Characterizations of quadratic differences

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Abstract. We characterize two-place functions of the form
\[ F(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y), \]
which we call a quadratic difference, by means of systems of functional equations. Our
functions map groups to groups.

1. Introduction

It is a well-known result of JESSEN, KARPF and THORUP [5] that the
Cauchy difference
\[ \Delta(x, y) = f(x + y) - f(x) - f(y), \]
for \( \Delta : G \times G \to H \) with \( G \) and \( H \) abelian groups and \( H \) divisible, is
characterized by the system of functional equations
\[
\begin{align*}
\Delta(x, y) &= \Delta(y, x) \\
\Delta(x, y) + \Delta(x + y, z) &= \Delta(x, y + z) + \Delta(y, z),
\end{align*}
\]
that is, by symmetry and the cocycle equation. (For an extensive dis-
cussion and bibliography, see [4], in which similar results are proved for
Cauchy differences of all orders.)

The problem of finding a similar characterization for the quadratic
difference
\[ F(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y) \]
was formulated a decade ago [6]. (The name quadratic difference derives
from the fact that any solution of \( f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0 \)
is called a quadratic function.) Several necessary conditions are known, one of which is the functional equation

\[(1.3) \quad F(x + y, z) + F(x - y, z) - 2F(y, z)
= F(x, y + z) + F(x, y - z) - 2F(x, y),\]

introduced by Székelhidi [6]. It was demonstrated in [2] that this equation is not sufficient to guarantee (1.2), even when \(G = H = R\) (the additive real group). On the other hand some sets of sufficient conditions for (1.2) are known. For instance, (1.3) and boundedness [6], or (1.3) and two times continuous differentiability [3] are sufficient when \(F : \mathbb{R}^2 \to \mathbb{R}\). Yet it is clear that analytic conditions are not necessary for (1.2).

The purpose of this paper is to present some sets of conditions which are both necessary and sufficient for \(F : G \to H\) to have the representation (1.2). To our knowledge, this has not been done until now. We assume throughout that \(G\) is an abelian group and that \(H\) is a uniquely divisible abelian group. (So \(H\) is a vector space over the rationals.) We shall note a few places where the divisibility hypotheses concerning \(H\) can be relaxed.

We shall need to make use of the following result on Cauchy differences.

**Theorem 1** [4: Theorem 2.2]. A map \(K : G^3 \to H\) is symmetric and satisfies the condition that

\[(1.4) \quad (x, y) \mapsto K(x, y, w)\] is a (1.1) Cauchy difference for each \(w \in G,\)

if and only if \(K\) is itself a Cauchy difference of order 2, i.e. there exists a map \(g : G \to H\) such that

\[K(x, y, z) = g(x + y + z) - g(x+y) - g(x+z) - g(y+z) + g(x) + g(y) + g(z)\]

for all \(x, y, z \in G.\)

2. Pieces of the solution

In this section, we give characterizations of (1.2) for even \(f\) and for odd \(f\), as well as other special solutions of (1.3). These will be combined in the following section to obtain the main results.
Lemma 2.1. Necessary and sufficient conditions for $F : G \times G \to H$ to have the decomposition (1.2) with odd $f : G \to H$ are

\begin{align}
(2.1) & \quad F(0, -y) = -F(0, y), \\
(2.2) & \quad F(x, y) = -\frac{1}{2}F(0, x + y) - \frac{1}{2}F(0, x - y) + F(0, x) + F(0, y). 
\end{align}

Proof. If $F$ has the form (1.2) with $f$ odd, then setting $x = 0$ yields

\begin{equation}
F(0, y) = -2f(y). 
\end{equation}

The oddness of $f$ gives (2.1), and (1.2) with (2.3) translates into (2.2).

 Conversely, suppose $F$ satisfies (2.1) and (2.2). Then clearly $F$ has the decomposition (1.2) with $f$ defined by (2.3). Moreover, the oddness of $f$ follows from (2.1). This completes the proof.

Remark. Lemma 2.1 holds when $H$ is any uniquely 2-divisible abelian group. The same is true of the following.

Lemma 2.2. $F : G \times G \to H$ has decomposition (1.2) with odd $f : G \to H$, if and only if $F$ is an odd solution of (1.3).

Proof. Let $F$ be an odd solution of (1.3). Putting $y = z = 0$ in (1.3), we get $F(x, 0) = F(0, 0)$. Since $F$ is odd, $F(0, 0) = 0$, so

\begin{equation}
F(x, 0) = 0. 
\end{equation}

Putting $y = 0$ in (1.3) and using (2.4), we obtain

\begin{equation}
F(x, z) - 2F(0, z) = F(x, -z). 
\end{equation}

Defining $f : G \to H$ by (cf. (2.3)) $f(x) := -\frac{1}{2}F(0, x)$, we have

\begin{align}
(2.5) & \quad F(x, z) - F(x, -z) = -4f(z). 
\end{align}

Also, the oddness of $F$ implies that $f$ is odd.

Now replace $(x, z)$ by $(-x, -z)$ in (1.3) and use the oddness of $F$ to write the result as

\begin{align}
- F(x - y, z) - F(x + y, z) - 2F(y, z) = -F(x, -y + z) - F(x, -y - z) + 2F(x, -y). 
\end{align}

Adding this to (1.3) and applying (2.5) several times, we arrive at (1.2), as desired. (This could also be derived from Lemma 2.1.)

Conversely, any $F$ of the form (1.2) satisfies (1.3), and if $f$ is odd, then so is $F$. This completes the proof.
Lemma 2.3. Necessary and sufficient conditions for $F : G \times G \to H$ to have a representation (1.2), where $f$ is even and $f(0) = 0$, are

\begin{align}
F(x + y, y) + F(x - y, y) - 2F(y, y) &= F(x, 2y) + F(x, 0) - 2F(x, y), \\
F(-x, y) &= F(x, y) = F(y, x), \\
F(x, 0) &= 0,
\end{align}

and that the map $K : G^3 \to H$ defined by

\begin{equation}
K(x, y, z) := 2F(y + z, x) + 2F(x + z, y) - 2F(x, y) - F(x - z, y) - F(z - y, x) - F(y - x, z)
\end{equation}

satisfies condition (1.4).

Proof. Suppose $F$ satisfies (1.2) with $f$ even and $f(0) = 0$. Then it is straightforward to verify that $F$ satisfies (2.6)–(2.8). Moreover, inserting (1.2) into (2.9), we compute that

\begin{align*}
K(x, y, w) &= 2[f(y + w + x) + f(y + w - x) - 2f(y + w) - 2f(x)] \\
&+ 2[f(x + w + y) + f(x + w - y) - 2f(x + w) - 2f(y)] \\
&- 2[f(x + y) + f(x - y) - 2f(x) - 2f(y)] \\
&- [f(x - w + y) + f(x - w - y) - 2f(x - w) - 2f(y)] \\
&- [f(w - y + x) + f(w - y - x) - 2f(w - y) - 2f(x)] \\
&- [f(y - x + w) + f(y - x - w) - 2f(y - x) - 2f(w)] \\
&= 4[f(x + y + w) - f(x + w) - f(y + w) + f(w)] \\
&- 2[f(x + y - w) - f(x - w) - f(y - w) + f(w)] \\
&- 2[f(x + y) - f(x) - f(y)],
\end{align*}

using the fact that $f$ is even. The last member of this equation is obviously a Cauchy difference in $x$ and $y$ for each fixed $w$, since it is the sum of three such Cauchy differences. Hence $K$ satisfies (1.4).

For the converse, let us define $L : G^3 \to H$ by

\[ L(x, y, z) := 2[F(x + y, z) - F(x, z) - F(y, z)] + K(x, y, z). \]
Then, by the hypothesis on $K$, we see that $L$ also satisfies condition (1.4). Moreover, by (2.9) and (2.7) we have
\[ L(x, y, z) = 2[F(x + y, z) + F(y + z, x) + F(z + x, y)] \]
\[ \quad - [F(x - z, y) + F(z - y, x) + F(y - x, z)] \]
\[ \quad - 2[F(x, y) + F(y, z) + F(z, x)]. \]

That is, $L$ is a symmetric function of its three variables. Hence, by Theorem 1, we obtain
\[ L(x, y, z) = g(x + y + z) \]
\[ \quad - [g(x + y) + g(x + z) + g(y + z)] + g(x) + g(y) + g(z), \]
for some function $g : G \rightarrow H$.

Furthermore, by (2.7), (2.8) and (2.10), $L$ vanishes when one variable is equal to 0. It follows then, from (2.11), that
\[ g(0) = 0. \]

Also, the evenness of $F$ yields $L(-x, -y, -z) = L(x, y, z)$, because of (2.10). So (2.11) shows that we may replace $g$ by its even part. Thus, without loss of generality, we may assume that
\[ g \text{ is even.} \]

Finally, consider a cross section of $L$ in (2.10). Using also (2.6)–(2.8), we calculate that
\[ L(x, y, -y) = 2F(x + y, -y) + 2F(-y + x, y) - F(x + y, y) - F(-2y, x) \]
\[ \quad - F(y - x, -y) - 2F(x, y) - 2F(y, -y) - 2F(-y, x) \]
\[ = F(x + y, y) + F(x - y, y) - 2F(y, y) - F(x, 2y) - 4F(x, y) \]
\[ = -6F(x, y). \]

Therefore $F$ is a cross section of $L$,
\[ F(x, y) = -\frac{1}{6}L(x, y, -y). \]

By (2.11), we have now
\[ F(x, y) = -\frac{1}{6}\{g(x) - g(x + y) - g(x - y) - g(0) + g(x) + g(y) + g(-y)\}. \]

Defining $f : G \rightarrow H$ by
\[ f(x) := \frac{1}{6}g(x) \]
and observing (2.12) and (2.13), this verifies (1.2) and finishes the proof.

Similar to Lemma 2.3 is the following
Lemma 2.4. Necessary and sufficient conditions for \( F : G \times G \to H \) to have representation (1.2) with \( f \) even and \( f(0) = 0 \) are (1.3), (2.7), (2.8), and that the map \( K : G^3 \to H \) defined by

\[
K(x, y, z) := F(x + y, z) + F(x + z, y) - F(z - y, x) - 2F(y, z)
\]

satisfies condition (1.4).

Proof. If \( F \) has representation (1.2) with \( f \) even and \( f(0) = 0 \), then it is clear that \( F \) satisfies (1.3), (2.7) and (2.8). In addition, for the map \( K \) defined by (2.14), we have

\[
K(x, y, w) = [f(x + y + w) + f(x + y - w) - 2f(x + y) - 2f(w)] + [f(x + w + y) + f(x + w - y) - 2f(x + w) - 2f(y)] - [f(w - y + x) + f(w - y - x) - 2f(w - y) - 2f(x)] - 2[f(y + w) + f(y - w) - 2f(y) - 2f(w)] = 2[f(x + y + w) - f(x + w) - f(y + w) + f(w)] - 2[f(x + y) - f(x) - f(y)],
\]

showing that \( (x, y) \mapsto K(x, y, w) \) is a Cauchy difference. Thus (1.4) is fulfilled.

Conversely, (2.14) and (2.7) show that \( K(x, y, z) = K(x, z, y) \). The following calculation shows that \( K \) is also symmetric in its first two arguments. By (1.3), (2.7) and (2.14), we deduce that indeed

\[
K(x, y, z) = F(x + y, z) + F(x + z, y) - F(z - y, x) - 2F(y, z) = F(y + x, z) + [F(y, z + x) - F(y - z, x) - 2F(y, z)] = F(y + x, z) + [F(y + z, x) - F(y, z - x) - 2F(z, x)] = F(y + x, z) + F(y + z, x) - F(z - x, y) - 2F(x, z) = K(y, x, z).
\]

Hence \( K \) is a symmetric function, and by Theorem 1 there exists \( h:G\to H \) for which

\[
K(x, y, z) = h(x + y + z) - [h(x + y) + h(x + z) + h(y + z)] + h(x) + h(y) + h(z).
\]

Because of (2.14), this means that

\[
F(x + y, w) + F(x + w, y) - F(w - y, x) - 2F(y, w) = h(x + y + w) - h(x + y) - h(x + w) - h(y + w) + h(x) + h(y) + h(w).
\]
With \( y = w, \ x = v - w \), this reduces by (2.8) to
\[
2F(v, w) - 2F(w, w) = h(v + w) - 2h(v) - h(2w) + h(v - w) + 2h(w),
\]
or
\[
(2.15) \quad 2F(v, w) = h(v + w) - 2h(v) + h(v - w) + k(w),
\]
where \( k : G \to H \) is defined by \( k(w) := 2F(w, w) - h(2w) + 2h(w) \).

Next, by (2.7) and (2.15), we have
\[
h(v + w) - 2h(v) + h(v - w) + k(w) = 2F(v, w)
\]
\[
= 2F(-v, w) = h(-v + w) - 2h(-v) + h(-v - w) + k(w).
\]
That is
\[
[h(v + w) - h(-v - w)] + [h(v - w) - h(-v + w)] = 2[h(v) - h(-v)],
\]
which means that the odd part of \( h \) is additive. So we can replace \( h \) by its even part in (2.15), and thus no generality is lost by assuming that
\[
h \text{ is even.}
\]

Now (2.15) and (2.7) also yield
\[
h(v + w) + h(v - w) - 2h(v) + k(w) = 2F(v, w)
\]
\[
= 2F(w, v) = h(w + v) + h(w - v) - 2h(w) + k(v),
\]
from which we get \( k(w) + 2h(w) = k(v) + 2h(v) = j \) (constant). Hence (2.15) takes the form
\[
2F(v, w) = h(v + w) + h(v - w) - 2h(v) + j - 2h(w).
\]
Finally, (2.8) gives \( j = 2h(0) \) and so, defining \( f : G \to H \) by
\[
f(x) := \frac{1}{2}[h(x) - h(0)],
\]
we have (1.2) with \( f \) even and \( f(0) = 0 \). This concludes the proof.

**Theorem 2.5.** In order for \( F : G \times G \to H \) to have representation (1.2) with even \( f : G \to H \), it is necessary and sufficient that \( F \) satisfy one of the following sets of conditions. Either

(i) \( F \) satisfies (2.6), (2.7), and the map \( K \) defined by (2.9) satisfies (1.4); or
(ii) \( F \) satisfies (1.3), (2.7), and the map \( K \) defined by (2.14) satisfies (1.4).

**Proof.** Given (1.2) with \( f \) even, we define \( f' : G \to H \) and \( F' : G \times G \to H \) by
\[
\begin{align*}
f'(x) &:= f(x) - f(0), \\
F'(x, y) &:= F(x, y) - F(0, 0).
\end{align*}
\]
Then \( F' \) is represented in the form (1.2) by \( f' \), which is even and \( f'(0) = 0 \). Thus, by Lemmas 2.3 and 2.4, \( F' \) satisfies conditions (i) and (ii) (as well as (2.8) \( F'(x, 0) = 0 \)). By (2.16), this means that also \( F \) satisfies conditions (i) and (ii).

Conversely, given \( F \) satisfying either set (i) or (ii) of conditions, we define \( F' \) by \( F'(x, y) = F(x, y) - F(0, 0) \) and claim (by Lemma 2.3 or 2.4) the representation (1.2) for \( F' \) by means of an even function \( f' \) satisfying \( f'(0) = 0 \).

Defining \( f : G \to H \) by
\[
f(x) := f'(x) - \frac{1}{2}F(0, 0)
\]
we have (1.2) for \( F \), with \( f \) even. This concludes the proof.

We need a few more preliminary results, and they are collected in the next two lemmas. A map \( F : G \times G \to H \) is called **skew-symmetric** (or **anti-symmetric**) if \( F(x, y) = -F(y, x) \) for all \( x, y \in G \).

**Lemma 2.6.** Let \( F : G \times G \to H \) be skew-symmetric. Then \( F \) satisfies (1.3), if and only if \( F \) is biquadratic (i.e. quadratic in each variable).

**Proof.** First, suppose \( F \) is a skew-symmetric solution of (1.3). By several applications of (1.3) and the skew-symmetry, we compute
\[
\begin{align*}
F(x + y, z) + F(x - y, z) - 2F(y, z) \\
= F(x, y + z) + F(x, y - z) - 2F(x, y) \\
= -\{F(y + z, x) + F(y - z, x) - 2F(z, x)\} - 2F(z, x) - 2F(x, y) \\
= -\{F(y, z + x) + F(y, z - x) - 2F(y, z)\} + 2F(x, z) - 2F(x, y) \\
= F(z + x, y) + F(z - x, y) - 2F(z, y) + 2F(x, z) - 2F(x, y) \\
= \{F(z + x, y) + F(z - x, y) - 2F(x, y)\} + 2F(x, z) + 2F(y, z) \\
= \{F(z, x + y) + F(z, x - y) - 2F(z, x)\} + 2F(x, z) + 2F(y, z) \\
= -F(x + y, z) - F(x - y, z) + 4F(x, z) + 2F(y, z).
\end{align*}
\]
Comparing the first and last members, therefore
\[2F(x + y, z) + 2F(x - y, z) = 4F(x, z) + 4F(y, z),\]
which shows that \(F\) is quadratic in its first argument. Since \(F\) is skew-symmetric, it is also quadratic in the second argument, hence \(F\) is biquadratic.

Conversely any biquadratic function \(F\) satisfies (1.3), for
\[
F(x + y, z) + F(x - y, z) - 2F(y, z) = 2F(x, z) = F(x, y + z) + F(x, y - z) - 2F(x, y).
\]
This completes the proof.

Remarks. (i) It was shown in [2] that biquadratic skew-symmetric functions are not in general of the form (1.2). In particular, not every solution of (1.3) has the form (1.2).

(ii) As in Lemmas 2.1 and 2.2, it is only necessary that \(H\) be uniquely 2-divisible in Lemma 2.6.

For the next and final lemma of this section, for a given \(F : G \times G \rightarrow H\) let \(F_o\) and \(F_e\) be the canonical odd and even parts of \(F\). That is, define \(F_o, F_e : G \times G \rightarrow H\) by
\[
F_o(x, y) = \frac{1}{2}[F(x, y) - F(-x, -y)],
F_e(x, y) = \frac{1}{2}[F(x, y) + F(-x, -y)].
\]
Furthermore, we decompose \(F_e\) into its symmetric and skew-symmetric parts \(F_{e+}\) and \(F_{e-}\), viz.
\[
F_{e+}(x, y) = \frac{1}{2}[F_e(x, y) + F_e(y, x)],
F_{e-}(x, y) = \frac{1}{2}[F_e(x, y) - F_e(y, x)].
\]

Lemma 2.7. Let \(F : G \times G \rightarrow H\) satisfy (1.3). Then:

(i) \(F_o\) and \(F_e\) satisfy (1.3);
(ii) \(F_e\) satisfies
\[
F_e(x, y) = F_e(-x, y) = F_e(x, -y);
\]
(iii) \(F_{e+}\) and \(F_{e-}\) satisfy (1.3); and
(iv) \(F_{e+}\) satisfies (2.7).
Proof. Let $F$ satisfy (1.3). Then part (i) clearly follows, by definition of $F_e$ and $F_o$ and the structure of (1.3).

Now, suppose $F_e$ satisfies (1.3), and put $x = 0$ to get

\[(2.19) \quad F_e(-y, z) - F_e(y, z) = F_e(0, y + z) + F_e(0, y - z) - 2F_e(0, y).\]

Applying the evenness of $F_e$ to the right hand side of this equation, since $F_e(0, t) = F_e(0, -t)$ we get

\[F_e(-y, z) - F_e(y, z) = -F_e(0, -y - z) + F_e(0, -y + z) - 2F_e(0, -y).\]

By (2.19), the right hand side of this equation is equal to $F_e(y, z) - F_e(-y, z)$, hence we have

\[2F_e(-y, z) = 2F_e(y, z).\]

Since $H$ is uniquely divisible by 2, we see that $F_e$ is even in its first variable. But since $F_e$ is even, we have

\[F_e(x, y) = F_e(-x, -y) = F_e(x, -y).\]

Thus $F_e$ is even in each variable separately, which proves part (ii).

As for part (iii), assume that $F$ satisfies (1.3), and define $T:G \times G \to H$ by $T(x, y) = F_e(y, x)$. Then, by parts (i) and (ii) of the proof, we calculate that

\[T(x + y, z) + T(x - y, z) - 2T(y, z)\]
\[= F_e(z, x + y) + F_e(z, x - y) - 2F_e(z, y)\]
\[= F_e(z, y + x) + F_e(z, y - x) - 2F_e(z, y)\]
\[= F_e(z + y, x) + F_e(z - y, x) - 2F_e(y, x)\]
\[= T(x, y + z) + T(x, y - z) - 2T(x, y).\]

That is, $T$ satisfies (1.3). Therefore $F_{e+} = \frac{1}{2}(F_e + T)$ and $F_e = \frac{1}{2}(F_{e-} - T)$ also satisfy (1.3).

Finally, $F_{e+}$ is by definition symmetric, and by part (ii) of this proof

\[F_{e+}(-x, y) = \frac{1}{2}[F_e(-x, y) + F_e(y, -x)]\]
\[= \frac{1}{2}[F_e(x, y) + F_e(y, x)] = F_{e+}(x, y).\]

Hence $F_{e+}$ satisfies (2.7).
3. Main results

Using the results of the previous section, we can construct several characterizations of quadratic differences. The characterizations are carried out by decompositions. The decompositions used are accomplished by the standard definitions of odd, even, symmetric and skew-symmetric parts of a function. Explicitly these are given by (2.17) and (2.18).

Theorem 3.1. Let $F : G \times G \rightarrow H$ be given, and decompose $F$ into the sum

\[
F = F_o + F_e
\]

using definition (2.17). In order for $F$ to have the representation (1.2) for some $f : G \rightarrow H$, it is necessary and sufficient that

(a) $F_o$ satisfies either (1.3) or the pair of conditions (2.1), (2.2); and
(b) $F_e$ satisfies either (i) or (ii) of Theorem 2.5.

Proof. First, we establish sufficiency. Suppose that $F_o$ and $F_e$ fulfill statements (a) and (b) respectively. Then, by applying Lemma 2.1 or 2.2 (according to (a)) to $F_o$, we deduce that $F_o$ has the representation

\[
F_o(x, y) = f_o(x + y) + f_o(x - y) - 2f_o(x) - 2f_o(y),
\]

with odd $f_o : G \rightarrow H$. Furthermore, for $F_e$, Theorem 2.5 gives

\[
F_e(x, y) = f_e(x + y) + f_e(x - y) - 2f_e(x) - 2f_e(y),
\]

with even $f_e : G \rightarrow H$. Hence, by (3.1), $F$ has the form (1.2) with $f = f_e + f_o$.

Conversely, suppose $F$ is of the form (1.2). Then $F_o$ and $F_e$ have the forms (3.2) and (3.3), respectively, where $f_o$ and $f_e$ are the odd and even parts (resp.) of $f$. By Lemmas 2.1 and 2.2, $F_o$ satisfies statement (a) of the theorem. For $F_e$, we get statement (b) from Theorem 2.5. This completes the proof.

We characterize a more general form in the next theorem, in which we do not assume that the even part of $F$ is symmetric.
Theorem 3.2. A map $F : G \times G \rightarrow H$ has the form

$$F(x,y) = f(x+y) + f(x-y) - 2f(x) - 2f(y) + \Sigma(x,y),$$

with $\Sigma : G \times G \rightarrow H$ skew-symmetric and biquadratic, and with arbitrary $f : G \rightarrow H$, if and only if $F$ fulfills (1.3) and the map $K_{e^+}$ defined by means of $F_{e^+}$ through either (2.9) or (2.14) satisfies (1.4). [Note: Then $\Sigma = F_{e^-}$.] 

Proof. Suppose that $F$ fulfills (1.3) and that $F_{e^+}$ fulfills one of the stated hypotheses. By Lemma 2.7, (1.3) carries over also to $F_{e^+}$, to $F_o$, and $F_{e^-}$. We write $F$ in the form

$$F = F_o + F_{e^+} + F_{e^-}.$$

By Lemma 2.2, we have (3.2) with $f_o$ odd. Also, by Lemma 2.6, we conclude that $F_{e^-}$ is biquadratic, and we put $\Sigma = F_{e^-}$. It only remains to consider $F_{e^+}$, and (3.4) will be established as soon as we show that $F_{e^+}$ has the form (3.3) with even $f_e$.

To this end, let us observe that $F_{e^+}$ satisfies (2.7), by Lemma 2.7. Thus (since (2.6) is a special case of (1.3)) $F_{e^+}$ satisfies either (i) or (ii) of Theorem 2.5, hence $F_{e^+}$ has the form (3.3) with even $f_e$.

For the converse, we observe first that any $F$ of the form (3.4), with $\Sigma$ biquadratic, satisfies (1.3). Next, we calculate $F_{e^+}$. Since any quadratic function is even, we have

$$F_e(x,y) = f_e(x+y) + f_e(x-y) - 2f_e(x) - 2f_e(y) + \Sigma(x,y),$$

where $f_e$ is the even part of $f$. Then

$$F_{e^+}(x,y) = \frac{1}{2}\{F_e(x,y) + F_e(y,x)\}$$

$$= \frac{1}{2}\{2f_e(x+y) + 2f_e(x-y) - 4f_e(x) - 4f_e(y) + \Sigma(x,y) + \Sigma(y,x)\}.$$ 

Since $\Sigma$ is skew-symmetric, this reduces to a representation of the form (3.3) for $F_{e^+}$. Therefore, by Theorem 2.5, the map $K_{e^+}$ defined in terms of $F_{e^+}$, through either (2.9) or (2.14), satisfies (1.4). This concludes the proof of the theorem.

Theorem 3.2 permits us to state a further characterization of quadratic differences, which is the following.
Theorem 3.3. In order for $F : G \times G \rightarrow H$ to have quadratic decomposition (1.2) in terms of an arbitrary $f : G \rightarrow H$, it is necessary and sufficient that

(a) $F$ satisfies (1.3),
(b) the map $K_{e^+}$ defined by means of $F_{e^+}$ through either (2.9) or (2.14) satisfies (1.4), and
(c) $F$ satisfies

\begin{equation}
F(x + y, x - y) + 2F(x, y) = F(x, x) + F(y, y) + F(0, 0).
\end{equation}

Proof. Suppose hypotheses (a), (b) and (c) are fulfilled. By Theorem 3.2, $F$ has the form (3.4) with arbitrary $f : G \rightarrow H$ and with skew-symmetric, biquadratic $\Sigma : G \times G \rightarrow H$. For such $F$, (3.5) is satisfied if and only if

\begin{equation}
\Sigma(x + y, x - y) + 2\Sigma(x, y) = 0,
\end{equation}

since $\Sigma(x, x) = 0$ by skew-symmetry. We have to show that $\Sigma = 0$. Now we employ the representation (see [1]) for the biquadratic $\Sigma$ in the form

\begin{equation}
\Sigma(x, y) = A(x, x; y, y),
\end{equation}

where $A : G^4 \rightarrow H$ is 4-additive and has the partial symmetries

\begin{equation}
A(x, y; u, v) = A(y, x; u, v) = A(x, y; v, u).
\end{equation}

Moreover, the skew-symmetry of $\Sigma$ implies

\begin{equation}
A(x, x; y, y) = -A(y, y; x, x),
\end{equation}

and in particular $A(x, x; x, x) = 0$. Inserting (3.7) into (3.6), expanding by additivity, and using all properties of $A$ we arrive at

\begin{equation}
2[A(x, y; x, x) - A(x, y; x, y) + A(x, y; y, y)] + A(x, x; y, y) = 0.
\end{equation}

Considering the terms which are even in $x$, we find that

\begin{equation}
A(x, x; y, y) = 2A(x, y; x, y).
\end{equation}

Here the left hand side is skew-symmetric as a function of $x$ and $y$, whereas the right hand side is symmetric. Hence both sides are zero. Therefore $\Sigma = 0$ by (3.7), and (3.4) reduces to (1.2).

Conversely, any map of the form (1.2) satisfies (a) and (b) by Theorem 3.2, and (c) is verified easily by direct substitution.

Finally, Theorem 3.2 has another consequence which offers a slight improvement over part of Theorem 3.1. Namely, if $F_e$ satisfies part (ii) of Theorem 2.5, then we can remove the assumption (which is a part of (2.7)) that $F_e(-x, y) = F_e(x, y)$. As shown in Lemma 2.7, this is a consequence of (1.3) for $F_e$. 
Corollary 3.4. In order for \( F : G \times G \rightarrow H \) to be of the form (1.2) with arbitrary \( f : G \rightarrow H \), it is necessary and sufficient that 
(a) \( F \) satisfies (1.3),  
(b) the map \( K_e \) defined in terms of \( F_e \) by (2.14) satisfies (1.4), and 
(c) \( F_e \) is symmetric.

Proof. Condition (c) means that \( F_e = F_{e+} \), so \( F_{e-} = 0 \). The rest follows from Theorem 3.2.

Open problem. It is still not known if (3.4), with \( \Sigma \) biquadratic and skew-symmetric, is the general solution of (1.3).

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