The properties of $T^*$-groups

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Abstract. This paper is a continuation of [1]. The aim of this work is the generalization of Zacher's theorem [2] for the solvable $T^*$-groups, a characterization of these groups by the normalizers of $p$-subgroups and the study of subnormal $p$-subgroups and of the Sylow subgroups of a solvable $T^*$-group.

Throughout this paper $G$ will denote a finite group. A $T$-group is a group $G$ whose subnormal subgroups are normal in $G$. In [1] M. ASAAD introduced the concept of $T^*$-group. A group $G$ is called $T^*$-group if every subnormal subgroup of $G$ is quasinormal in $G$. A subgroup of $G$ is quasinormal in $G$ if it permutes with every Sylow subgroup of $G$. In [1] we proved theorems concerning $T$-groups for $T^*$-groups. Now we are continuing the study of solvable $T^*$-groups. The notation used in this paper is standard.

First we generalize a theorem of ZACHER [2] for solvable $T^*$-groups.

**Theorem 1.** Let $G$ be a solvable group, the prime divisors of its order $p_1 > p_2 > \cdots > p_k$, and let $P_1, \ldots, P_k$ be a Sylow system with $P_i \in \text{Syl}_{p_i}(G)$. $G$ is a $T^*$-group if and only if it satisfies the following conditions:

(i) If $1 \leq i < j \leq k$, then $P_j \leq N_G(P_i)$.

(ii) For all $1 \leq i < j \leq k$, if $x \in P_i$, $y \in P_j$ then there exists a natural number $n$ such that $x^y = x^n$.

**Proof.** 1. Suppose $G$ is a solvable $T^*$-group. Then by Lemma 1 of [1] $G$ is supersolvable whence it has a Sylow tower. So $G$ satisfies (i). As every subgroup of a solvable $T^*$-group is again a $T^*$-group by Theorem 1.

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of [1], it follows that \( N_G(P_i) \) is a solvable \( T^* \)-group. We have that \( \langle x \rangle \) is subnormal in \( N_G(P_i) \), and using Lemma 2 of [1] \( P_j < N_G(\langle x \rangle) \) is true. Thus \( G \) satisfies (ii).

2. Conversely, assume \( G \) satisfies (i) and (ii). We show that every subgroup of \( P_i \) is quasinormal in \( N_G(P_i) \). Let \( B \) be an arbitrary subgroup of \( P_i \). By the conditions \( P_j < N_G(B) \) for all \( j > i \). As \( P_j^y < N_G(B^y) \) for every \( y \in N_G(P_i) \), clearly any Sylow \( p_j \)-subgroup of \( N_G(P_i) \) normalizes any subgroup of \( P_i \). Let \( \ell < i \) and let \( D \) be an arbitrary Sylow \( p_\ell \)-subgroup of \( N_G(P_i) \). By Hall’s theorems \( (P_i^zD)^{\ell} \leq P_i^z \) for some \( z \in G \). As \( (P_i^z)^{\ell} = P_i^{\ell z}D^{\ell} \), \( D^{\ell} \leq P_i \) and \( P_i < N_G(P_\ell) \) clearly \( P_i^{\ell z} < N_G(D^{\ell}) \) follows, furthermore \( D^{\ell} \leq N_G(P_i^{\ell}) \), whence \( D \) centralizes \( P_i \).

Thus every subgroup of \( P_i \) is quasinormal in \( N_G(P_i) \). Using Lemma 4 of [1] it follows that either \( P_i \leq G' \) or each Sylow \( q \)-subgroup \( (q \neq p_i) \) of \( N_G(P_i) \) centralizes \( P_i \). So we can repeat the first part of the proof of Theorem 2 in [1]. Consequently \( G = HK \), where \( H \) is a nilpotent normal Hall subgroup of \( G \), \( K \) is a nilpotent Hall subgroup of \( G \), \( H \cap K = 1 \), furthermore for arbitrary \( x \in H \), \( y \in K \) there exists a natural number \( i \) such that \( x^y = x^i \). Then \( G \) is a solvable \( T^* \)-group by Theorem 2 of [1].

We need the following

**Lemma.** Let \( U \) be a \( p \)-subgroup of \( G \), \( a \in N_G(U) \) such that \( (|a|, |U|) = 1 \) furthermore \( a \) normalizes every subgroup of \( U \). If there is an element \( b \neq 1 \) of \( U \) such that \( ab = ba \), then \( a \in C_G(U) \) follows.

**Proof.** Clearly \( C_U(a) \neq 1 \). Assume \( C_U(a) \neq U \).

(a) \( Z(\Omega_1(U)) \neq C_U(a) \).

Denote \( W = Z(\Omega_1(U))C_U(a) \). Clearly \( \langle a \rangle \) normalizes every subgroup of \( W \) and each element of \( \langle a \rangle \) induces the identity on \( W/Z(\Omega_1(U)) \) by conjugation. Applying Lemma 3 of [1] \( \langle a \rangle \leq C_G(W) \) follows, a contradiction.

(b) \( Z(\Omega_1(U)) > C_U(a) \).

Clearly there exists a subgroup \( T \neq 1 \) such that \( Z(\Omega_1(U)) = C_U(a) \times T \).

Let \( b \in T \), \( b \neq 1 \) and \( u \in C_U(a) \), \( u \neq 1 \). Clearly \( a \) normalizes \( b \) and \( \langle bu \rangle \), consequently \( (bu)^a = (bu)^m \) where \( 2 \leq m \leq p-1 \). As \( (bu)^a = b^au = b^m u^m \), \( u^{m-1} = (b^m)^{-1} b^a \) follows. We have \( b^a = b^n \) where \( 2 \leq n \leq p-1 \) so \( u^{m-1} = b^{n-m} \) is true, but \( \langle u \rangle \cap \langle b \rangle = 1 \) thus \( u^{m-1} = 1 \), a contradiction.

So \( Z(\Omega_1(U)) = C_U(a) \).

(c) \( Z(\Omega_1(U)) < \Omega_1(U) \).

Similarly to case (b) we can show that this case is impossible too.

Thus \( C_U(a) = Z(\Omega_1(U)) = \Omega_1(U) \). Let \( \ell \in U \setminus \Omega_1(U) \). By the conditions \( a \) normalizes \( \langle \ell \rangle \). As \( a \) centralizes \( \Omega_1(\langle \ell \rangle) \) consequently \( a \) centralizes \( \langle \ell \rangle \). A contradiction.
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Theorem 2. $G$ is a solvable $T^*$-group if and only if every $p$-subgroup $A$ (for all prime divisors $p$ of the order of $G$) is quasinormal in $N_G(P_0)$ where $P_0$ is a $p$-subgroup containing the subgroup $A$.

Proof. Assume $G$ is a solvable $T^*$-group. Then by Theorem 1 of [1] $N_G(P_0)$ is a solvable $T^*$-group too. Clearly $A$ is subnormal in $N_G(P_0)$, whence $A$ is quasinormal in $N_G(P_0)$.

Conversely, let $p_1$ be the smallest prime divisor of the order of $G$. We show that $G$ has a normal $p_1$-complement. Let $P_1$ be a Sylow $p_1$-subgroup of $G$ and let $H$ be an arbitrary subgroup of $P_1$. We prove that $N_G(H)/C_G(H)$ is a $p_1$-group. Assume there is an element $b$ of $N_G(H) \setminus C_G(H)$ of order $q$ with $q \neq p_1$. Let $a$ be an element of $H$ of order $p_1$. By the conditions, $\langle a \rangle$ is quasinormal in $N_G(H)$. It is easy to see $b \in N_H(\langle a \rangle)$. As $q > p_1$, $b \in C_G(a)$ follows. Clearly every subgroup of $H$ is quasinormal in $N_G(H)$ by the conditions, whence $b$ normalizes every subgroup of $H$. Using our Lemma $b \in C_G(H)$ is true, a contradiction. Thus $N_G(H)/C_G(H)$ is a $p_1$-group, consequently $G$ has a normal $p_1$-complement. So $G = P_1K$, $K \triangleleft G$ and $P_1 \cap K = 1$. Consider the smallest prime divisor $p_2$ of the order of $K$. Similarly we can prove that $K$ has a normal $p_2$-complement.

Thus $G$ has a tower such that the prime divisors of the order of $G$ are $p_1 < p_2 < \cdots < p_k$ and for arbitrary $1 \leq i \leq k$ there is a Sylow $p_i$-subgroup such that $P_i < N_G(P_j)$ for all $1 \leq i < j \leq k$. If $i$ and $j$ are such as above and $x \in P_j$, $y \in P_i$, then $\langle x \rangle$ is quasinormal in $N_G(P_j)$ by the conditions, whence it is easy to see that $y \in N_G(\langle x \rangle)$, consequently $x^y = x^n$ for some natural number $n$. Applying Theorem 1 $G$ is a solvable $T^*$-group.

Theorem 3. Let $G$ be a solvable $T^*$-group. Then an arbitrary subnormal $p$-subgroup of $G$ (for all prime divisors $p$ of the order of $G$) is either normal or it is centralized by all Sylow $q$-subgroups of $G$ with $q \neq p$.

Proof. Let $A$ be a subnormal $p$-subgroup of $G$. By Theorem 7 of [1] $G = MN$ where $M$ is a nilpotent normal Hall subgroup of $G$, $N$ is a nilpotent Hall subgroup of $G$, $M \cap N = 1$ furthermore every subgroup of prime power order of $M$ is normal in $G$

(a) $A \leq M$. By the above $A$ is normal in $G$

(b) $A \leq N^y$ for some $y \in G$.

Let $Q$ be a Sylow $q$-subgroup of $M$ with $q \neq p$. By the subnormality of $A$ there is a chain $A < A_1 < \cdots < A_\ell < A_{\ell+1} < \cdots < A_m = G$. 

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\[ \text{Theorem 2. } G \text{ is a solvable } T^*\text{-group if and only if every } p\text{-subgroup } A \text{ (for all prime divisors } p \text{ of the order of } G \text{) is quasinormal in } N_G(P_0) \text{ where } P_0 \text{ is a } p\text{-subgroup containing the subgroup } A. \]

\[ \text{Proof. } \text{Assume } G \text{ is a solvable } T^*\text{-group. Then by Theorem 1 of [1] } N_G(P_0) \text{ is a solvable } T^*\text{-group too. Clearly } A \text{ is subnormal in } N_G(P_0), \text{ whence } A \text{ is quasinormal in } N_G(P_0). \]

\[ \text{Conversely, let } p_1 \text{ be the smallest prime divisor of the order of } G. \text{ We show that } G \text{ has a normal } p_1\text{-complement. Let } P_1 \text{ be a Sylow } p_1\text{-subgroup of } G \text{ and let } H \text{ be an arbitrary subgroup of } P_1. \text{ We prove that } N_G(H)/C_G(H) \text{ is a } p_1\text{-group. Assume there is an element } b \text{ of } N_G(H) \setminus C_G(H) \text{ of order } q \text{ with } q \neq p_1. \text{ Let } a \text{ be an element of } H \text{ of order } p_1. \text{ By the conditions, } \langle a \rangle \text{ is quasinormal in } N_G(H). \text{ It is easy to see } b \in N_H(\langle a \rangle). \text{ As } q > p_1, b \in C_G(a) \text{ follows. Clearly every subgroup of } H \text{ is quasinormal in } N_G(H) \text{ by the conditions, whence } b \text{ normalizes every subgroup of } H. \text{ Using our Lemma } b \in C_G(H) \text{ is true, a contradiction. Thus } N_G(H)/C_G(H) \text{ is a } p_1\text{-group, consequently } G \text{ has a normal } p_1\text{-complement. So } G = P_1K, K \triangleleft G \text{ and } P_1 \cap K = 1. \text{ Consider the smallest prime divisor } p_2 \text{ of the order of } K. \text{ Similarly we can prove that } K \text{ has a normal } p_2\text{-complement.} \]

\[ \text{Thus } G \text{ has a tower such that the prime divisors of the order of } G \text{ are } p_1 < p_2 < \cdots < p_k \text{ and for arbitrary } 1 \leq i \leq k \text{ there is a Sylow } p_i\text{-subgroup such that } P_i < N_G(P_j) \text{ for all } 1 \leq i < j \leq k. \text{ If } i \text{ and } j \text{ are such as above and } x \in P_j, y \in P_i, \text{ then } \langle x \rangle \text{ is quasinormal in } N_G(P_j) \text{ by the conditions, whence it is easy to see that } y \in N_G(\langle x \rangle), \text{ consequently } x^y = x^n \text{ for some natural number } n. \text{ Applying Theorem 1 } G \text{ is a solvable } T^*\text{-group.} \]

\[ \text{Theorem 3. Let } G \text{ be a solvable } T^*\text{-group. Then an arbitrary subnormal } p\text{-subgroup of } G \text{ (for all prime divisors } p \text{ of the order of } G \text{) is either normal or it is centralized by all Sylow } q\text{-subgroups of } G \text{ with } q \neq p. \]

\[ \text{Proof. } \text{Let } A \text{ be a subnormal } p\text{-subgroup of } G. \text{ By Theorem 7 of [1] } G = MN \text{ where } M \text{ is a nilpotent normal Hall subgroup of } G, N \text{ is a nilpotent Hall subgroup of } G, M \cap N = 1 \text{ furthermore every subgroup of prime power order of } M \text{ is normal in } G\]

(a) $A \leq M$. By the above $A$ is normal in $G$

(b) $A \leq N^y$ for some $y \in G$.

\[ \text{Let } Q \text{ be a Sylow } q\text{-subgroup of } M \text{ with } q \neq p. \text{ By the subnormality of } A \text{ there is a chain } A < A_1 < \cdots < A_\ell < A_{\ell+1} < \cdots < A_m = G. \]
Let $A_{\ell}$ be such that $Q \leq A_{\ell}$ but $Q \nless A_{\ell-1}$. $A$ normalizes every subgroup of $Q$. Since $A_{\ell-1} \triangleleft A_{\ell}$, $Q \triangleleft A_{\ell}$ and $A \leq A_{\ell-1}$ it follows that each element of $A$ induces the identity on $Q/Q \cap A_{\ell-1}$ by conjugation. Using Lemma 3 of [1] $A \leq C_G(Q)$ follows. As $Q$ is an arbitrary Sylow subgroup of $M$, $A \leq C_G(M)$ is true. We have $G = M \cdot N^y$, $N^y$ is a nilpotent Hall subgroup of $G$ and $N^y = P \times T$ where $P$ is a Sylow $p$-subgroup of $G$, whence $C_G(M) \geq M \cdot T$. As $MT \lhd G$ it is easy to see that $A$ is centralized by an arbitrary Sylow $q$-subgroup of $G$ with $q \neq p$.

**Theorem 4.** $G$ is a solvable $T^*$-group if and only if every Sylow subgroup $P$ satisfies one of the following conditions:

(a) every subgroup of $P$ is normal in $G$

(b) every Sylow subgroup of $N_G(P)$ different from $P$ centralizes $P$.

**Proof.** Assume $G$ is a solvable $T^*$-group. By the Theorem 7 of [1] $G = MN$ where $M$ is a nilpotent normal Hall subgroup of $G$, $N$ is a nilpotent Hall subgroup of $G$, $M \cap N = 1$ and every subgroup of prime power order of $M$ is normal in $G$. Let $R$ be an arbitrary Sylow subgroup of $G$.

Assume $R \leq M$. By the above every subgroup of $R$ is normal in $G$.

Assume $R \leq N^y$ for some $y \in G$. Clearly $N_G(R) = N^y \cdot (N_G(R) \cap M)$. The structure of $G$ yields $B = N_G(R) \cap M \leq C_G(R)$ and $N^y = R \times L$ where $L$ is a nilpotent Hall subgroup of $G$, consequently $N_G(R) = R \times (L \cdot B)$ so $R$ satisfies (b).

Conversely, let $G$ be a counterexample of smallest order. Let $M_0$ be the product of every Sylow subgroup of $G$ each subgroup of which is normal in $G$. Clearly $M_0$ is a nilpotent normal Hall subgroup of $G$. By the Theorem of Zassenhaus there is a subgroup $N_0$ such that $M_0 \cdot N_0 = G$ and $M_0 \cap N_0 = 1$. Clearly $N_0$ is a Hall subgroup in $G$ and it satisfies the conditions of our theorem. By the minimality of $G$ $N_0$ is a solvable $T^*$-group. Using Theorem 2 of [1] $N_0 = A \cdot B$ where $A$ is a nilpotent normal Hall subgroup of $G$, $B$ is a nilpotent Hall subgroup of $G$ and $A \cap B = 1$ furthermore for arbitrary $a \in A$, $b \in B$ there is a natural number $i$ such that $a^b = a^i$. Assume $A \neq 1$. Let $P$ be a Sylow subgroup of $A$. Using $N_{N_0}(P) = N_0$, $B \leq C_G(P)$ follows by the conditions whence $B \leq C_G(A)$. Thus $N_0 = A \times B$ and $N_0$ is a nilpotent normal Hall subgroup of $G$. Using Theorem 2 of [1] it follows that $G$ is a solvable $T^*$-group.
References


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