Abstract. Some structural descriptions of semigroups which are semilattices of nil-extensions of rectangular groups were considered in [1], [3] and [4]. In this paper some new, indicatorial (i.e. in terms of forbidden members) and other descriptions for these semigroups will be given.

Decomposition of semigroups into a semilattice of completely Archimedean semigroups is a very important topic investigated in the papers of M. S. Putcha [16], M. L. Veronesi [22], and more intensive in the serials of papers of L. N. Shevrin [17], [18], [19], [20] and of the authors of this paper [1], [4], [5], [6]. If the components in these decompositions are orthodox (i.e. their idempotents form subsemigroups), then we have semilattices of nil-extensions of rectangular groups. These semigroups also form a very interesting class of semigroups. Recently X. Tang [21] solved the problem of characterization of semigroups with congruence extension property in this class. Some structural descriptions of semigroups which are semilattices of nil-extensions of rectangular groups were considered in [1], [3] and [4]. In this paper we will give some new, indicatorial (i.e. in terms of forbidden members) and other descriptions for these semigroups. Also, we will study chains of completely Archimedean semigroups and chains of nil-extensions of rectangular groups.

Throughout this paper, \( \mathbb{Z}^+ \) will denote the set of all positive integers. If \( S \) is a semigroup, \( \text{Reg}(S) \) will denote the set of all regular elements of \( S \), \( E(S) \) the set of all idempotents of \( S \), for \( a \in \text{Reg}(S) \), \( V(a) \) the set of all inverses of \( a \), i.e. \( V(a) = \{ x \in S \mid a = axa, \ x = xax \} \), and for \( e \in E(S) \), \( G_e \) will denote the maximal subgroup of \( S \) with \( e \) as its identity.

\textit{Mathematics Subject Classification}: Primary 20M10.
Supported by Grant 0401B of RFNS through Math. Inst. SANU.
A semigroup $S$ is $\pi$-regular if for every $a \in S$ there exist $n \in \mathbb{Z}^+$, $x \in S$ such that $a^n = a^nx^a$. A semigroup $S$ is completely $\pi$-regular if for every $a \in S$ there exist $n \in \mathbb{Z}^+$, $x \in S$ such that $a^n = a^nx^a$, $a^n.x = xa^n$, i.e. if $a^n$ lies in a subgroup of $S$. Note that L. N. SHEVRIN [20] used the name epigroups for these semigroups. If $S$ is a completely $\pi$-regular semigroup, $x \in S$ and $x^n$ belongs to some subgroup $G$ of $S$, then $x^0$ will denote the identity of $G$.

A semigroup $S$ with a zero 0 is a nil-semigroup if for every $a \in S$ there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$. An ideal extension $S$ of a semigroup $T$ is a nil-extension of $T$ if the factor semigroup $S/T$ is a nil-semigroup. A semigroup $S$ is Archimedean if for all $a, b \in S$ there exists $n \in \mathbb{Z}^+$ such that $a^n \in SbS$. A semigroup $S$ is completely Archimedean if $S$ is Archimedean and has a primitive idempotent, or equivalently, if it is a nil-extension of a completely simple semigroup.

Let $X_1$ and $X_2$ be classes of semigroups. By $X_1 \circ X_2$ we denote the Mal’cev product of the classes $X_1$ and $X_2$, i.e. the class of all semigroups $S$ on which there exists a congruence $\rho$ such that $S/\rho$ is in $X_2$ and each $\rho$-class which is a subsemigroup is in $X_1$ [14]. If $\mathcal{X}$ is a class of semigroups, by $\mathbf{S}(\mathcal{X})$ and $\mathbf{H}(\mathcal{X})$ we will denote the class of all subsemigroups of semigroups from $\mathcal{X}$ and the class of all homomorphic images of semigroups from $\mathcal{X}$, respectively. If $\mathcal{X} = \{S\}$, then we write $\mathbf{S}(S)$ and $\mathbf{H}(S)$ instead of $\mathbf{S}(\mathcal{X})$ and $\mathbf{H}(\mathcal{X})$.

We introduce the following notations:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Class of semigroups</th>
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<tbody>
<tr>
<td>$\mathcal{C}A$</td>
<td>completely Archimedean</td>
<td>$\mathcal{M} \times \mathcal{G}$</td>
<td>rectangular groups</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>semilattices</td>
<td>$\mathcal{C}$</td>
<td>chains</td>
</tr>
<tr>
<td>$\mathcal{C}\pi\mathcal{R}$</td>
<td>completely $\pi$-regular</td>
<td>$\mathcal{N}$</td>
<td>nil-semigroups</td>
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Clearly $\mathcal{X} \circ \mathcal{S}$ is the class of all semilattices of semigroups from the class $\mathcal{X}$. If $\mathcal{X}_2$ is a subclass of the class $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is a class of all semigroups that are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

In this paper we will use the following semigroups given by the presentations:

$$A_2 = \langle a, e \mid a^2 = 0, e^2 = e, aea = a, eae = e \rangle,$$

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle,$$

$$S_2 = \langle a, b \mid ab = ba \rangle,$$

and $L_2$ and $R_2$ will denote the twoelement left zero and right zero semigroups, respectively.
For undefined notions and notations we refer to [3], [8] and [11].

We start with the following construction: Let $R$ be the ring $\mathbb{Z}$ of integers or the ring $\mathbb{Z}_p$ of residues mod $p$, $p \geq 2$, and let $I = \{0, 1\} \subseteq R$. Define a multiplication on $R \times I \times I$ by:

$$(m; i, \lambda)(n; j, \mu) = (m + n - (i - j)(\lambda - \mu); i, \mu),$$

$m, n \in R$, $i, j, \lambda, \mu \in I$. Then $R \times I \times I$ with this multiplication is a semigroup, and we will use the notations:

- $l_{\text{ent}}: \mathbb{Z}$
- $\text{matrix}$ semigroup over the additive group of the ring $\mathbb{Z}$
- $\alpha$ $\text{xy}$, $\text{yx}$

Then $E = \{0\}$ and for $\alpha$ $\text{xy}$, $\text{yx}$ $\in E$.

Now, if either one of the semigroups $E$ is completely simple subsemigroup of $S$, then $E \in \mathbb{Z}$, $p \geq 2$, is in $S(S)$, then $(1; 0, 0)$ is an inverse of the idempotent $(0; 1, 1)$ in $E \in \mathbb{Z}$, and $(1; 0, 0)$ is not an idempotent. Thus, (iii) holds.

(iii) $\Rightarrow$ (i). To prove (i), it is sufficient to prove that every completely simple subsemigroup of $S$ is a rectangular group. Let $K$ be some completely simple subsemigroup of $S$. Assume that $K$ is not a rectangular group, i.e. that there exists $e, f \in E(K)$ such that $ef \notin E(K)$. Thus, $ef$ is a group element of $K$ of order $p \geq 2$ or of infinite order, and it is easy to
verify that \(ef,efe, fef\) and \(fe\) are mutually different elements of the same (finite or infinite) order. Also, it can be checked directly that the mapping

\[
(ef)^n \phi = (n; 0, 0), \quad \text{and} \quad (efe)^n \phi = (n; 0, 1),
\]

\[
(fef)^n \phi = (n; 1, 1), \quad \text{and} \quad (fef)^n \phi = (n; 1, 0),
\]

is an isomorphism between the semigroup \(\langle ef, f \rangle\) and the semigroup \(E(\alpha)\), where \(\alpha\) is the order of \(ef\) (finite or infinite). Since this assertion is in contradiction with (iii), then we obtain that \(\bar{K}\) is a rectangular group.

(iii) \(\Rightarrow\) (iv). By Theorem [19], \(S \in C \pi R\) and \(A_2, B_2 \notin S(H(S))\), and by the proof of (iii) \(\Rightarrow\) (i), \(S \in ((M \times G) \circ N) \circ S\). Now, by Theorem 1 [4], Corollary 2 [2] and Theorem 2.1 [1], \(H(S) \subseteq ((M \times G) \circ N) \circ S\) and \(S(H(S)) \subseteq S(((M \times G) \circ N) \circ S)\), so there are no \(E(\infty)\) and \(E(p), p \in Z^+, p \geq 2, \text{in} \ S(H(S))\).

(iv) \(\Rightarrow\) (iii). This follows by Theorem [19].

(i) \(\Rightarrow\) (v). By Theorem [19], \(S \in C \pi R\) and \(A_2, B_2 \notin H(S(S) \cap C \pi R)\). By Theorem 1 [9], \(S(CA) \cap C \pi R \subseteq CA\), so it can be checked directly that \(S(X) = CA \circ S\) \(\subseteq C \pi R \subseteq \mathcal{X} = CA \circ S\). By this and by Theorem 2.1 [1] (i) \(\Rightarrow\) (iii), \(S(X) \cap C \pi R \subseteq \mathcal{X}\), where \(\mathcal{X} = ((M \times G) \circ N) \circ S\), whence

\[
H(S(S) \cap C \pi R) \subseteq H(S(X) \cap C \pi R) \subseteq H(X) \subseteq X,
\]

so \(E(p) \notin H(S(S) \cap C \pi R)\), for any \(p \in Z^+, p \geq 2\). Thus, (v) holds.

(v) \(\Rightarrow\) (iii). By Theorem [19], \(S \in CA \circ S\). Clearly, for each \(p \in Z^+, p \geq 2\), \(E(p) \notin S(S)\). Since \(E(p) \notin H(E(\infty))\) for each \(p \in Z^+, p \geq 2\), then \(E(\infty) \notin S(S)\), by (v). Thus, (iii) holds. \(\square\)

Similarly we can prove the following

**Theorem 2.** The following conditions on a semigroup \(S\) are equivalent:

(i) \(S\) is a semilattice of nil-extensions of left groups;

(ii) \(S \in C \pi R\) and \((xy)^0 = (xy)^0(yx)^0\), for all \(x, y \in S\);

(iii) \(S \in C \pi R\) and there are no \(A_2, B_2\) and \(R_2\) in \(S(H(S))\);

(iv) \(S \in C \pi R\) and there are no \(A_2, B_2\) and \(R_2\) in \(H(S(S) \cap C \pi R)\).

**Corollary 1.** The following conditions on a semigroup \(S\) are equivalent:

(i) \(S\) is a semilattice of nil-extensions of groups;

(ii) \(S \in C \pi R\) and \((xy)^0 = (yx)^0\), for all \(x, y \in S\);

(iii) \(S \in C \pi R\) and there are no \(A_2, B_2, R_2\) and \(L_2\) in \(S(H(S))\);
(iv) \( S \in C_\pi R \) and there are no \( A_2, B_2, R_2 \) and \( L_2 \) in \( H(S(S) \cap C_\pi R) \).

Further we will consider chains of nil-extensions of rectangular groups. Some descriptions of these semigroups, and more generally, of chains of (completely) Archimedean semigroups were given by the authors in [1], [5] and [7]. Here we give some new characterizations of these semigroups. First we prove one more general theorem.

**Theorem 3.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is a chain of completely Archimedean semigroups;

(ii) \( S \in C_\pi R \) and \( \text{Reg}(S) \) is a chain of completely simple semigroups;

(iii) \( S \in C_\pi R \) and there are no \( A_2, B_2 \) and \( S_2 \) in \( H(S(S) \cap C_\pi R) \);

(iv) \( S \in C_\pi R \) and there are no \( A_2, B_2 \) and \( S_2 \) in \( S(H(S)) \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( S \) be a chain \( Y \) of completely Archimedean semigroups \( S_\alpha, \alpha \in Y \), and for \( \alpha \in Y \), let \( S_\alpha \) be a nil-extension of a completely simple semigroup \( K_\alpha \).

Assume \( a, b \in \text{Reg}(S) \), \( x \in V(a) \) and \( y \in V(b) \). Then \( xa, by \in E(S) \) and \( xa \in S_\alpha, by \in S_\beta \), for some \( \alpha, \beta \in Y \). Since \( Y \) is a chain, then \( \alpha \beta = \alpha \) or \( \alpha \beta = \beta \). Assume that \( \alpha \beta = \alpha \). Then \( xaby \in S_\alpha \) and \( xa \in E(S_\alpha) \), so by Theorem 1 [9] and Lemma 1 [9], \( xabyxa \in xasxa = G_{xa} \), whence \( xa \in xabyxaG_{xa}xabyxa \subseteq xabySbyxa \). Now

\[
ab = axab \in axabySbyxab = abySbyxab \subseteq abSab.
\]

Therefore, \( ab \in \text{Reg}(S) \). Similarly we prove that \( \alpha \beta = \beta \) implies \( ab \in \text{Reg}(S) \). Hence, \( \text{Reg}(S) \) is a subsemigroup of \( S \) and clearly

\[
\text{Reg}(S) = \bigcup_{\alpha \in Y} \text{Reg}(S_\alpha) = \bigcup_{\alpha \in Y} K_\alpha,
\]

so \( \text{Reg}(S) \) is a chain \( Y \) of completely simple semigroups \( K_\alpha, \alpha \in Y \).

(ii) \( \Rightarrow \) (i). Let \( S \in C_\pi R \) and let \( \text{Reg}(S) \) be a chain of completely simple semigroups. Then clearly \( \text{Reg}(S) = G(S) \), so by Theorem 1 [17], \( S \) is a semilattice of completely Archimedean semigroups and by Theorem 5.1 [1], \( S \) is a chain of completely Archimedean semigroups.

(iii) \( \Rightarrow \) (i). By Theorem [19], \( S \) is a semilattice \( Y \) of completely Archimedean semigroups \( S_\alpha, \alpha \in Y \). If \( Y \) is not a chain, then \( Y \) contains a subsemigroup \( Z \) isomorphic to \( S_2 \), so \( S_2 \in H(T) \) and \( T \in S(S) \cap C_\pi R \), where \( T = \bigcup_{\alpha \in Z} S_\alpha \), which contradicts (iii). Thus, \( Y \) is a chain, so (i) holds.
(iv) ⇒ (i). By Theorem [19], $S$ is a semilattice $Y$ of completely Archimedean semigroups $S_\alpha$, $\alpha \in Y$. Since $Y \in \mathbb{H}(S)$, then $S_2 \notin S(Y)$, by (vi), so $Y$ is a chain.

(i) ⇒ (iii) and (i) ⇒ (iv). By Theorem [19], $S \in C\pi \mathcal{R}$ and $A_2, B_2 \notin \mathbb{H}(S(S) \cap C\pi \mathcal{R}) \cup \mathbb{S}(H(S))$. Also, it can be checked directly that $\mathbb{H}(\mathcal{X}) \subseteq \mathcal{X}$ and $\mathbb{S}(\mathcal{X}) \subseteq \mathcal{X}$, whence $S_2 \notin H(S(S) \cap C\pi \mathcal{R}) \cup S(H(S))$. □

**Corollary 2.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a chain of nil-extensions of rectangular groups;
(ii) $S \in C\pi \mathcal{R}$ and $\text{Reg}(S)$ is a chain of rectangular groups;
(iii) $S \in C\pi \mathcal{R}$ and $E(S)$ is a chain of rectangular bands;
(iv) $S \in C\pi \mathcal{R}$ and there are no $A_2, B_2, E(\infty), E(p)\; p \in \mathbb{Z}^+$, $p \geq 2$ and $S_2$ in $S(H(S))$;
(v) $S \in C\pi \mathcal{R}$ and there are no $A_2, B_2, E(p)\; p \in \mathbb{Z}^+$, $p \geq 2$ and $S_2$ in $H(S(S) \cap C\pi \mathcal{R})$.

**Acknowledgement.** The authors are indebted to the referee for several useful comments and suggestions concerning the presentation of this paper.

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*(Received May 3, 1994; revised October 3, 1994)*