Triangular systems on discrete subgroups of simply connected nilpotent Lie groups

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Abstract. We show that for discrete subgroups $\Gamma$ of simply connected nilpotent Lie groups, limit laws of commutative infinitesimal triangular systems of probability measures on $\Gamma$ are infinitely divisible (and thus embeddable into a Poisson semigroup).

1. Introduction

In [4] we proved that for discrete subgroups $\Gamma$ of simply connected step 2-nilpotent Lie groups $G$ limit laws of commutative infinitesimal triangular systems of probability measures on $\Gamma$ are infinitely divisible. This assertion (for not necessarily discretely supported measures) is a classical theorem for $G = \mathbb{R}$ and $\mathbb{R}^d$. See the introduction of [4] for the history of its carrying over; recently, also RIDDHI SHAH [7] treated the problem. The purpose of this note is to get rid of the step 2-assumption. The method will be (as in [4]) to verify the conditions of WEHN [9] and SIEBERT [8]. But here we will use the fact that limit theorems for convolution semigroups on $G$ are equivalent to limit theorems for their generating distributions.

2. Preliminaries

Throughout this work we use the notation of HAZOD, SCHEFFLER [2] in general. [2] and the literature cited there can be consulted for further information and background material. For a locally compact group $G$ with neutral element $e$ let $\mathcal{U}(e)$ be the system of Borel neighbourhoods

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A Poisson measure is a probability measure \( \mu \in M^1(G) \) of the form
\[
\mu = \exp \lambda (\nu - \varepsilon_e) \quad (\nu \in M^1(G), \lambda \geq 0).
\]
Let \( G \) be a simply connected nilpotent Lie group. Via the exponential map \( G \) may be identified with its Lie algebra \( \mathfrak{g} \), the product on \( G \) being given by the Campbell-Hausdorff-formula
\[
x \cdot y = x + y + \frac{1}{2} [x, y] + \frac{1}{12} \{ [[x, y], y] + [[y, x], y] \} + \ldots,
\]
where due to the nilpotency only the terms up to some fixed order \( m \) arise. \( G \) is then called step \( m \)-nilpotent.

3. Triangular systems

A commutative infinitesimal triangular system (c.i.t.s.) on the locally compact group \( G \) is a double array \( \Delta = \{ \mu_{n,j} \}_{n \geq 1; 1 \leq j \leq k(n)} \subset M^1(G) \) \((k(n) \to \infty \ (n \to \infty))\) of probability measures on \( G \) such that
\[
\mu_{n,i} * \mu_{n,j} = \mu_{n,j} * \mu_{n,i} \quad (n \geq 1; \ 1 \leq i, j \leq k(n)) \quad (\text{commutativity}),
\]
\[
\min_{1 \leq j \leq k(n)} \mu_{n,j}(U) \to 1 \quad (n \to \infty) \quad (U \in \mathcal{U}(e)) \quad (\text{infinitesimality}).
\]
The c.i.t.s. \( \Delta \) is said to converge resp. to be relatively compact if the sequence of “row” products
\[
\{ \mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k(n)} \}_{n \geq 1}
\]
has this property (with respect to the weak topology). For a c.i.t.s. \( \Delta \) the accompanying Poisson system is defined as the c.i.t.s.
\[
\tilde{\Delta} := \{ \exp(\mu_{n,j} - \varepsilon_e) \}_{n \geq 1; 1 \leq j \leq k(n)}
\]
(cf. Siebert [8], Section 8; note that in contrast to the classical case no additional centering is performed). Now one can show (cf. Siebert [8], Remarks 1 and 4 on pp. 148 f.) that for a Lie group \( G \) \( \Delta \) converges to \( \mu \in M^1(G) \) iff \( \tilde{\Delta} \) converges to \( \mu \) provided Wehn’s conditions
\[
\begin{align*}
\text{(W1)} \quad & \limsup_{n \to \infty} \sum_{j=1}^{k(n)} \int_G \Phi(x) \mu_{n,j}(dx) < \infty, \\
\text{(W2)} \quad & \limsup_{n \to \infty} \sum_{j=1}^{k(n)} \left| \int_G \xi_\ell(x) \mu_{n,j}(dx) \right| < \infty \quad (1 \leq \ell \leq d)
\end{align*}
\]
hold, where $d = \dim G, \{\xi_1, \xi_2, \ldots, \xi_d\} \subset C^\infty(G)$ is a system of canonical coordinates with compact support such that $\xi_\ell(x^{-1}) = -\xi_\ell(x)$ ($1 \leq \ell \leq d$, $x \in G$) and $\Phi$ is a co-called Hunt function, i.e. $\Phi \in C^\infty(G)$, $\Phi(x) = \Phi(-x) \geq 0$ ($x \in G$), $\Phi(x) = \sum_{\ell=1}^d \xi_\ell(x)^2$ ($x \in U_0$), $\Phi(x) \equiv 1$ ($x \in \text{cpl} U_1$) for some $U_0, U_1 \in U(e)$ with $U_0 \subset \text{int} U_1$. Our method will be to verify (W1) and (W2) in order to get our result. A continuous convolution semigroup (c.c.s.) $\{\mu_t\}_{t \geq 0}$ on $G$ is a continuous monoid homomorphism

$$(0, \infty [, +, 0) \ni t \mapsto \mu_t \in (M^1(G), \ast, \varepsilon, \varepsilon_e).$$

**Theorem 1.** Let $G$ be a simply connected nilpotent Lie group, $\Gamma \subset G$ a discrete subgroup. Assume $\Delta = \{\mu_{n,j}\}_{n \geq 1; 1 \leq j \leq k(n)}$ is a c.i.t.s. on $\Gamma$ converging to $\mu \in M^1(\Gamma)$. Then also $\tilde{\Delta}$ converges to $\mu$ and $\mu$ is embeddable into a Poisson semigroup on $\Gamma$.

**Proof.** It suffices to prove (W1) and (W2), for in this case (by Siebert [8], Remarks 1 and 4 on pp. 148 f.) $\tilde{\Delta}$ converges to $\mu$ and then the embeddability in a c.c.s. follows from the aperiodicity, the strong root compactness of $G$ (cf. Nobel [5], 2.2; Heyer [3], Theorem 3.1.17), and Nobel [5], Theorem 1; that the c.c.s. has to be Poisson follows from the aperiodicity and Heyer [3], 3.1.11 and Theorems 3.1.13, 6.1.10. W.l.o.g. we may assume that the canonical coordinates $\xi_1, \xi_2, \ldots, \xi_d$ are adapted to a Jordan-Hölder basis of $G \cong \mathcal{G}$. (W2) holds trivially by the discreteness. Now (W1) is verified by induction on the step of nilpotency $m$: For $m = 1$ (W1) follows from the classical convergence conditions for infinitesimal triangular systems on $\mathbb{R}$ (cf. Gnedenko, Kolmogorov [1], Theorem 23.2) and the discreteness. Now assume (W1) holds for $m$. We show that it holds also for $m + 1$. Assume $G$ is step $(m + 1)$-nilpotent. Consider the quotient group $\bar{G} \cong \bar{\mathcal{G}} := G/G_m$, where

$$\mathcal{G} =: G_0 \supsetneq [G_0, G] =: G_1 \supsetneq [G_1, G] =: G_2 \supsetneq \cdots \supsetneq [G_m, G] =: G_{m+1} = \{0\}$$

is the descending central series and let $M := G_m$, i.e.

$$(1) \quad G \cong \bar{G} \oplus M.$$ 

The notation $(y, z) \in G$ and so on will be understood with respect to (1), i.e. $y \in \bar{G}$, $z \in M$. Consider the projections

$$\pi : G \cong \bar{G} \oplus M \ni (y, z) \mapsto y \in \bar{G},$$

$$p : G \cong \bar{G} \oplus M \ni (y, z) \mapsto z \in M.$$ 

Clearly, $\bar{G}$ is a simply connected step $m$-nilpotent Lie group and $\pi$ is the canonical homomorphism. Observe that by Raghunath [6], Theorem II.2.10 $\Gamma$ is finitely generated; so it is easy to see that by the
nilpotency it follows that also $\pi(\Gamma)$ and $p(\Gamma)$ are discrete. Since $\pi(\Delta)$ converges to $\pi(\mu)$, we then have by the induction hypothesis that (W1) holds on $\bar{G}$, hence

$$\limsup_{n \to \infty} \sum_{j=1}^{k(n)} \int_{\bar{G}} \Phi(y,0)\pi(\mu_{n,j})(dy) < \infty.$$  

So by the discreteness of $\pi(\Gamma)$, (2) and SIEBERT [8], Remarks 1 and 4 on pp. 148 f, we have that $\pi(\bar{\Delta})$ converges to $\pi(\mu)$, which, by the aperiodicity, the strong root compactness, NOBEL [5], Theorem 1 and Remark 2 (bottom), and HAZOD, SCHEFFLER [2], Theorem 2.1 a) implies in the obvious way that the sequence of generating distributions $\{A_n\}_{n \geq 1}$ on $\bar{G}$, where

$$A_n(f) := \sum_{j=1}^{k(n)} \int_{\bar{G}} [f(y) - f(0)]\pi(\mu_{n,j})(dy) \quad (f \in \mathcal{E}(\bar{G})),$$

is relatively compact with respect to the topology of convergence for every $f \in \mathcal{E}(\bar{G})$ (where $\mathcal{E}(G)$ is the space of bounded complex-valued $C^\infty$-functions on $G$). So the same holds for the sequence $\{A_n\}_{n \geq 1}$ of generating distributions on $G$, where

$$A_n(f) := \sum_{j=1}^{k(n)} \int_{G} [f(x) - f(0)](\pi,0)(\mu_{n,j})(dx) \quad (f \in \mathcal{E}(G)).$$

Hence again by (2), the discreteness of $\pi(\Gamma)$, SIEBERT [8], Remarks 1 and 4 on pp. 148 f., and HAZOD, SCHEFFLER [2], Proposition 2.1 the c.i.t.s. $\hat{\Delta}$, where

$$\hat{\Delta} := \{(\pi,0)(\mu_{n,j})\}_{n \geq 1; 1 \leq j \leq k(n)} \subset M^1(G),$$

is relatively compact. Let $\{X_{n,j}\}_{n \geq 1; 1 \leq j \leq k(n)}$ be a system of $\Gamma$-valued random variables with $\mathcal{L}(X_{n,j}) = \mu_{n,j}$ ($n \geq 1; 1 \leq j \leq k(n)$) such that $X_{n,1}, X_{n,2}, \ldots, X_{n,k(n)}$ are independent ($n \geq 1$). Then the relative compactness and thus uniform tightness of $\Delta$ and $\hat{\Delta}$ implies that the sequence

$$\left\{ \mathcal{L} \left( \prod_{j=1}^{k(n)} X_{n,j} - \prod_{j=1}^{k(n)} (\pi(X_{n,j}),0) \right) \right\}_{n \geq 1} = \left\{ \mathcal{L} \left( 0, \sum_{j=1}^{k(n)} p(X_{n,j}) \right) \right\}_{n \geq 1}$$

is uniformly tight and thus weakly relatively compact, which implies, since $p(\Gamma)$ is discrete, as in the induction basis,

$$\limsup_{n \to \infty} \sum_{j=1}^{k(n)} \int_{M} \Phi(0,z)p(\mu_{n,j})(dz) < \infty.$$
Now (2), (3) imply (W1) on $G$. □

Remark 1. The same proof works also if the $\mu_{n,j}$ are symmetric on $G$, yielding a result offered by Riddhi Shah [7], who refers to the theory of so-called Hun semigroups.

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