A Frobenius-type theorem for supersolvable groups

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Abstract. Frobenius’ Theorem for $p$-nilpotent groups is one of the most fundamental theorems in finite group theory. In this paper a Frobenius-type Theorem for supersolvable groups is given by applying strictly $p$-closed groups, and some applications are obtained.

Throughout, all groups mentioned are assumed to be finite groups. The terminology and notations employed agree with standard usage.

Let $p$ be a prime. A group $G$ is said to be strictly $p$-closed whenever $G_p$, a Sylow $p$-subgroup of $G$, is normal in $G$ with $G/G_p$ Abelian of exponent dividing $p - 1$.

Let $P$ be a Sylow $p$-subgroup of a group $G$; Frobenius’ Theorem [1, Theorem 10.3.2] states that: a group $G$ is $p$-nilpotent, if and only if $N_G(P_1)/C_G(P_1)$ is a $p$-group for every subgroup $P_1$ of $P$. If the condition that $N_G(P_1)/C_G(P_1)$ is a $p$-group is replaced by the weaker condition that $N_G(P_1)/C_G(P_1)$ is a strictly $p$-closed group, we can obtain a generalization of Frobenius’ Theorem for supersolvable groups.

First we prove the following

Theorem 1. Let $G$ be a $p$-solvable group, $N$ a normal subgroup of $G$ such that $G/N$ is a $p$-supersolvable group. If $N_G(P)/C_G(P)$ is strictly $p$-closed for every $p$-subgroup $P$ of $N$, then $G$ is $p$-supersolvable.

Proof. Let $K$ be a minimal normal subgroup of $G$ contained in $N$. Then $K$ is an elementary Abelian $p$-group or a $p'$-group since $G$ is a $p$-solvable group. Set $\overline{G} = G/K$, and $\overline{N} = N/K$. If $K$ is an elementary Abelian $p$-group, then, for every $p$-subgroup $\overline{P} = P/K$ of $\overline{N}$, $P$ is a $p$-subgroup of $N$, and so $N_G(P)/C_G(P)$ is strictly $p$-closed. Since the quotient group of a strictly $p$-closed group is also a strictly $p$-closed
group, \((N_G(P)/K)/(C_G(P)K/K)\) is a strictly \(p\)-closed group. It follows from \(N_G(P)/K = N_{G/K}(P/K)\) and \(C_{G/K}(P/K) \geq C_G(P)K/K\) that \(N_{\overline{\Gamma}}(\overline{P})/C_{\overline{\Gamma}}(\overline{P})\) is a strictly \(p\)-closed group. If \(K\) is a \(p\)'-group, then, for every \(p\)-subgroup \(\overline{P} = H/K\) of \(N\), \(H = PK\), where \(P \in \text{Syl}_p H\). By the condition \(N_G(P)/C_G(P)\) is strictly \(p\)-closed, and so \(N_G(P)K/C_G(P)K\) is also strictly \(p\)-closed. It is clear that \(C_{\overline{\Gamma}}(\overline{P}) \geq C_G(P)K/K\). Using \([3, \text{Theorem 3.16}]\) \(N_{\overline{\Gamma}}(\overline{P}) = N_G(P)K/K\) we have that \(N_{\overline{\Gamma}}(\overline{P})/C_{\overline{\Gamma}}(\overline{P})\) is strictly \(p\)-closed. Hence we conclude by induction that \(G/K\) is \(p\)-supersolvable.

If \(K\) is a \(p\)'-group, then \(G\) is \(p\)-supersolvable. If \(K\) is an elementary Abelian \(p\)-group, set \(C = C_G(K)\). By the condition \(G/C\) is strictly \(p\)-closed. Let \(A/C \in \text{Syl}_p(G/C)\), then \(A/C \triangleleft G/C\), and the semidirect product \(A/C \rtimes K\) is a \(p\)-group. Hence \(Z(A/C \rtimes K) \cap K \neq 1\). Since \(G/C\) can act on \(Z(A/C \rtimes K) \cap K\), by conjugation and since the action of \(G/C\) on \(K\) is irreducible we have \(Z(A/C \rtimes K) \cap K = K\). Hence the action of \(A/C\) on \(K\) is trivial and \(A/C = 1\). Therefore \(G/C\) is Abelian of exponent dividing \(p - 1\). By \([2, \text{Theorem I.1.4}]\) \(|K| = p\), and \(G\) is \(p\)-supersolvable. The proof of Theorem 1 is complete.

**Theorem 2.** Let \(N\) be a normal subgroup of a group \(G\), and \(G/N\) be a supersolvable group. Then \(G\) is a supersolvable group if and only if for every prime \(p \mid |N|, N_G(P)/C_G(P)\) is a strictly \(p\)-closed group for every \(p\)-subgroup \(P\) of \(N\).

The proof of Theorem 2 needs the following

**Lemma 1.** Let \(P\) be a \(p\)-subgroup of a group \(G\), and \(N_G(P)/C_G(P)\) a strictly \(p\)-closed group. If \(H\) is a subgroup of \(G\), and \(P \leq H\), then \(N_H(P)/C_H(P)\) is a strictly \(p\)-closed group too.

**Proof.** Since \(N_H(P) = H \cap N_G(P)\) and \(C_H(P) = H \cap C_G(P)\), we have

\[
N_H(P)/C_H(P) = H \cap N_G(P)/H \cap C_G(P) \simeq [H \cap N_G(P)]C_G(P)/C_G(P).
\]

Noticing that subgroups of a strictly \(p\)-closed group are strictly \(p\)-closed groups, \(N_H(P)/C_H(P)\) is strictly \(p\)-closed.

**Proof of Theorem 2.** Assume first that \(G\) is a supersolvable group. Let \(p\) be a prime, \(P\) a \(p\)-subgroup of \(N\), \(H = N_G(P)\). Since \(P \triangleleft H\), we have a chief series of \(H\) passing through \(P\):

\[
1 = P_0 < P_1 < \cdots < P_S = P \leq \cdots \leq H.
\]
As a subgroup of the supersolvable group $G$, $H$ itself is supersolvable, and so $|P_j/P_{j-1}| = p$ $(j = 1, 2, \ldots, s)$. By [2, Theorem I.1.4]

$$\text{Aut}_H(P_j/P_{j-1}) \simeq H/C_H(P_j/P_{j-1})$$

is Abelian of exponent dividing $p - 1$. Set $L = \bigcap_{j=1}^s C_H(P_j/P_{j-1})$ and $C = C_G(P)$, then $L \triangleleft H$ and $H/L$ is also Abelian of exponent dividing $p - 1$, and moreover, $L \geq C$. We claim that $L/C$ is a $p$-group. Suppose to the contrary that some $Cx \in L/C$ has order $n$ relatively prime to $p$. Let $\alpha \in \text{Aut}(P)$ be the automorphism induced by $x$, i.e., $\alpha(g) = x^{-1}gx$ $(g \in P)$, then the order of $\alpha$ in $\text{Aut}(P)$ divides $n$, hence it is also relatively prime to $p$. Also note that $x \in L$ implies $[P_j, \alpha] \leq P_{j-1}$ for $1 \leq j \leq s$, so that [2, Lemma I.1.11] applies to show $\alpha$ is trivial. Hence so is $Cx$ too, proving the claim. It follows that $N_G(P)/C_G(P)$ is strictly $p$-closed with Sylow $p$-subgroup $L/C$.

Suppose now that for every prime $p \mid |N|$, $N_G(P)/C_G(P)$ is a strictly $p$-closed group for every $p$-subgroup $P$ of $N$. Let $K$ be a minimal normal subgroup of $G$ contained in $N$. Then $K$ is a $p$-group for some prime $p$. In fact, assume that $p$ is the smallest prime dividing $|K|$; by Lemma 1 and $(p - 1, |K|) = 1$, $N_K(P)/C_K(P)$ is a $p$-group for every $p$-subgroup $P$ of $K$. Using Frobenius’ Theorem [1, Theorem 10.3.2], $K$ has a normal $p$-complement, say $L$. Noticing that $L \triangleleft G$, $L \triangleleft K$ and that $K$ is a minimal normal subgroup of $G$, we have $L = 1$, and hence $K$ is an elementary Abelian $p$-group.

Set $\overline{G} = G/K$ and $\overline{N} = N/K$. Similarly to the proof of Theorem 1 we have that for every prime $q \mid |\overline{N}|$, $N_{\overline{G}}(\overline{R})/C_{\overline{G}}(\overline{R})$ is strictly $q$-closed for every $q$-subgroup $\overline{Q}$ of $\overline{N}$. Hence we conclude by induction that $G/K$ is supersolvable. By the condition and Theorem 1 $G$ is $p$-supersolvable. Noticing that $K$ is a minimal normal $p$-subgroup of $G$, we have that $K$ is a cyclic group of order $p$. It follows that $G$ is supersolvable. The proof of Theorem 2 is complete.

**Corollary 1.** A group $G$ is supersolvable if and only if, for every prime $p \mid |G|$, $N_G(P)/C_G(P)$ is strictly $p$-closed for every $p$-subgroup $P$ of $G$.

**Theorem 3.** Let $N$ be a normal subgroup of a group $G$, and $G/N$ a supersolvable group. Then $G$ is supersolvable if and only if, for every prime $p \mid |N|$, $[N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are $p$-groups for every $p$-subgroup $P$ of $N$.

From Theorem 2 and the following Lemma 2 Theorem 3 is immediate.
Lemma 2. A group $G$ is strictly $p$-closed if and only if $G'$ and $G^{p-1}$ are $p$-groups.

Proof. If $G$ is strictly $p$-closed, then $G/G_p$ is Abelian, where $G_p \in \text{Syl}_p G$. Hence $G' \leq G_p$ and $G'$ is a $p$-group. It follows from the exponent of $G/G_p$ dividing $p-1$ that $g^{p-1} \in G_p$ for every $g \in G$, therefore $G^{p-1}$ is also a $p$-group.

Suppose now that $G'$ and $G^{p-1}$ are $p$-groups. Let $G_p \in \text{Syl}_p G$. Since $G' \lhd G$, we have $G' \leq G_p$ and so $G_p \lhd G$ and $G/G_p$ is Abelian. By using that $G^{p-1}$ is a $p$-group we have $G^{p-1} \leq G_p$. Hence $G/G_p$ is Abelian of exponent dividing $p-1$.

Corollary 2. A group $G$ is supersolvable if and only if, for every prime $p \mid |G|$, $[N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are $p$-groups for every $p$-subgroup $P$ of $G$.

As an application of Theorem 2, we prove the following

Theorem 4. Let $N$ be a normal subgroup of a group $G$, and $G/N$ be a supersolvable group. If every minimal subgroup of $N$ is pronormal in $G$, and either the Sylow 2-subgroups of $N$ are Abelian or every cyclic subgroup of $N$ of order 4 is pronormal in $G$, then $G$ is supersolvable.

The proof of Theorem 4 needs the following

Lemma 3. Let $A_1, A_2, \ldots, A_s; \ B_1, B_2, \ldots, B_s$ be subgroups of the group $G$, and $B_i \lhd A_i, (i = 1, 2, \ldots, s)$. If $A_i/B_i$ is Abelian of exponent dividing $m$, then $(A_1 \cap A_2 \cap \cdots \cap A_s)/(B_1 \cap B_2 \cap \cdots \cap B_s)$ is also Abelian of exponent dividing $m$.

Proof. We only prove Lemma 3 when $s = 2$. Clearly $B_1 \cap B_2 \lhd A_1 \cap A_2$. For any $g_1, g_2 \in A_1 \cap A_2$, since $A_1/B_1$ and $A_2/B_2$ are Abelian and $g_1(B_1 \cap B_2) = g_1B_1 \cap g_1B_2$, we have $g_1g_2(B_1 \cap B_2) = g_2g_1(B_1 \cap B_2)$, i.e., $A_1 \cap A_2/B_1 \cap B_2$ is Abelian. From $g_1^m \in B_1, g_2^m \in B_2$ we have $g_1^m \in B_1 \cap B_2$. Hence the exponent of $A_1 \cap A_2/B_1 \cap B_2$ divides $m$.

Proof of Theorem 4. For any prime $p \mid |N|$, if $P$ is a subgroup of $N$ of order $p$, then $N_G(P)/C_G(P)$ is Abelian of exponent dividing $p-1$ since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\text{Aut}(P)$. Hence $N_G(P)/C_G(P)$ is strictly $p$-closed. If $P$ is a cyclic subgroup of $N$ of order 4, it follows from $|\text{Aut}(P)|=2$ that $N_G(P)/C_G(P)$ is Abelian of exponent dividing 2. Hence $N_G(P)/C_G(P)$ is strictly 2-closed.

Let $A$ be any $p$-subgroup of $N$, and $x$ be an element of $A$ of order $p$. Then $\langle x \rangle$ is subnormal in $N_G(A)$. Using [1, exercise 10.3.3] $\langle x \rangle \triangleleft N_G(A)$. 

A Frobenius-type theorem for supersolvable groups

Since $\Omega_1(A) \triangleleft N_G(A) = H$, $C_H(\Omega_1(A)) \triangleleft H$, it is clear that $C = C_G(A) \leq C_H(\Omega_1(A))$. We claim that $C_H(\Omega_1(A))/C$ is a $p$-subgroup of $H/C$ if $p \neq 2$, or $p = 2$ and $A$ is Abelian. In fact, let $gC \in C_H(\Omega_1(A))/C$ and the order of $gC$ be a $p'$-number. Noticing that $\langle gC \rangle$ can act on $A$ by conjugation, and that the action of $\langle gC \rangle$ on $\Omega_1(A)$ is trivial, the action of $\langle gC \rangle$ on $A$ is trivial by [3, Theorem 7.26] if $p \neq 2$ or by [4, Theorem 5.2.4] if $p = 2$ and $A$ is Abelian. Hence $gC = C$, i.e., $C_H(\Omega_1(A))/C$ is a $p$-group.

Noticing that $C_H(\Omega_1(A)) = \bigcap_{x \in \Omega_1(A)} (C_H(\langle x \rangle))$, $H \subseteq \bigcap_{x \in \Omega_1(A)} N_H(\langle x \rangle)$ and that $N_H(\langle x \rangle)/C_H(\langle x \rangle)$ is Abelian of exponent dividing $p − 1$ (when $x$ has order $p$), $H/C_H(\Omega_1(A))$ is Abelian of exponent dividing $p − 1$ by Lemma 3. Hence $H/C = N_G(A)/C_G(A)$ is strictly $p$-closed if $p \neq 2$ or $p = 2$ and $A$ is Abelian.

If $A$ is a 2-subgroup of $N$ and $A$ is not Abelian, by considering the subgroup $\Omega_2(A)$ and using [3, Theorem 7.26], similar to the above proof we have that $C_H(\Omega_2(A))/C_G(A)$ is a 2-group, and that $H/C_H(\Omega_2(A))$ is Abelian of exponent dividing 2. Hence $H/C = N_G(A)/C_G(A)$ is a 2-group, and so strictly 2-closed. By Theorem 1 $G$ is supersolvable. The proof of Theorem 4 is complete.

Remark. The statement of Theorem 4 for the case when $N$ has odd order has been proved by M. Asaad in [5].

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