Second order parallel tensors on $P$-Sasakian manifolds

By U. C. DE (Kalyani)

Dedicated to the memory of Professor K. Yano

Abstract. The object of the present paper is to study the symmetric and skew-symmetric properties of a second order parallel tensor in a $P$-Sasakian manifold.

Introduction. In 1926 H. Levy ([1]) proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers ([2]) R. Sharma generalized Levy’s result and also studied a second order parallel tensor on Kähler space of constant holomorphic sectional curvature as well as on contact manifolds ([3]), ([4]).

In this paper it is shown that in a $P$-Sasakian manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. Further, it is shown that on a $P$-Sasakian manifold there is no non-zero parallel 2-form.

1. Preliminaries. Let $(M, g)$ be an $n$-dimensional Riemannian manifold admitting a 1-form $\eta$ which satisfies the conditions

1. \[(\nabla_X \eta) (Y) - (\nabla_Y \eta) (X) = 0,\]

2. \[(\nabla_X \nabla_Y \eta) (Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z),\]
where $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. If moreover $(M, g)$ admits a vector field $\xi$ and a $(1,1)$ tensor field $\varphi$ such that

\[ g(X, \xi) = \eta(X), \]
\[ \eta(\xi) = 1, \]
\[ \nabla_X \xi = \varphi X, \]

then such a manifold is called a para-Sasakian manifold or briefly a $P$-Sasakian manifold by T. ADATI and K. MATSUMOTO ([5]) which are considered as special cases of an almost paracontact manifold introduced by I. SATO ([6]).

It is known that in a $P$-Sasakian manifold the following relations hold ([5], [6]):

\[ \varphi^2 X = X - \eta(X)\xi, \]
\[ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \]
\[ \eta(\varphi X) = 0. \]

The above result will be used in the next section.

**Definition.** A tensor $T$ of second order is said to be a second order parallel tensor if $\nabla T = 0$ where $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$.

2. Let $\alpha$ denotes a $(0, 2)$-symmetric tensor field on a $P$-Sasakian manifold $M$ such that $\nabla \alpha = 0$. Then it follows that

\[ \alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0 \]

for arbitrary vector fields $W, X, Y, Z$ on $M$.

Substitution of $W = Y = Z = \xi$ in (2.1) gives us

\[ \alpha(\xi, R(\xi, Y)\xi) = 0 \]

(because $\alpha$ is symmetric).

As the manifold is $P$-Sasakian, using (7) in the above equation we get

\[ g(X, \xi)\alpha(\xi, \xi) - \alpha(X, \xi) = 0. \]

Differentiating (2.2) covariantly along $Y$, we get

\[ [g(\nabla_Y X, \xi) + g(X, \nabla_Y \xi)] \alpha(\xi, \xi) + 2g(X, \xi)\alpha(\nabla_Y \xi, \xi) \]
\[ - [\alpha(\nabla_Y X, \xi) + \alpha(X, \nabla_Y \xi)] = 0. \]
Putting \( X = \nabla_Y X \) in (2.2), we get
\[
(2.4) \quad g(\nabla_Y X, \xi)\alpha(\xi, \xi) - \alpha(\nabla_Y X, \xi) = 0.
\]
From (2.3) and (2.4) we get
\[
(2.5) \quad g(X, \varphi Y)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\varphi Y, \xi) - \alpha(X, \varphi Y) = 0.
\]
Replacing \( X \) by \( \varphi Y \) in (2.2) and using (9) gives
\[
(2.6) \quad \alpha(\varphi Y, \xi) = 0.
\]
From (2.5) and (2.6) it follows that
\[
(2.7) \quad g(X, \varphi Y)\alpha(\xi, \xi) - \alpha(X, \varphi Y) = 0.
\]
Replacing \( Y \) by \( \varphi Y \) in (2.7) and using (3), (6) and (2.2) we get
\[
(2.8) \quad \alpha(X, Y) = \alpha(\xi, \xi)g(X, Y).
\]
Differentiating (2.8) covariantly along any vector field on \( M \), it can be easily seen that \( \alpha(\xi, \xi) \) is constant. Hence we can state the following theorem:

**Theorem 1.** On a \( P \)-Sasakian manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.

As an immediate corollary of Theorem 1, we have the following result:

**Corollary.** If the Ricci tensor field is parallel in a \( P \)-Sasakian manifold, then it is an Einstein manifold.

The above corollary is proved by T. Adati and T. Miyazawa ([7]) in another way.

Next, let \( M \) be a \( P \)-Sasakian manifold and \( \alpha \) a parallel 2-form. Putting \( Y = W = \xi \) in (2.1) and using (7) and (8), we obtain
\[
(2.9) \quad \alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X) + g(X, Z)\alpha(\xi, \xi).
\]
Since \( \alpha \) is a 2-form, that is, \( \alpha \) is a \((0,2)\) skew-symmetric tensor therefore \( \alpha(\xi, \xi) = 0 \). Hence (2.9) reduces to
\[
(2.10) \quad \alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X).
\]
Now, let \( A \) be a \((1,1)\) tensor field which is metrically equivalent to \( \alpha \), i.e., \( \alpha(X, Y) = g(AX, Y) \). Then, from (2.10) we have
\[
g(AX, Z) = \eta(X)g(A\xi, Z) - \eta(Z)g(A\xi, X),
\]
and thus,

\[ (2.11) \quad AX = \eta(X)A\xi - g(A\xi, X)\xi. \]

Since \( \alpha \) is parallel, then \( A \) is parallel. Hence, using that \( \nabla_X\xi = \varphi X \), it follows that

\[ \nabla_X(A\xi) = A(\nabla_X\xi) = A(\varphi X). \]

Thus

\[ (2.12) \quad \nabla_{\varphi X}(A\xi) = A(\varphi^2 X) = AX - \eta(X)A\xi. \]

Therefore, we have from (2.11) and (2.12)

\[ (2.13) \quad \nabla_{\varphi X}(A\xi) = -g(A\xi, X)\xi. \]

Now, from (2.11) we get

\[ (2.14) \quad g(A\xi, \xi) = 0. \]

From (2.13) and (2.14) it follows that

\[ (2.15) \quad g(\nabla_{\varphi X}(A\xi), A\xi) = 0. \]

Replacing \( X \) by \( \varphi X \) in (2.15) and since \( \nabla_{\xi}\xi = 0 \), it follows that

\[ (2.16) \quad g(\nabla_X(A\xi), A\xi) = 0, \]

for any \( X \) and thus \( \|A\xi\| = \text{constant on } M \).

From (2.16) we deduce

\[ g(A(\nabla_X\xi), A\xi) = -g(\nabla_X\xi, A^2\xi) = 0. \]

Replacing \( X \) by \( \varphi X \) in the above equation, it follows

\[ g(\nabla_{\varphi X}\xi, A^2\xi) = g(\varphi^2 X, A^2\xi) = g(X - \eta(X)\xi, A^2\xi) = 0. \]

Thus, \( g(X, A^2\xi) = g(\eta(X)\xi, A^2\xi) \).

Hence

\[ (2.17) \quad A^2\xi = -\|A\xi\|^2\xi. \]

Differentiating the above equation covariantly along \( X \), it follows that

\[ \nabla_X(A^2\xi) = A^2(\nabla_X\xi) = A^2(\varphi X) = -\|A\xi\|^2(\nabla_X\xi) = -\|A\xi\|^2(\varphi X). \]

Hence \( A^2(\varphi X) = -\|A\xi\|^2(\varphi X) \).
Replacing $X$ by $\varphi X$, we have (2.17)

$$A^2X = -\|A\xi\|^2X.$$ 

Now, if $\|A\xi\| \neq 0$, then $J = \frac{1}{\|A\xi\|}A$ is an almost complex structure on $M$. In fact, $(J, g)$ is a Kähler structure on $M$. The fundamental 2-form is $g(JX, Y) = \lambda g(AX, Y) = \lambda \alpha(X, Y)$, with $\lambda = 1/\|A\xi\| = \text{constant}$. But, (2.11) means

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X),$$

and thus $\alpha$ is degenerate, which is a contradiction. Therefore $\|A\xi\| = 0$ and hence $\alpha = 0$.

Hence we can state the following theorem:

**Theorem 2.** On a $P$-Sasakian manifold there is no non-zero parallel 2-form.

**References**


U. C. DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI – 741 235
WEST BENGAL, INDIA

(Received October 24, 1994; revised November 20, 1995)