Nonoscillation in half-linear differential equations

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Abstract. We establish some necessary conditions on the nonoscillation of the following half-linear second order differential equation

\[ [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \quad t \geq t_0, \]

where \( p > 1 \) is a constant, \( r(t) \) and \( c(t) \) are continuous functions from \( [t_0, \infty) \) to \( [0, \infty) \) with \( r(t) > 0 \).

1. Introduction

This paper is concerned with the half-linear second order differential equation

\[ (E) \quad [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \quad t \geq t_0, \]

where \( p > 1 \) is a constant, \( r(t) \) and \( c(t) \) are continuous functions on \( [t_0, \infty) \) for some \( 0 \geq 0 \). Throughout the paper, we assume that

\begin{align*}
(A_1) & \quad \frac{1}{p} + \frac{1}{q} = 1; \\
(A_2) & \quad r(t) > 0 \text{ for } t \geq t_0 \text{ and } \int_{t_0}^{\infty} r^{-q}(s)ds = \infty; \\
(A_3) & \quad c(t) \geq 0 \text{ for } t \geq t_0 \text{ and } c(t) \neq 0 \text{ on any interval of the form } [t, \infty), \\
& \quad t \geq t_0.
\end{align*}

By a solution of (E) we mean a function \( u \in C^1[t_0, \infty) \) such that \( r|u'|^{p-2}u' \in C^1[t_0, \infty) \) and that satisfies (E). In [1], ELBERT established the existence, uniqueness and extension to \( [t_0, \infty) \) of solutions to the initial value problem for (E). We will say that a nontrivial solution \( u \) of (E) is

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nonoscillatory if there exists a number \( N > 0 \) such that \( u(t) \neq 0 \) for all \( t \geq N \). Equation (E) is nonoscillatory if all its solutions are nonoscillatory.

Kusano, Naito and Ogata [2], and Li and Yeh [3] independently showed that if (E) is nonoscillatory then

\[
\int_{t_0}^{\infty} c(s) ds < \infty
\]

and

\[
\lim_{t \to \infty} \pi^{p-1}(t) \int_{t_0}^{\infty} c(s) ds \leq 1,
\]

where

\[
\pi(t) = \int_{t_0}^{t} r^{1-q}(s) ds, \quad t \geq t_0.
\]

It follows from (2) that if (E) is nonoscillatory then

\[
\int_{t_0}^{\infty} r^{1-q}(s) \left( \int_{s}^{\infty} c(\tau) d\tau \right)^{q} ds < \infty.
\]

The purpose of this paper is to improve the results (1), (2), (3), and hence extend the result of Lovelady [4].

2. Main results

In order to prove our main theorem, we need the following lemma.

**Lemma 2.1.** If \( u(t) \) is a nonoscillatory solution of (E) which is not eventually a constant, then \( u(t)u'(t) > 0 \) for all large \( t \).

**Proof.** Without loss of generality, we may assume that \( u(t) > 0 \) on \([T_0, \infty)\) for some \( T_0 \geq t_0 \). It follows from (E) that

\[
[r(t)|u'(t)|^{p-2}u'(t)]' \leq 0 \quad \text{for} \quad t \geq T_0,
\]

which implies that \( r(t)|u'(t)|^{p-2}u'(t) \) is nonincreasing on \([T_0, \infty)\). Suppose there exists a \( T_1 \geq T_0 \) such that \( u'(T_1) \leq 0 \). Then \( r(T_1)|u'(T_1)|^{p-2}u'(T_1) \leq 0 \). Since \( r(t)|u'(t)|^{p-2}u'(t) \) is decreasing and not identically zero on \([T_0, \infty)\), there exists a \( T_2 \geq T_1 \) such that

\[
r(t)|u'(t)|^{p-2}u'(t) \leq r(T_2)|u'(T_2)|^{p-2}u'(T_2) = -k < 0 \quad \text{for} \quad t \geq T_2,
\]
which implies

\begin{equation}
(2) \quad u'(t) \leq -k^{q-1}r^{1-q}(t) \quad \text{for } t \geq T_2.
\end{equation}

Integrating (2) from $T_2$ to $t$, we obtain by $(A_2)$

\[ u(t) \leq u(T_2) - k^{q-1} \int_{T_2}^{t} r^{1-q}(s)ds \to -\infty \quad \text{as } t \to \infty, \]

which contradicts to $u(t) > 0$ on $[T_0, \infty)$. Thus $u'(t) > 0$ on $[T_0, \infty)$. This completes our proof.

**Theorem 2.2.** Let

\[ f(t) = \int_{t}^{\infty} c(s)ds, \quad t \in [t_0, \infty). \]

If (E) is nonoscillatory, then there exist a number $T_0 \geq t_0$ and a sequence $\{w_k\}_{k=0}^{\infty}$ of continuous functions from $[T_0, \infty)$ to $(0, \infty)$ with the following properties:

(a) $w_1 = f$.

(b) $w_k(t) \leq w_{k+1}(t)$ for $t \geq T_0$ and each integer $k \geq 1$.

(c) $\int_{t}^{\infty} r^{1-q}(s)f^{q-1}(s)w_k(s)ds < \infty$ for $t \geq T_0$ and each integer $k \geq 0$; and

\[ w_{k+1}(t) = f(t) + (p-1) \int_{t}^{\infty} r^{1-q}(s)f^{q-1}(s)w_k(s)ds \quad \text{for } t \geq T_0 \quad \text{and each integer } k \geq 1. \]

(d) If $t \geq T_0$, then $w_0(t) = \lim_{k \to \infty} w_k(t)$, and the convergence is uniform in each compact subset of $[T_0, \infty)$.

(e) $\limsup_{t \to \infty} t^{p-1}(t)w_k(t) \leq 1$ for each integer $k \geq 0$.

(f) $w_0(t) = f(t) + (p-1) \int_{t}^{\infty} r^{1-q}(s)f^{q-1}(s)w_0(s)ds$ for $t \geq T_0$.

**Proof.** Let $u(t)$ be a nonoscillatory solution of (E). By Lemma 2.1, without loss of generality, we may assume that $u(t) > 0$ and $u'(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Let

\[ w(t) = \frac{r(t)|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)} \quad \text{for } t \geq T_0. \]

Then $w(t) > 0$ and

\begin{equation}
(3) \quad w'(t) = -c(t) - (p-1)r^{1-q}(t)w^q(t) < 0
\end{equation}
for $t \geq T_0$. This implies that $w(t)$ is decreasing and $\lim_{t \to \infty} w(t)$ exists. Integrating (3) from $t$ to $T$, we obtain

$$w(T) - w(t) = - \int_t^T c(s)ds - (p - 1) \int_t^T r^{1-q}(s)w^q(s)ds$$

for $T \geq t \geq T_0$. It follows from (1) and the existence of $\lim_{t \to \infty} w(t)$ that

$$\int_{T_0}^\infty r^{1-q}(s)w^q(s)ds < \infty.$$  

It follows from $(A2)$ and the decrease of $w(t)$ that $\lim_{T \to \infty} w(T) = 0$. This implies

$$w(t) = f(t) + (p - 1) \int_t^\infty r^{1-q}(s)w^q(s)ds \quad \text{for} \quad t \geq T_0.$$  

It is clear from (5) that $w \geq f$ on $[T_0, \infty)$, and hence (4) and (5) imply that

$$\int_{T_0}^\infty r^{1-q}f^{q-1}(s)w(s)ds \leq \int_{T_0}^\infty r^{1-q}(s)w^q(s)ds < \infty$$

and

$$w(t) \geq f(t) + (p - 1) \int_t^\infty r^{1-q}f^{q-1}(s)w(s)ds$$

for $t \geq T_0$, respectively. It follows from (E) that $r^{q-1}(t)u'(t)$ is decreasing on $[T_0, \infty)$. Then

$$\frac{u(t)}{r^{1-q}(t)u'(t)\pi(t)} = \frac{u(T_0) + \int_{T_0}^t u'(s)ds}{r^{q-1}(t)u'(t)\pi(t)}$$

$$= \frac{u(T_0) + \int_{T_0}^t r^{1-q}(s)r^{q-1}(s)u'(s)ds}{r^{q-1}(t)u'(t)\pi(t)}$$

$$\geq \frac{u(T_0) + r^{q-1}(t)u'(t)\int_{T_0}^t r^{1-q}(s)ds}{r^{q-1}(t)u'(t)\pi(t)}$$

$$\geq \frac{\pi(t) - \pi(T_0)}{\pi(t)}$$

for $t \geq T_0$. This implies that

$$\pi^{p-1}(t)w(t) \leq \left(\frac{\pi(t)}{\pi(t) - \pi(T_0)}\right)^{p-1},$$
thus,

(8) \[ \limsup_{t \to \infty} \pi^{n-1}(t) w(t) \leq 1. \]

Let \( w_1(t) = f(t) \) on \([T_0, \infty)\), and let\n
\[ w_2(t) = f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_1(s) ds \quad \text{for } t \geq T_0. \]

Then \( w_2(t) \geq w_1(t) \) and\n
\[ w_2(t) \leq f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w(s) ds \leq w(t) \]

for \( t \geq T_0 \). It follows from (8) that \( \limsup_{t \to \infty} \pi^{n-1}(t) w_2(t) \leq 1 \). Suppose \( n \) is a positive integer and \( w_1, w_2, \ldots, w_n \) are defined such that \( w_1 \leq w_2 \leq \cdots \leq w_n \leq w \) on \([T_0, \infty)\), then\n
\[ \int_{T_0}^\infty r^{1-q}(s) f^{q-1}(s) w_k(s) ds < \infty \]

whenever \( 1 \leq k \leq n \), and\n
\[ w_{k+1}(t) = f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s) ds \]

whenever \( 1 \leq k \leq n - 1 \) and \( t \geq T_0 \). Let \( w_{n+1} \) be given by\n
\[ w_{n+1}(t) = f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_n(s) ds. \]

Now\n
\[ w_n(t) = f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_{n-1}(s) ds \]

\[ \leq f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_n(s) ds \]

\[ \leq f(t) + (p - 1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w(s) ds \]

\[ \leq w(t), \]

this implies that \( w_n(t) \leq w_{n+1}(t) \leq w(t) \) for \( t \geq T_0 \). It is clear from (8) that\n
\[ \int_{T_0}^\infty r^{1-q}(s) f^{q-1}(s) w_{n+1}(s) ds \leq \int_{T_0}^\infty r^{1-q}(s) f^{q-1}(s) w(s) ds < \infty. \]
We now see that there is a sequence \( \{ w_k \}_{k=1}^{\infty} \) satisfying (a), (b), (c), and (9)\n\[ w_k(t) \leq w(t) \]
whenever \( k \geq 1 \) and \( t \geq T_0 \). Now (8) and (9) give (e). From (c) we see that the family \( \{ w_1, w_2, \ldots \} \) is equicontinuous, so (9) says that there is a subsequence \( \{ w_{k_j} \}_{j=1}^{\infty} \) with a locally uniformly limit on \( [T_0, \infty) \). This and (b) say that \( \{ w_k \}_{k=1}^{\infty} \) has a locally uniform limit, say \( w_0 \), on \( [T_0, \infty) \). Clearly, \( w_0 \leq w \), so that\n\[ \int_{T_0}^{\infty} r^{1-q}(s) f^{q-1}(s) w_0(s) ds < \infty. \]

Now, Lebesgue’s Dominated Convergence Theorem yields\n\[ \int_{t}^{\infty} r^{1-q}(s) f^{q-1}(s) w_0(s) ds = \lim_{k \to \infty} \int_{t}^{\infty} r^{1-q}(s) f^{q-1}(s) w_k(s) ds \]
for \( t \geq T_0 \). This implies (d), and (f) is clear from the above discussion, so that the proof is complete.

**Corollary 2.3.** If (E) is nonoscillatory, then\n\[ \limsup_{t \to \infty} \pi^{p-1}(t) \left\{ \int_{t}^{\infty} c(s) ds + (p-1) \int_{t}^{\infty} r^{1-q}(s) \left( \int_{s}^{\infty} c(\tau) d\tau \right)^q ds \right\} \leq 1. \]

**Proof.** As in the proof of Theorem 2.2, we have\n\[ \limsup_{t \to \infty} \pi^{p-1}(t) w_2(t) \leq 1, \]
and\n\[ w_2(t) = f(t) + (p-1) \int_{t}^{\infty} r^{1-q}(s) f^{q-1}(s) w_1(s) ds \]
\[ = f(t) + (p-1) \int_{t}^{\infty} r^{1-q}(s) f^{q}(s) ds, \]
where \( f(t) = \int_{t}^{\infty} c(s) ds \). Hence, the proof is complete.

**Corollary 2.4.** If (E) is nonoscillatory, then\n\[ \int_{t_0}^{\infty} c(s) \exp \left( (p-1) \int_{t_0}^{s} r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty \]
and\n\[ \int_{t_0}^{\infty} r^{1-q}(s) f^{q}(s) \exp \left( (p-1) \int_{t_0}^{s} r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty. \]
Proof. As in the proof of Theorem 2.2, there is a number $T_0 \geq t_0$ and a function $w_0$ on $[T_0, \infty)$ such that

$$w_0(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_0(s)ds,$$

where $f(t) = \int_t^\infty c(s)ds$. This implies that

$$w_0'(t) = -c(t) - (p-1)r^{1-q}(t)f^{q-1}(t)w_0(t).$$

Its solution is

$$w_0(T_0) - \int_{T_0}^t c(s) \exp \left( (p-1) \int_s^T r^{1-q}(\tau)f^{q-1}(\tau)d\tau \right) ds = w_0(t) \exp \left( (p-1) \int_{T_0}^t r^{1-q}(\tau)f^{q-1}(\tau)d\tau \right) > 0.$$

Hence,

$$w_0(T_0) > \int_{T_0}^t c(s) \exp \left( (p-1) \int_s^T r^{1-q}(\tau)f^{q-1}(\tau)d\tau \right) ds.$$

This implies

$$\int_{T_0}^\infty c(s) \exp \left( (p-1) \int_s^T r^{1-q}(\tau)f^{q-1}(\tau)d\tau \right) ds < \infty.$$

Clearly, (13) is equivalent to (10). Let $z$ be given on $[T_0, \infty)$ by

$$z(t) = \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_0(s)ds.$$

Then

$$z'(t) = -r^{1-q}(t)f^{q-1}(t)w_0(t) = -r^{1-q}(t)f^{q}(t) - (p-1)r^{1-q}(t)f^{q-1}(t)z(t),$$

which implies that (11) holds. Hence, the proof is complete.

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