Remark on Ky Fan convexity

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Abstract. In the paper is proved that the Nakaido-Isolda’s theorem fails to hold if concavity is replaced by Ky Fan concavity.

Let $X, Y$ be arbitrary sets. The function

$$f : X \times Y \to \mathbb{R}$$

is called Ky Fan concave in the variable $x$ if

$$\forall x_1, x_2 \in X \quad \forall \lambda \in [0, 1] \quad \exists x_3 \in X \quad \forall y. \quad f(x_3, y) \geq \lambda f(x_1, y) + (1 - \lambda)f(x_2, y).$$

The concavity with respect to $y$ is defined symmetrically. In [1] the authors stated the following

Theorem. There exists functions $f_1, f_2 \in C^\infty([0, 1] \times [0, 1])$ such that $f_1$ is Ky Fan concave in the variable $x$, $f_2$ is Ky Fan concave in the variable $y$ and the pair $f_1, f_2$ has no saddle point i.e. there is no point $(x_0, y_0)$ satisfying

$$f_1(x_0, y_0) \geq f_1(x_1, y_0) \quad \forall x$$
$$f_2(x_0, y_0) \geq f_2(x_0, y) \quad \forall y.$$

This is a counterexample showing that the Nikaido-Isolda theorem fails to hold if concavity is replaced by Ky Fan concavity. The proof given in [1] was not correct; it suggested that there are polynomials $f_1, f_2$
satisfying Theorem. In fact we do not know whether there are analytical functions $f_1, f_2$ satisfying Theorem.

**Proof of Theorem.** Define the functions $k_1, k_2 : [0, 1] \times [0, 1]$ as follows. Let $0 < \delta < \frac{1}{4}$ be fixed. For $0 \leq x \leq \frac{1}{4}$ the function $k_1((x), k_2(x))$ varies linearly from $(0, 1)$ to $(\frac{1}{2} + \delta, \frac{1}{2} - \delta)$; for $\frac{1}{4} \leq x \leq \frac{1}{2}$ it goes linearly from $(\frac{1}{2} + \delta, \frac{1}{2} - \delta)$ to $(0, 0)$, for $\frac{1}{2} \leq x \leq \frac{3}{4}$ from $(0, 0)$ to $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ and for $\frac{3}{4} \leq x \leq 1$ from $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ to $(1, 0)$.

We can suppose that $((k_1(x), k_2(x))$ is extended linearly from $[0, \frac{1}{4}]$ to $(-\infty, \frac{1}{4}]$ and from $[\frac{3}{4}, 1]$ to $[\frac{3}{4}, \infty)$. Consider a function
\[
\varphi \in C_0^\infty(\mathbb{R}), \text{ supp } \varphi = [-\delta, \delta], \quad \varphi \geq 0, \quad \varphi(x) = \varphi(-x)\forall x, \int_{-\infty}^{\infty} \varphi = 1.
\]
The existence of a such a function is widely known.

Define
\[
\hat{k}_1 = k_1 * \varphi \quad \hat{k}_2 = k_2 * \varphi.
\]
Introduce the sets
\[
A = \{(k_1(x), k_2(x)) : x \in [0, 1]\},
\]
\[
\hat{A} = \{((k_1(x), \hat{k}_2(x)) : x \in [0, 1]\}.
\]
Since the convolution by $\varphi$ gives an average of the values $k_i(x)$, all points of $\hat{A}$ belongs to the (closed) convex hull of $A$. Hence $\hat{A}$ lies in the triangle of vertices $(0, 0), (1, 0), (0, 1)$. On the other hand, $\int_{-\infty}^{\infty} x\varphi(x)dx = 0$ implies that $\hat{k}_i(x) = k_i(x)$ whenever $k_i$ varies linearly in $[x-\delta, x+\delta]$. Consequently $\hat{A}$ contains the side $[(1,0)(0,1)]$ of the above mentioned triangle. This means that the function
\[
f_1(x, y) = (1-y)\hat{k}_1(x) + y\hat{k}_2(x)
\]
is Ky Fan-concave in $x$; this follows easily from the fact that
\[
\forall x_1, x_2 \in [0, 1] \quad \forall \lambda \in [0, 1] \quad \exists x_3 \in [0, 1] : \lambda \hat{k}_1(x) + (1-\lambda)\hat{k}_1(x_2) \leq \hat{k}_1(x_3),
\]
\[
\lambda \hat{k}_2(x_1) + (1-\lambda)\hat{k}_2(x_2) \leq \hat{k}_2(x_3);
\]
see [1], p. 138 or [2] p. 204-205 for more details. Investigate the set
\[
C_1 = \{(x_0, y_0) : f_1(x_0, y_0) = \max_{x \in [0,1]} f_1(x, y_0)\}.\]
For given $y_0$, those values $x_0$ are involved for which the perpendicular projection of $(\hat{k}_1(x_0), \hat{k}_2(x_0))$ to the line along the vector $(1 - y_0, y_0)$ is the farthest from the origin. Keeping in mind what has been proved about the set $\hat{A}$ we see that for $0 \leq y_0 < \frac{1}{2}$ only the point $(1, 0)$ is projected, for $y_0 = \frac{1}{2}$ the whole segment $[(1, 0), (0, -1)]$ and for $\frac{1}{2} < y_0 \leq 1$ the only point $(0, 1)$. Consequently (using $\text{supp } \varphi = [-\delta, \delta]$)

$$C_1 = \{1\} \times \left[0, \frac{1}{2}\right] \cup \left[0, \frac{1}{4}\right] \times \left\{\frac{1}{2}\right\} \cup \left[\frac{3}{4}, 1\right] \times \left\{\frac{1}{2}\right\} \cup \{0\} \times \left(\frac{1}{2}, 1\right].$$

Considered the function

$$f_2(x, y) = -(x - y)^2;$$

it is obviously concave hence also Ky Fan-concave in $y$. On the other hand

$$C_2 = \{(x_0, y_0) : f_2(x_0, y_0) = \max_{y \in [0, 1]} f_2(x_0, y)\}$$

is the line segment $y = x, \ 0 \leq x \leq 1$ which does not meet $C_1$,

$$C_1 \cap C_2 = \emptyset$$

which proves Theorem. \hfill \Box

References