Abstract. In a recent paper Imdad and Ahmad [1] proved several fixed point theorems for set-valued mappings satisfying conditions weaker than commuting, such as weakly commuting, quasi-commuting, and slightly commuting. In this paper we show that two of these definitions are special cases of $\delta$-compatibility, and prove a fixed point theorem for four maps satisfying the $\delta$-compatibility condition.

Let $(X, d)$ be a complete metric space, $B(X)$ the collection of all nonempty bounded subsets of $X$. For $A, B \in B(X)$, define $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$. Let $F$ be a multivalued map from $X$ into $B(X)$, $I$ a single-valued selfmap of $X$. From [4] and [2], the pair $(F, I)$ is weakly commuting on $X$ if for any $x$ in $X$,

$$\delta(FIx, IFx) \leq \max \{\delta(Ix, Fx), \text{diam } IFx\}$$

quasi-commuting on $X$ if, for any $x$ in $X$ $IFx \subseteq FIx$,

slightly commuting on $X$ if for any $x$ in $X$,

$$\delta(FIx, IFx) \leq \max \{\delta(Ix, Fx), \text{diam } Fx\}.$$  

From [3] the pair $(F, I)$ is $\delta$-compatible if $\delta(IFx_n, FIx_n) \to 0$ whenever \{x_n\} is a sequence in $x$ such that $Ix_n \to t$ and $Fx_n \to \{t\}$ for some $t \in X$. As noted in [3], weakly commuting and slightly commuting imply $\delta$-compatibility.

Let $\Phi$ be the set of all real-valued functions $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ which are semi-continuous from the right and nondecreasing in each of the coordinate variables such that $\phi(t, t, t, at, bt) < t$ for each $t > 0$, $a, b \geq 0$, $a + b \leq 4$.

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Let $\Phi$ denote the set of real-valued functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which are upper semi-continuous from the right and nondecreasing with $\psi(t) < t$ for $t > 0$.

**Theorem.** Let $F, G : X \to B(X)$, $I, J$ two selfmaps of $X$. Suppose that $(F, I)$ and $(G, J)$ are $\delta$-compatible, one of the four maps is continuous, $F(X) \subseteq J(X)$, $G(X) \subseteq I(X)$, and $\phi \in \Phi$. Suppose that

\[ \delta(Fx, Gy) \leq \phi(\delta(Ix, Fx), \delta(Jy, Gy), \delta(Ix, Gy), \delta(Jy, Fx), d(Ix, Jy)) \]

$\psi \in \Phi$, satisfies, for $t > 0$, $a, b > 0$, $a + b \leq 4$,

\[ \psi(t) = \max \left\{ \phi(t, t, t, bt), \phi(t, 0, 0, t), \phi(0, 0, t, t) \right\} < t. \]

Then $F, G, I$ and $J$ have a unique common fixed point $z$ such that $Iz = Jz = z$ and $Fz = Gz = \{z\}$. Also, $z$ is the unique common fixed point of $F$ and $I$, and of $G$ and $J$; i.e., $z = Iz = Jz = Fz = Gz$.

**Proof.** Let $x_0 \in X$, $y_1$ an arbitrarily chosen point in $X_1 \coloneqq Fx_0$. Since $F(X) \subseteq J(X)$, there exists a point $x_1 \in X$ such that $Jx_1 = y_1$. Choose an arbitrary point $y_2$ in $X_2 \coloneqq Gx_1$. Since $G(X) \subseteq I(X)$, there exists a point $x_2 \in X$ with $Ix_2 = y_2$. In general, for any $x_{n+1} \in X_2 \subseteq Fx_{2n}$, there exists an $x_{2n+1} \in X$ satisfying $Jx_{2n+1} = y_{2n+1}$. For $y_{2n+2} \in X_{2n+2} \coloneqq Gx_{2n+1}$, there exists an $x_{2n+2} \in X$ such that $Ix_{2n+2} = y_{2n+2}$. Set $V_n \coloneqq \delta(X_n, X_{n+1})$.

**Case 1.** Suppose there exists an $n$ for which $V_n = 0$. Then, if $n$ is even we have $\delta(Gx_{2n-1}, Fx_{2n}) = 0$. This implies that $Fx_{2n} = y_{2n+1} = Jx_{2n+1} = Gx_{2n+1} = y_{2n+2} = Ix_{2n+2}$. Since $G$ and $J$ are $\delta$-compatible, from Proposition 3.1 of [3],

\[ GJx_{2n+1} = JGx_{2n+1} = CGx_{2n+1}. \]

From (1),

\[ \delta(Fx_{2n+2}, Gx_{2n+1}) \leq \phi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) \leq \psi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) < \delta(Fx_{2n+2}, Gx_{2n+1}) \]

which implies that $Fx_{2n+2} = Gx_{2n+1}$. From the $\delta$-compatibility of $F$ and $I$,

\[ IFx_{2n+2} = FIx_{2n+2} = FFx_{2n+2}. \]
Again from (1),
\[
\delta(FFx_{2n+2}, Fx_{2n+2}) = \delta(FFx_{2n+2}, Gx_{2n+1})
\]
\[
\leq \phi(0, 0, \delta(FFx_{2n+2}, Fx_{2n+2}), \delta(FFx_{2n+2}, Fx_{2n+2})),
\]
\[
\delta(FFx_{2n+2}, Fx_{2n+2})
\]
\[
\leq \psi(0, 0, \delta(FFx_{2n+2}, Fx_{2n+2}), \delta(FFx_{2n+2}, Fx_{2n+2})),
\]
\[
< \delta(FFx_{2n+2}, Fx_{2n+2}),
\]
and \(FFx_{2n+2} = Fx_{2n+2}\). Therefore \(Fx_{2n+2}\) is a fixed point of \(F\). Using (3), \(Fx_{2n+2}\) is a fixed point of \(I\). Since \(Fx_{2n+2} = Gx_{2n+1}\), using (1),
\[
\delta(Gx_{2n+1}, GGx_{2n+1}) = \delta(Gx_{2n+2}, GGx_{2n+1})
\]
\[
\leq \phi(0, 0, \delta(Gx_{2n+1}, GGx_{2n+1}), \delta(Gx_{2n+1}, GGx_{2n+1})),
\]
\[
\delta(Gx_{2n+1}, GGx_{2n+1})
\]
\[
\leq \psi(0, 0, \delta(Gx_{2n+1}, GGx_{2n+1}), \delta(Gx_{2n+1}, GGx_{2n+1})),
\]
\[
< \delta(Gx_{2n+1}, GGx_{2n+1})
\]
which yields that \(Gx_{2n+1} = GGx_{2n+1}\), and hence \(Gx_{2n+1}\) is a fixed point of \(G\) and, from (2), a fixed point of \(J\). Thus \(Fx_{2n+2}\) is a common fixed point of \(F, G, I, \) and \(J\).

Case 2. Suppose that \(V_n > 0\) for all \(n\). Then, using the same argument as in [2], it follows that \(\{y_n\}\) is a Cauchy sequence and hence converges to a point \(z\) in \(X\). Thus \(\lim y_{2n} = \lim Ix_{2n} = \lim y_{2n+1} = \lim Jx_{2n+1} = z\) and \(\lim Fx_{2n} = \lim Gx_{2n+1} = \{z\}\). From Proposition 3.1 of [3], \(\lim (FIx_{2n}, IFx_{2n}) = 0\).

Suppose that \(I\) is continuous. Then \(\lim Iy_{2n} = Iz\), and, from the argument in [2], \(Iz = z\), \(Fz = \{z\}\), and \(Gz' = \{z\}\), \(Jz' = z\). Since \(G\) and \(J\) are \(\delta\)-compatible, it follows that \(GJz' = JGz'\), and hence that \(Gz = Jz\), which leads to \(z\) being a common fixed point of the four maps.

Suppose now that \(F\) is continuous. Then, as in [2], \(\lim Fy_{2n} = Fz\), and \(Fz = \{z\}\). Since \(F(X) \subseteq J(X)\), there exists a point \(z'\) in \(X\) such that \(Jz' = z\). Applying (1) to \(\delta(Gz', Fx_{2n})\) and then taking the limit as \(n \to \infty\) it follows that \(Gz' = \{z\}\). Since \(J\) and \(G\) are \(\delta\)-compatible, \(GJz' = JGz'\), which leads to \(Gz = Jz\). Applying (1) to \(\delta(Fx_{2n}, Gz)\) and letting \(n \to \infty\), we obtain that \(Gz = \{z\}\). Thus \(Jz = Gz = \{z\}\).
Since $G(X) \subseteq I(X)$, there exists a point $z''$ in $X$ such that $Iz'' = z$. Thus
\[
\delta(Fz'', z) = \delta(Fz'', Gz) \leq \phi(\delta(Fz'', z), 0, 0, \delta(Fz'', z), 0) \\
\leq \psi(\delta(Fz'', z), 0, 0, \delta(Fz'', z), 0) \leq \delta(Fz'', z),
\]
which implies that $Fz'' = \{z\}$. Using the fact that $F$ and $I$ are $\delta$-compatible it follows that $Fz = Iz$. Therefore $z$ is a common fixed point of the four maps.

The proofs for $J$ or $G$ continuous are similar and will be omitted.

The uniqueness of $z$ follows from (1). \qed

Theorems 3.1–3.3 of [2] are special cases of the theorem of this paper.

We remark that Theorem 1 of [1] can also be extended to compatible maps.

References


