On the diophantine equation $D_1 x^2 + D_2 = k^n$

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**Abstract.** Let $D, D_1, D_2, k$ be positive integers such that $D = D_1 D_2$, $D_1 > 1$, $D_2 > 1$, $k > 1$ and $\gcd(D_1, D_2) = \gcd(D, k) = 1$. Let $\omega(k)$ be the number of distinct prime factors of $k$. Further, let $N(D_1, D_2, k)$ be the number of positive integer solutions $(x, n)$ of the equation $D_1 x^2 + D_2 = k^n$. In this paper, we prove that if $2 \nmid k$ and $\max(D_1, D_2) > \exp \exp \exp 105$, then $N(D_1, D_2, k) \leq 2^{\omega(k)} - 1$ or $2^{\omega(k)} - 1$ according as the triple $(D_1, D_2, k)$ is exceptional or not. The above upper bound is the best possible if $k$ is a prime.

1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers, respectively. Let $D, D_1, D_2, k \in \mathbb{N}$ be such that $D = D_1 D_2$, $D_1 > 1$, $D_2 > 1$, $k > 1$ and $\gcd(D_1, D_2) = \gcd(D, k) = 1$. Let $\omega(k)$ be the number of distinct prime factors of $k$. Further let $N(D_1, D_2, k)$ be the number of solutions $(x, n)$ of the equation

$$D_1 x^2 + D_2 = k^n, \quad x, n \in \mathbb{N}. \quad (1)$$

In [2], BENDER and HERZBERG proved that if $2 \nmid k$ and $k > \Gamma(D)$, where $\Gamma(D)$ is an effectively computable constant depending on $D$, then $N(D_1, D_2, k) \leq 2^{\omega(k)} - 1$. In [6] and [7], the author proved that if $k$ is a prime and $\max(D_1, D_2) > C$, where $C$ is an effectively computable absolute constant, then $N(D_1, D_2, k) \leq 1$ except in some explicit cases.

For any $m \in \mathbb{Z}$ with $m \geq 0$, let $F_m$ be the $m$-th Fibonacci number. A triple $(D_1, D_2, k)$ will be called *exceptional* if $D_1, D_2$ and $k$ satisfy either

$$3D_1 s_1^2 - D_2 = \delta, \quad 4D_1 s_1^2 - \delta = k^{r_1}, \quad 4D_2 + \delta = 3k^{r_1}, \quad \delta \in \{-1, 1\}, \quad r_1, s_1 \in \mathbb{N}, \quad (2)$$

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or

\[
D_1 s_2^2 = \frac{F_{6m}}{4}, \quad D_2 = \begin{cases} 
\frac{3}{4}F_{6m} - F_{6m-1}, \\
\frac{3}{4}F_{6m} + F_{6m+1}, 
\end{cases}
\]

\[
k^{r_2} = \begin{cases} 
F_{6m-2}, \\
F_{6m+2}, 
\end{cases} \quad m, r_2, s_2 \in \mathbb{N}.
\]

In this paper, we prove the following general result:

**Theorem.** If \(2 \nmid k\) and \(\max(D_1, D_2) > \exp \exp \exp 105\), then we have

\[
N(D_1, D_2, k) \leq \begin{cases} 
2^{\omega(k)-1} + 1, & \text{if } (D_1, D_2, k) \text{ is exceptional}, \\
2^{\omega(k)-1}, & \text{otherwise}.
\end{cases}
\]

Moreover, all solutions \((x, n)\) of (1) satisfy \(n < 10\sqrt{D}\log 2e\sqrt{D}/\pi\).

If \((D_1, D_2, k)\) is exceptional, then (1) has at least two solutions, namely

\[
(x, n) = \begin{cases} 
(s_1, r_1), \quad (s_1|D_1 s_1^2 - 3D_2|, 3r_1), \\
(s_2, r_2), \quad (s_2|D_2 s_2^4 - 10D_1 D_2 s_2^2 + 5D_2^2|, 5r_2),
\end{cases}
\]

if (2) holds,

if (3) holds.

The upper bound (4) is the best possible if \(k\) is a prime.

2. Preliminaries

Let \(h(-4D)\) be the class number of the primitive binary quadratic forms with discriminant \(-4D\).

**Lemma 1.** \(h(-4D) < 4\sqrt{D}\log 2e\sqrt{D}/\pi\).

**Proof.** By [4, Theorem 12.10.1], we have \(h(-4D) = 2\sqrt{D}K(-4D)/\pi\), and by [4, Theorem 12.14.2], \(K(-4D) < \log 4D + 2\). This implies the lemma.

**Lemma 2 ([8, Theorems 1 and 3]).** If \(2 \nmid k\) and the equation

\[
D_1 X^2 + D_2 Y^2 = kZ^2, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0,
\]

has solutions \((X, Y, Z)\), then all solutions of (6) belong to at most \(2^{\omega(k)-1}\) classes. Further, for any fixed class \(S\), there exists a unique solution
On the diophantine equation \( D_1 x^2 + D_2 = k^n \) in \( S \) such that \( X_1 > 0, Y_1 > 0, Z_1 \leq Z \) and \( h(-4D) \equiv 0 \pmod{2Z_1} \), where \( Z \) runs through all solutions in \( S \). Further, every solution \((X, Y, Z)\) in \( S \) can be expressed as

\[
Z = Z_1 t, X \sqrt{D_1} + Y \sqrt{-D_2} = \lambda_1 \left( X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2} \right)^t
\]

\[
t \in \mathbb{N}, 2 \nmid t, \; \lambda_1, \lambda_2 \in \{-1, 1\}.
\]

The solution \((X_1, Y_1, Z_1)\) is called the least solution of \( S \).

**Lemma 3** ([5, the proof of the Theorem]). Let \( \varepsilon = X_1 \sqrt{D_1} + \sqrt{-D_2} \) and \( \bar{\varepsilon} = X_1 \sqrt{D_1} - \sqrt{-D_2} \), where \( X_1 \in \mathbb{N} \). If

\[
|\varepsilon^t - \bar{\varepsilon}^t| \leq |\varepsilon - \bar{\varepsilon}|,
\]

for some \( t \in \mathbb{N} \), then \( t < 8 \cdot 10^6 \). Moreover, if \( t \geq 7 \) and \( 2 \nmid t \), then \( \max(D_1, D_2) < \exp \exp \exp 105 \).

### 3. Proof of the Theorem

Let \((x, n)\) be a solution of (1). Then \((X, Y, Z) = (x, 1, n)\) is a solution of (6). By Lemma 2, we may assume that \((x, 1, n)\) belongs to a certain class \( S \). Let \((X_1, Y_1, Z_1)\) be the least solution of \( S \). Then we have

\[
n = Z_1 t, \; t \in \mathbb{N}, \; 2 \nmid t,
\]

\[
x \sqrt{D_1} + \sqrt{-D_2} = \lambda_1 \left( X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2} \right)^t, \; \lambda_1, \lambda_2 \in \{-1, 1\}.
\]

From (9), we get

\[
1 = \lambda_1 \lambda_2 Y_1 \left( \left( \frac{t}{1} \right) (D_1 X_1^2)^{(t-1)/2} + \left( \frac{t}{3} \right) (D_1 X_1^2)^{(t-3)/2} (-D_2 Y_1^2) + \ldots \right. + \left. \left( \frac{t}{t} \right) (-D_2 Y_1^2)^{(t-1)/2} \right),
\]

whence we obtain \( Y_1 = 1 \). This implies that \((x, n) = (X_1, Z_1)\) is a solution of (1). Moreover, if (1) has another solution \((x, n)\) such that \((x, n) \neq (X_1, Z_1)\) and \((x, 1, n)\) also belongs to \( S \), then \( x \) and \( n \) satisfy (8) and (9) for \( Y_1 = 1 \) and \( t > 1 \).

Let \( \varepsilon = X_1 \sqrt{D_1} + \sqrt{-D_2} \) and \( \bar{\varepsilon} = X_1 \sqrt{D_1} - \sqrt{-D_2} \). Since \( Y_1 = 1 \), if the other solution exists, (9) implies that (7) holds for \( t > 1 \). Therefore, by Lemma 3, if \( \max(D_1, D_2) > \exp \exp \exp 105 \), then we have \( t \leq 7 \). Since \( 2 \nmid t \), we get \( t = 3 \) or 5.
For any nonnegative integer \(m\), let \(L_m\) and \(F_m\) be the \(m\)-th Lucas number and Fibonacci number, respectively. For \(t = 3\) and \(t = 5\), by (9) we get (2) and (3), respectively. Since \(D_1, D_2\) and \(k\) are fixed, the integers \(r_1, s_1, \delta, r_2, s_2, m\) in (2) and (3) are given. Now we proceed to prove that (2) and (3) cannot hold at the same time. Notice that

\[
F_{6m-2} = L_{3m-1}F_{3m-1}
\]

and

\[
3F_{6m} - 4F_{6m-1} + \delta = 3F_{6m-2} - F_{6m-1} + \delta
\]

\[
= \begin{cases} 
(L_{3m-1} + L_{3m-3})F_{3m-1}, & \text{if } \delta = 1 \text{ and } 2 \mid m \text{ or } \\
\delta = -1 \text{ and } 2 \nmid m, & \\
(F_{3m-1} + F_{3m-3})L_{3m-1}, & \text{if } \delta = 1 \text{ and } 2 \nmid m \text{ or } \\
\delta = -1 \text{ and } 2 \mid m.
\end{cases}
\]

If (2) and (3) were hold at the same time together with \(k^{r_2} = F_{6m-2}\), then we would have

\[
3^{r_2}(L_{3m-1}F_{3m-1})^{r_1} = ((L_{3m-1} + L_{3m-3})F_{3m-1})^{r_2} \quad \text{or}
\]

\[
(F_{3m-1} + F_{3m-3})L_{3m-1})^{r_1}F_{3m-1})^{r_2}, \quad r_1, r_2 \in \mathbb{N}.
\]

Since \(\gcd(L_{3m-1}, L_{3m-2}L_{3m-3}) = \gcd(F_{3m-1}, F_{3m-2}F_{3m-3}) = 1\), (10) is impossible. Using the same method, we can prove a contradiction in the case \(k^{r_2} = F_{6m+2}\). Therefore, if \((D_1, D_2, k)\) is not exceptional, then (1) has at most one solution \((x, n)\) such that \((x, 1, n)\) belongs to a fixed class. Moreover, if \((D_1, D_2, k)\) is exceptional, then there exists exactly one class, say \(S\), such that (1) has exactly two solutions (5) with \((x, 1, n)\) belonging to \(S\), and the other classes have most one. Thus, by Lemma 2, (4) is proved.

On the other hand, by (8), we have \(n \leq 5Z\), and from Lemma 2, \(2Z_1 \leq h(-4D)\). Thus, by Lemma 1, we obtain \(n < 10\sqrt{D} \log 2e\sqrt{D}/\pi\). This completes the proof.

**Remark 1.** The proof of the condition “\(\max(D_1, D_2) > \exp \exp \exp 105\)” in Lemma 3 involves an upper bound (of BAKER [1]) for the solutions of Thue’s equations. Using the sharper bounds GYÖRY and PAPP [3], the condition could be improved.

**Remark 2.** Using a similar argument as in the proof of our Theorem, we can obtain an analogous result for the equation

\[
D_1x^2 + D_2 = 4k^n, \quad x, n \in \mathbb{N}.
\]

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