \(e/x\), respectively, \(x \setminus e\) is the left inverse, respectively, the right inverse of \(x \in L\). If the left and the right inverses of \(x\) coincide, then \(x^{-1} = e/x = x \setminus e\) is the inverse of \(x \in L\). If any element \(x \in L\) has an inverse \(x^{-1}\) then we say that \(L\) has two-sided inverse property. \(L\) satisfies the left, respectively, the right inverse property if there exists a bijection \(\iota : L \rightarrow L\), such that \(\iota(x) \cdot xy = y\), respectively, \(yx \cdot \iota(x) = y\) holds for all \(x, y \in L\). It is well-known that in loops with left or right inverse property all elements have inverses (cf. [7, I.4.2 Theorem]), hence \(\iota(x) = x^{-1}\).

We say that a loop \(L\) has a weak inverse property if it has one of the following properties:
two-sided inverse: \( x^{-1} = e/x = x\backslash e \) for all \( x \in L \),
left inverse: \( x^{-1} \cdot xy = y \) for all \( x, y \in L \),
right inverse: \( yx \cdot x^{-1} = y \) for all \( x, y \in L \).

A loop \( L \) defined on a differentiable manifold is called \( C^r \)-differentiable loop, 0 \( < r \in \mathbb{N} \) or \( r = \infty \), if the multiplication \((x, y) \mapsto x \cdot y\), the left division \((x, y) \mapsto x\backslash y\) and the right division \((x, y) \mapsto x/y\) are \( C^r \)-differentiable \( L \times L \to L \) maps.

3. Linear abelian extensions

Before the introduction of a natural loop-multiplication on the tangent bundle of differentiable loops we consider a more general class of abstract non-associative extensions. If \((A, +)\) is an abelian group, \( \varphi, \psi \) are automorphisms of \( A \) and \( c \in A \) is a constant, then the binary operations

\[
x \cdot y = \varphi(x) + \psi(y) + c, \quad x \backslash y = \psi^{-1}(y - \varphi(x) - c), \quad y / x = \varphi^{-1}(y - \psi(x) - c)
\]

on \( A \) determine a quasigroup \((A, \cdot, \backslash, /)\), called \( T \)-quasigroup or central quasigroup over the abelian group \( A \), investigated intensively in the last half century (cf. e.g. [3], [4], [9] and [10]). The study of non-associative abelian extensions of abelian groups by loops, such that the quasigroups, induced between cosets of the kernel subgroups, are \( T \)-quasigroups, is initiated by D. ŠTANOVSKÝ and P. VOJTECHOVSKÝ in [11]. We investigated in a recent paper [1] the two-sided inverse, left inverse, right inverse and inverse properties of a class of abelian extensions, called abelian linear extensions.

Let \( A = (A, +) \) be an abelian group and \( L = (L, \cdot, \backslash, /) \) a loop with the identity element \( e \in L \). A pair \((P, Q)\) is called a loop cocycle if \( P, Q \) are mappings \( L \times L \to \text{Aut}(A) \) satisfying \( P(\xi, e) = \text{Id} = Q(e, \eta) \) for every \( \xi, \eta \in L \).

**Definition 3.1.** If \((P, Q)\) is a loop cocycle, then the binary operations on \( L \times A \)

\[
(\xi, x) \cdot (\eta, y) = (\xi \eta, P(\xi, \eta)x + Q(\xi, \eta)y),
(\xi, x) \backslash (\eta, y) = (\xi \eta, Q(\xi, \xi \eta)^{-1}(y - P(\xi, \xi \eta)x)),
(\eta, y) / (\xi, x) = (\eta / \xi, P(\eta / \xi, \xi)^{-1}(y - Q(\eta / \xi, \xi)x))
\]

define the linear abelian extension \( F(P, Q) \) of the group \( A \) by the loop \( L \). The linear abelian extension \( F(P, Q) \) is a loop with the identity element \((e, 0)\).
Definition 3.1 implies that if a linear abelian extension $F(P,Q)$ has one of the weak inverse or weak associative properties, then the loop $L$ necessarily has the corresponding property. The left and right inverses in $F(P,Q)$ are

$$(\xi, x)\backslash(\epsilon, 0) = (\xi\backslash\epsilon, -Q(\xi, \xi)\epsilon^{-1}P(\xi, \xi)\epsilon x),$$

$$(\epsilon, 0)/(\xi, x) = (\epsilon/\xi, -P(\epsilon/\xi, \xi)^{-1}Q(\epsilon/\xi, \xi)x),$$

hence we obtain:

**Proposition 3.1.** If the loop $L$ has one of the weak inverse or weak associative properties, the extension $F(P,Q)$ has the corresponding property if and only if the loop cocycle $(P,Q)$ fulfills the identities given in the following list for each property:

(A) two-sided inverse:

$$P(\xi, \xi^{-1}) = Q(\xi, \xi^{-1})P(\xi^{-1}, \xi)^{-1}Q(\xi^{-1}, \xi),$$

(B) left inverse:

$$Q(\xi^{-1}, \xi\eta) = Q(\xi, \eta)^{-1}, \quad P(\xi^{-1}, \xi\eta) = Q(\xi, \eta)^{-1}P(\xi, \eta)Q(\xi^{-1}, \xi)^{-1}P(\xi^{-1}, \xi),$$

(C) right inverse:

$$P(\xi\eta, \eta^{-1}) = P(\xi, \eta)^{-1}, \quad Q(\xi, \eta^{-1}) = P(\xi, \eta)^{-1}Q(\xi, \eta)P(\eta, \eta^{-1})^{-1}Q(\eta, \eta^{-1}),$$

(D) monoassociative:

$$P(\xi, \xi^2) + Q(\xi, \xi) \quad (P(\xi, \xi) + Q(\xi, \xi)) = P(\xi^2, \xi)(P(\xi, \xi) + Q(\xi, \xi)) + Q(\xi^2, \xi),$$

(E) left alternative:

$$Q(\xi, \xi\eta)Q(\xi, \eta) = Q(\xi^2, \eta),$$

$$P(\xi, \xi\eta) + Q(\xi, \xi\eta)P(\xi, \eta) = P(\xi^2, \eta)(P(\xi, \xi) + Q(\xi, \xi)),$$

(F) right alternative:

$$P(\eta\xi, \xi)P(\eta, \xi) = P(\eta, \xi^2),$$

$$P(\eta\xi, \xi)Q(\eta, \xi) + Q(\eta\xi, \xi) = Q(\eta, \xi^2)(P(\xi, \xi) + Q(\xi, \xi)),$$

(G) flexible:

$$Q(\xi, \eta\xi)P(\eta, \xi) = P(\xi\eta, \xi)Q(\xi, \eta),$$

$$P(\xi, \eta\xi) + Q(\xi, \eta\xi)Q(\eta, \xi) = P(\xi\eta, \xi)P(\xi, \eta) + Q(\xi\eta, \xi),$$
(H) left Bol:

\[ Q(\xi, \eta \cdot \xi \zeta) Q(\eta, \xi \zeta) Q(\xi, \zeta) = Q(\xi \cdot \eta \xi, \zeta), \]

\[ Q(\xi, \eta \cdot \xi \zeta) P(\eta, \xi \zeta) = P(\xi \cdot \eta \xi, \zeta) Q(\xi, \eta \xi) P(\eta, \xi), \]

\[ P(\xi, \eta \cdot \xi \zeta) + Q(\xi, \eta \cdot \xi \zeta) Q(\eta, \xi \zeta) P(\xi, \zeta) = P(\xi \cdot \eta \xi, \zeta) (P(\xi, \eta \xi) + Q(\eta, \xi \xi) Q(\eta, \xi)). \]

(J) right Bol:

\[ P(\zeta, \eta \cdot \xi) = P(\zeta \cdot \eta \xi, \xi) P(\zeta, \eta \xi), \]

\[ Q(\zeta, \eta \cdot \xi) P(\xi, \eta \xi) Q(\xi, \eta) = P(\zeta \cdot \eta \xi, \xi) P(\xi, \eta \xi) Q(\xi, \zeta) + Q(\xi \cdot \eta, \xi \xi). \]

Proof. Assertions (A), (B), (C) with respect to the weak inverse properties of the extension \( F(P, Q) \) are proved by Propositions 3, 5 and 7 in [1].

Assertion (D) follows from the equations

\[ (\xi, x) \cdot (\xi, x)^2 = (\xi \cdot \xi^2, P(\xi, \xi^2) x + Q(\xi, \xi^2) x) = (\xi, x^2) \cdot (\xi, x) \]

\[ = (\xi^2 \cdot \xi, P(\xi^2, \xi)(P(\xi, \xi) x + Q(\xi, \xi) x) + Q(\xi^2, \xi) x), \]

for all \( \xi \in L \) and \( x \in A \).

The components of the left alternative identity of \( F(P, Q) \) give the identity

\[ P(\xi, \xi \eta) x + Q(\xi, \xi \eta)(P(\xi, \xi) x + Q(\xi, \xi) y) = P(\xi^2, \eta)(P(\xi, \xi) x + Q(\xi, \xi) y) + Q(\xi^2, \eta) y \]

for all \( \xi, \eta \in L \) and \( x, y \in A \). Putting \( x = 0 \), respectively, \( y = 0 \), we get the identities (E).

Considering the opposite loop of \( L \) we obtain (F). Similar computations give the condition (G) of flexible property.

The components of the left Bol identity in \( F(P, Q) \) give the equation

\[ P(\xi, \eta \cdot \xi \zeta) x + Q(\xi, \eta \cdot \xi \zeta)[P(\eta, \xi \zeta) x + Q(\eta, \xi \zeta)(P(\xi, \xi) x + Q(\xi, \zeta) z)] = P(\xi \cdot (\eta \xi), \zeta)[P(\xi, \eta \xi) x + Q(\xi, \eta \xi)(P(\eta, \xi) y + Q(\eta, \xi) x)] + Q(\xi \cdot \eta \xi, \zeta) z \]

for all \( \xi, \eta, \zeta \in L \) and \( x, y, z \in A \). Putting \( x = y = 0 \) into the identity (4), we obtain the first identity in assertion (H), the substitutions \( x = z = 0 \) and \( y = z = 0 \) give the second and the third identities in (H).

We obtain condition (J) from (H) using the opposite loop of \( L \). \( \square \)
4. Tangent prolongation

We extend the construction (1) of the tangent prolongation of Lie groups to \(C^r\)-differentiable loops \(L, r \geq 1\). Similarly to Lie groups, the map \(\mathcal{F} : (\xi, x) \mapsto d_x \lambda_x^r \) for any \((\xi, x) \in L \times T_r(L)\) gives an identification \(\mathcal{F} : L \times T_r(L) \to T(L)\) of the Cartesian product manifold \(L \times T_r(L)\) with the tangent bundle \(T(L)\).

Definition 4.1. Let \(\alpha(t), \beta(t)\) be differentiable curves in the \(C^r\)-differentiable loop \(L (r \geq 1)\) with initial data \(\alpha(0) = \beta(0) = \epsilon\) and \(\alpha'(0) = x, \beta'(0) = y, x, y \in T_r(L)\). We define the multiplication of the tangent prolongation \(\mathcal{F}(L \times T_r(L))\) of \(L \times T_r(L)\) by

\[
(\xi, x) \cdot (\eta, y) = \left(\xi \eta, d_\xi \lambda_{\xi \eta}^{-1} \frac{d}{dt} \bigg|_{t=0} (\xi \alpha(t) \cdot \eta \beta(t))\right) = (\xi \eta, d_\xi \lambda_{\xi \eta}^{-1} \rho_\eta \lambda_x^r + d_x \lambda_{\xi \eta}^{-1} \lambda_x y).
\]

The identity element of \(\mathcal{F}(L \times T_r(L))\) is \((\epsilon, 0)\).

Clearly, the maps \(\lambda_{\xi \eta}^{-1} \rho_\eta \lambda_x^r\) and \(\lambda_{\xi \eta}^{-1} \lambda_x y\) are inner mappings of the loop \(L\). They are differentiable maps and the assignment \(\phi \mapsto d_\phi\) is a homomorphism \(d_\varepsilon : \text{Inn}(L) \to \text{Aut}(T_r(L))\). Naturally, the map \(\varphi \mapsto d_\varphi \xi\) with arbitrary \(\xi \in L\) acts on all elements \(\varphi \in \text{Mlt}(L)\) of the multiplication group \(\text{Mlt}(L)\) of \(L\).

Lemma 4.1. Let \(L\) be a \(C^r\)-differentiable loop and \(T_r(L)\) its tangent vector space. The tangent prolongation \(\mathcal{F}(L \times T_r(L))\) of \(L\) is the linear abelian extension \(F(P, Q)\) of the abelian group \(T_r(L)\) by \(L\) determined by the loop cocycle \((P, Q)\) of \(\mathcal{F}(L \times T_r(L))\) given by the maps

\[
P(\xi, \eta) := d_\xi (\lambda_{\xi \eta}^{-1} \rho_\eta \lambda_x^r), \quad Q(\xi, \eta) := d_\xi (\lambda_{\xi \eta}^{-1} \lambda_x y).
\]

Proposition 4.2. The tangent prolongation \(\mathcal{F}(L \times T_r(L))\) of a \(C^r\)-differentiable loop \(L (r \geq 1)\) is a \(C^r\)-differentiable loop.

Proof. Since the multiplication, the left and right divisions of \(L\) are \(C^r\)-differentiable maps, and \(\lambda_{\sigma \tau} = \sigma \tau, \rho_{\sigma \tau} = \tau \sigma, \lambda_{\sigma}^{-1} \tau = \sigma^{-1} \tau, \rho_{\sigma}^{-1} \tau = \tau / \sigma\), the left and right translations and their inverses are \(C^r\)-differentiable maps, too. According to Lemma 4.1, the loop cocycle \((P, Q)\) of \(\mathcal{F}(L \times T_r(L))\) is expressed by the first derivative of the products of left and right translations and of their inverses. Hence \(P(\xi, \eta), Q(\xi, \eta)\), and the multiplication of \(\mathcal{F}(L \times T_r(L))\) determined by (3) are \(C^r\)-differentiable. According to (3), the left and right divisions of the tangent prolongation \(\mathcal{F}(L \times T_r(L))\) are \(C^r\)-differentiable, since the maps

\[
P(\xi, \eta)^{-1} := d_\xi (\lambda_{\xi}^{-1} \rho_\eta^{-1} \lambda_{\xi \eta}), \quad Q(\xi, \eta)^{-1} := d_\xi (\lambda_{\eta}^{-1} \lambda_{\xi}^{-1} \lambda_{\xi \eta})
\]

are \(C^r\)-differentiable. It follows that the tangent prolongation \(\mathcal{F}(L \times T_r(L))\) is a \(C^r\)-differentiable loop. \(\Box\)
5. Tangent-like extension

Now, we treat the properties of the tangent prolongation in a more general abstract setting, considering an arbitrary homomorphism $\Phi : \text{Inn}(L) \to \text{Aut}(A)$ instead of the assignment $d_e : \text{Inn}(L) \to \text{Aut}(T_e(L))$, furthermore we apply the obtained conditions to the tangent prolongation of $L$.

**Definition 5.1.** Let $L$ be a loop, $A$ an abelian group, and $\Phi : \text{Inn}(L) \to \text{Aut}(A)$ a homomorphism. The linear abelian extension $F(P, Q)$ defined by the loop cocycle $(P, Q)$

$$P(\xi, \eta) := \Phi(\lambda_{\xi\eta}^{-1}\rho_{\eta}\lambda_{\xi}), \quad Q(\xi, \eta) := \Phi(\lambda_{\xi\eta}^{-1}\lambda_{\xi}\lambda_{\eta})$$

is called a tangent-like extension of the group $A$ by the loop $L$ and will be denoted by $\Phi(L, A)$.

**Proposition 5.1.** If the loop $L$ has one of the weak inverse or weak associative properties, then the tangent-like extension $\Phi(L, A)$, respectively, the tangent prolongation $\mathcal{T}(L \times T_e(L))$ has the corresponding property if and only if the loop cocycle fulfills the identities given in the following list for each property:

(A) two-sided inverse:

(i) $\Phi(L, A)$:

$$\Phi(\rho_{\xi^{-1}}\lambda_{\xi}) = \Phi(\lambda_{\xi}\rho_{\xi}^{-1}\lambda_{\xi}^{-1})$$

(ii) $\mathcal{T}(L \times T_e(L))$:

$$d_\xi\rho_{\xi^{-1}} = d_\xi\lambda_{\xi}\rho_{\xi}^{-1}\lambda_{\xi}^{-1},$$

(B) left inverse:

(i) $\Phi(L, A)$:

$$\Phi(\lambda_{\xi}^{-1}\rho_{\xi}\lambda_{\xi}^{-1}) = \Phi(\lambda_{\xi}\rho_{\xi}^{-1}\lambda_{\xi}^{-1})$$

(ii) $\mathcal{T}(L \times T_e(L))$:

$$d_\xi\lambda_{\xi}\rho_{\xi} = d_\xi\lambda_{\xi}$$

(C) right inverse:

(i) $\Phi(L, A)$:

$$\Phi(\lambda_{\xi}^{-1}\lambda_{\xi}^{-1}\lambda_{\xi}^{-1}) = \Phi(\lambda_{\xi}^{-1}\rho_{\eta}\lambda_{\xi}^{-1}\lambda_{\xi}^{-1})$$

(ii) $\mathcal{T}(L \times T_e(L))$:

$$d_\eta\rho_{\eta}\lambda_{\eta} = d_\eta\lambda_{\xi}$$

(D) monoassociative:

(i) $\Phi(L, A)$:

$$\Phi(\lambda_{\xi}^{-1}\rho_{\xi}^{2}\lambda_{\xi}) + \Phi(\lambda_{\xi}^{-1}\lambda_{\xi}\rho_{\xi}\lambda_{\xi}) + \Phi(\lambda_{\xi}^{-1}\lambda_{\xi}^{2}) = \Phi(\lambda_{\xi}^{-1}\rho_{\xi}^{2}\lambda_{\xi}) + \Phi(\lambda_{\xi}^{-1}\rho_{\xi}\lambda_{\xi}^{2}) + \Phi(\lambda_{\xi}^{-1}\lambda_{\xi}^{2}\lambda_{\xi})$$

(ii) $\mathcal{T}(L \times T_e(L))$:

$$d_\xi\rho_{\xi}^{2} + d_\xi\lambda_{\xi}\rho_{\xi} + d_\xi\lambda_{\xi}^{2} = d_\xi\rho_{\xi}^{2} + d_\xi\rho_{\xi}\lambda_{\xi} + d_\xi\lambda_{\xi}^{2}$$

(E) left alternative:

(i) $\Phi(L, A)$:

$$\Phi(\lambda_{\xi}^{-1}\rho_{\xi}\lambda_{\xi}^{-1}) + \Phi(\lambda_{\xi}^{-1}\lambda_{\xi}\rho_{\xi}\lambda_{\xi}) = \Phi(\lambda_{\xi}^{-1}\rho_{\xi}\lambda_{\xi}^{-1}) + \Phi(\lambda_{\xi}^{-1}\rho_{\xi}\lambda_{\xi}^{2})$$

(ii) $\mathcal{T}(L \times T_e(L))$:

$$d_\xi\rho_{\xi} + d_\xi\lambda_{\xi}\rho_{\xi} = d_\xi\rho_{\xi} + d_\xi\rho_{\xi}\lambda_{\xi}$$
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(F) right alternative:

\begin{align*}
(\text{i}) & \quad \Phi(L, A) : \\
& \quad \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\eta, \xi}^{-1} \lambda_{\eta} \lambda_{\xi}) \\
& \quad = \Phi(\lambda_{\eta, \xi}^{-1} \lambda_{\eta} \rho_{\xi} \lambda_{\xi}) + \Phi(\lambda_{\eta, \xi}^{-1} \lambda_{\eta} \lambda_{\xi}), \\
(\text{ii}) & \quad \mathcal{T}(L \times T_{\epsilon}(L)) : \quad d_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\eta} \lambda_{\xi} = d_{\xi} \lambda_{\xi} \rho_{\xi} + d_{\xi} \lambda_{\eta} \lambda_{\xi},
\end{align*}

(G) flexible:

\begin{align*}
(\text{i}) & \quad \Phi(L, A) : \\
& \quad \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}) \\
& \quad = \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}), \\
(\text{ii}) & \quad \mathcal{T}(L \times T_{\epsilon}(L)) : \quad d_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\xi} \lambda_{\eta} = d_{\xi} \lambda_{\xi} \rho_{\xi} + d_{\xi} \lambda_{\eta} \lambda_{\eta},
\end{align*}

(H) left Bol:

\begin{align*}
(\text{i}) & \quad \Phi(L, A) : \\
& \quad \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}) \\
& \quad = \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}), \\
(\text{ii}) & \quad \mathcal{T}(L \times T_{\epsilon}(L)) : \quad d_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\xi} \lambda_{\eta} = d_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\xi} \lambda_{\eta},
\end{align*}

(J) right Bol:

\begin{align*}
(\text{i}) & \quad \Phi(L, A) : \\
& \quad \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}) \\
& \quad = \Phi(\lambda_{\xi, \eta}^{-1} \rho_{\xi} \lambda_{\eta} \lambda_{\xi}) + \Phi(\lambda_{\xi, \eta}^{-1} \lambda_{\eta} \lambda_{\xi}), \\
(\text{ii}) & \quad \mathcal{T}(L \times T_{\epsilon}(L)) : \quad d_{\xi} \lambda_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\eta} \lambda_{\xi} \eta = d_{\xi} \lambda_{\xi} \rho_{\xi} \lambda_{\eta} + d_{\xi} \lambda_{\xi} \lambda_{\xi} \eta.
\end{align*}

**Proof.** Substituting the expression (5) of the loop cocycle of the tangent-like extension $\Phi(L, A)$ into the first identities of (B), (C), (E), (F), (G), respectively, into the first two identities of (H), (J) in Proposition 3.1, we get relationships which are equivalent to the identities that characterize the corresponding weak inverse or weak associative property of $L$. The replacement of (5) into the further identities results in non-trivial conditions for each property of the tangent-like extension $\Phi(L, A)$, which are listed in items (i) of Proposition 5.1.

The inner mapping group $\text{Inn}(L)$ is a subgroup of the group generated by all left and right translations. The map $\Phi : \text{Inn}(L) \to \text{Aut}(A)$ is defined only on $\text{Inn}(L)$, but the assignment $\phi \mapsto d_{\xi} \phi$, $\xi \in L$ is defined on all left and right translations of the $\mathcal{C}^r$-differentiable loop $L$. Hence we can simplify the equations obtained by the application of the conditions for the tangent-like extension to the tangent prolongation. We list for each property the corresponding necessary and sufficient conditions of the tangent prolongation $\mathcal{T}(L \times T_{\epsilon}(L))$ in items (ii) of Proposition 5.1. \qed
6. Properties of the tangent prolongation

**Theorem 6.1.** The tangent prolongation $\mathcal{T}(L \times T_{\epsilon}(L))$ of a $C^r$-differentiable loop $L$ ($r \geq 1$) is a $C^{r-1}$-differentiable loop that has a weak inverse or weak associative property if and only if $L$ has the corresponding property.

**Proof.** According to Proposition 4.2, the tangent prolongation $\mathcal{T}(L \times T_{\epsilon}(L))$ of $L$ is $C^{r-1}$-differentiable. It follows from Lemma 4.1 and Definition 3.1 that a weak inverse or weak associative property of the tangent prolongation $\mathcal{T}(L \times T_{\epsilon}(L))$ implies the corresponding property of $L$. Let us consider a differentiable curve $\xi(t)$ in $L$ with initial values $\xi(0) = \xi$ and $\frac{d\xi}{dt}(0) = \dot{\xi}$ defined on an open interval $I \subset \mathbb{R}$ containing $0 \in I$.

If all elements of $L$ have two-sided inverses, then the derivation of the identity $\xi(t)^{-1} \cdot \xi(t) = \epsilon$ at $t = 0$ gives

$$d_{\xi^{-1}} \rho_{\xi}(\dot{\xi}^{-1}) + d_{\xi} \lambda_{\xi^{-1}}(\dot{\xi}) = 0,$$

where $\dot{\xi}^{-1} := \frac{d_{\xi^{-1}}(\Omega)}{dt}(0)$. Hence we can express

$$\dot{\xi}^{-1} = -d_{\xi} \rho_{\xi}^{-1} \lambda_{\xi^{-1}}(\dot{\xi}). \tag{6}$$

The derivation of the identity $\xi(t) \cdot \xi(t)^{-1} = \epsilon$ at $t = 0$ implies

$$\dot{\xi}^{-1} = -d_{\xi} \lambda_{\xi}^{-1} \rho_{\xi^{-1}}(\dot{\xi}). \tag{7}$$

It follows

$$d_{\xi} \rho_{\xi^{-1}} = d_{\xi} \lambda_{\xi} \rho_{\xi^{-1}} \lambda_{\xi^{-1}},$$

giving condition (A)(ii) in Proposition 5.1.

Assume that $L$ has the left inverse property, and differentiate the identity

$$\xi(t)^{-1} \cdot \xi(t) = \eta \quad \text{at} \ t = 0.$$

We obtain

$$d_{\xi^{-1}} \rho_{\xi \eta}(\dot{\xi}) + d_{\xi} \lambda_{\xi}^{-1} \rho_{\eta}(\dot{\xi}) = 0. \tag{8}$$

Replacing (6) into (8) gives

$$d_{\xi} \rho_{\xi \eta} \rho_{\xi}^{-1} \lambda_{\xi}^{-1} = d_{\xi} \lambda_{\xi}^{-1} \rho_{\eta},$$

which is equivalent to condition (B)(ii).
Similarly, if \( L \) has the right inverse property, then it follows from the identity 
\[ \eta \xi(t) \cdot (\xi(t))^{-1} = \eta \]
that 
\[ d_\xi \rho_\xi^{-1} \lambda_\eta(\xi) + d_\xi \lambda_\eta \xi(\xi^{-1}) = 0. \tag{9} \]
Putting (7) into (9), we get 
\[ d_\xi \rho_\xi^{-1} \lambda_\eta = d_\xi \lambda_\eta \lambda_\xi^{-1} \rho_\xi^{-1}, \]
equivalently to the condition (C)(ii).

For monoassociative loop \( L \) we differentiate the identity 
\[ \xi(t) \cdot \xi(t)^2 = \xi(t)^2 \cdot \xi(t) \]
at \( t = 0 \). We obtain 
\[ d_\xi \rho_\xi \xi^2(\xi) + d_\xi \lambda_\xi \rho_\xi(\xi) + d_\xi \lambda_\xi \xi^2(\xi) = d_\xi \rho_\xi \xi(\xi) + d_\xi \rho_\xi \lambda_\xi \xi(\xi) + d_\xi \lambda_\xi \xi^2(\xi), \]
giving the condition (D)(ii).

If \( L \) is left alternative, right alternative or flexible, then for any \( \eta \in L \), we differentiate the identities 
\[ \xi(t) \cdot \eta(\xi(t)) = \xi(t) \cdot \eta, \]
\[ (\xi(t) \cdot \eta)^2 = \eta(t) \eta, \]
at \( t = 0 \), and we obtain the conditions (E)(ii), (F)(ii), respectively, (G)(ii).

If \( L \) is a left Bol loop or right Bol loop, then for any \( \eta, \zeta \in L \), we differentiate 
\[ (\xi(t) \cdot \eta(\xi(t))) \zeta = \xi(t) \eta \cdot \xi(t) \zeta \]
respectively, 
\[ (\zeta \xi(t) \cdot \eta)^2 = \eta(\zeta \xi(t)), \]
and we get the conditions (H)(ii), respectively, (J)(ii). Hence the assertion is true. \( \square \)

References

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