On Riemannian manifolds endowed with a $\mathcal{T}$-parallel almost contact 4-structure

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Abstract. $\mathcal{T}$-parallel almost contact 4-structures on a Riemannian manifold are studied. It is proved that such a manifold is a local Riemannian product of two totally geodesic submanifolds, one of them being a space form. Additional results are obtained when the manifold is endowed with a framed $f$-structure.

1. Introduction

In the last two decades, contact, almost contact, paracontact and almost cosymplectic manifolds carrying $\mathcal{r}$ ($\mathcal{r} > 1$) Reeb vector fields $\xi_\mathcal{r}$ have been studied by a certain number of authors, as for instance: M. Kobayashi [11], A. Bucki [4], S. Tachibana and W. N. Yu [22], K. Yano and M. Kon [25], V. V. Goldberg and R. Rosca [8] and some others.

In the present paper we consider a $(2m + 4)$ dimensional Riemannian manifold carrying 4 structure vector fields $\xi_r$ ($r, s \in \{2m + 1, \ldots, 2m + 4\}$) and with a distinguished vector field $\mathcal{T}$, such that the vertical connection forms define a $\mathcal{T}$-parallel connection and the Reeb vector fields are $\mathcal{T}$-parallel (this structure is called a $\mathcal{T}$-parallel almost contact 4-structure and it will be defined in Definition 3.1). Then we shall prove that such a manifold is a local Riemannian product of two totally geodesic submanifolds, $M = M^\top \times M^\perp$, where $M^\perp$ is a space form tangent to the distribution generated by the Reeb vector fields, and that the vector field $\mathcal{T}$ is closed torse forming (Theorem 3.3).

In section 4 we shall study conformal-type structures induced by a $\mathcal{T}$-parallel almost contact 4-structure. Finally, in section 5 we assume that the manifold under consideration is endowed with a framed $f$-structure, proving that $M^\top$ is a Kählerian submanifold (Theorem 5.2).

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2. Preliminaries

Let \((M, g)\) be a Riemannian \(C^\infty\)-manifold and let \(\nabla\) be the covariant differential operator defined by the metric tensor \(g\). We assume that \(M\) is oriented and \(\nabla\) is the Levi-Civita connection. Let \(\Gamma TM\) be the set of sections of the tangent bundle \(TM\) and \(\flat : TM \to T^* M, X \to X^\flat\), the musical isomorphism defined by \(g\). Next, following a standard notation, we set: \(A^q(M, TM) = \text{Hom}(\Lambda^q TM, TM)\) and notice that elements of \(A^q(M, TM)\) are vector valued \(q\)-forms \((q \leq \dim M)\). Denote by \(d\nabla : A^q(M, TM) \to A^{q+1}(M, TM)\) the exterior covariant derivative operator with respect to \(\nabla\) (it should be noticed that generally \(d\nabla^2 = d\nabla \circ d\nabla \neq 0\), unlike \(d^2 = d \circ d = 0\)). The identity tensor field \(I\) of type \((1,1)\) can be considered as a vector valued 1-form \(I \in A^1(M, TM)\) (and it is also called the soldering form [7]).

We shall remember the following

Definition 2.1. (1) (see [10]) The operator \(d\omega = d + e(\omega)\) acting on \(\Lambda M\) is called the cohomology operator, where \(e(\omega)\) means the exterior product by the closed 1-form \(\omega \in \Lambda^1 M\), i.e., \(d\omega u = du + \omega \wedge u\) for any \(u \in \Lambda M\). One has \(d \circ d\omega = 0\), and if \(d\omega u = 0\), \(u\) is said to be \(d\omega\)-closed. If \(\omega\) is exact, then \(u\) is said to be \(d\omega\)-exact.

(2) (see [18], [16]) Any vector field \(X \in \Gamma TM\) such that: \(d\nabla(\nabla X) = \nabla^2 X = \pi \wedge I \in A^2(M; TM)\) for some 1-form \(\pi\), is called an exterior concurrent vector field and the 1-form \(\pi\), which is called the concurrence form, given by \(\pi = fX^\flat, f \in C^\infty(M)\).

(3) (see [23], [16]) A vector field \(T\) whose covariant differential satisfies \(\nabla T = rI + \alpha \otimes T\); \(r \in C^\infty(M)\) where \(\omega = T^\flat\) is a closed form, is called a closed torse forming.

If \(\Re\) denotes the Ricci tensor of \(\nabla\) and \(X\) an exterior concurrent vector field, one has \(\Re(X, Z) = -(n-1)fg(X, Z), Z \in \Gamma TM, n = \dim M\).

Let \(C\) be any conformal vector field on \(M\) (i.e., the conformal version of Killing’s equations). As is well known, \(C\) satisfies

\[(2.1) \quad L_C g(C, Z) = \rho g(C, Z) \text{ or } g(\nabla_Z C, Z') + g(\nabla_{Z'} C, Z) = \rho g(Z, Z')\]

\((Z, Z' \in \Gamma TM)\) where the conformal scalar \(\rho\) is defined by \(\rho = \frac{2}{n}(\text{div} C)\).

We recall the following basic formulas (see [3])
On Riemannian manifolds endowed with a $T$-parallel \ldots
horizontal component of $Z$ (resp. the vertical component of $Z$). We recall that setting $A, B \in \{1, 2, \ldots, 2m\}$ the connection forms $\vartheta^A_B$, $\vartheta^r_B$ and $\vartheta^s_r$ are called the horizontal, the transversal and the vertical connection forms respectively (see also [21]).

With the above notation, one has the following

**Definition 3.1** ([17], [9]). Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional oriented Riemannian manifold carrying 4 Reeb vector fields $\xi_r$ such that the vertical connection forms verifies $\vartheta^r_s = \langle T, \xi_s \wedge \xi_r \rangle$, where $T$ is a certain vertical vector field. Then, we say that vertical connection forms $\vartheta^r_s$ define on $D^\perp$ a $T$-parallel connection and $T$ is called the generator of the considered $(T.P.)$-connection. Moreover, if the Reeb vector fields are $T$-parallel, i.e., $\nabla_T \xi_r = 0$, then the manifold $M(\xi_r, \eta^r, g)$ is said to be endowed with a $T$-parallel almost contact 4-structure (abr. $T.P.A.C.$ 4-structure).

In the present paper we shall deal with these manifolds.

**Remark 3.2.** If we set $T = \sum t_r \xi_r$; $t_r \in C^\infty(M)$ then the vertical connection forms are expressed by $\vartheta^r_s = t_s \eta^r - t_r \eta^s$. Since the vertical connection forms satisfy $\vartheta^r_s(T) = 0$, then by reference to [13] we may say that $\vartheta^r_s$ are relations of integral invariance for the vector field $T$.

Similarly one may decompose in an unique fashion the soldering form $I$ of $M$ as $I = I^\top + I^\perp$ where $I^\top = \omega^A \otimes e_A$ and $I^\perp = \eta^r \otimes \xi_r$ mean the line element of $D^\top$ and the line element of $D^\perp$ respectively.

We can state

**Theorem 3.3.** Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional Riemannian manifold endowed with a $T$-parallel almost contact 4-structure and let $T$ be the generator vector field of this structure.

For such a manifold the structure covectors $\eta^r (r \in \{2m+1, \ldots, 2m+4\})$ are of class 2 and cohomologically exact, i.e., $d^\omega \eta^r = 0$, where $\omega$ is the dual form of the generator $T$ which enjoys the property to be a closed torse forming and to define a relative infinitesimal conformal transformation of the almost contact structure of $M$.

Any manifold $M$ which carries a $(T.P.A.C.)$ 4-structure may be viewed as the local Riemannian product $M = M^\top \times M^\perp$ such that:

(i) $M^\perp$ is a totally geodesic submanifold of $M$, tangent to the vertical distribution $D^\perp = \{\xi_r\}$ which enjoys the property to be a space form of curvature $-2a$ ($a = \text{const}$).
(ii) $M^\top$ is a totally geodesic submanifold of $M$, tangent to the horizontal distribution $D^\top = \{\xi_r\}^\perp$ of $M$.

**Proof.** Making use of the structure equations of Remark 2.4(2) and taking account of Remark 3.2 one derives:

\[(3.1) \quad \nabla \xi_r = t_r I^\perp - \eta^r \otimes T.\]

Hence if $Z^\perp_1, Z^\perp_2 \in D^\perp_p$ are any vertical vector fields, it quickly follows from (3.1) $\nabla Z^\perp_1 Z^\perp_2 \in D^\perp_p$. This, as is known, proves that $D^\perp_p$ is an autoparallel foliation and that the leaves $M^\perp$ of $D^\perp_p$ are totally geodesic submanifolds of $M$ (in our case, $\dim M^\perp = 4$). Next making use of the structure equations of Remark 2.4(3) one finds

\[(3.2) \quad d\eta^r = \omega \wedge \eta^r\]

where $\omega = T^\flat$ denotes the dual form of the generator vector field $T$.

By reference to [7], equations (3.2) show that all the Reeb covectors $\eta^r$ are exterior recurrent and by a simple argument it follows that the recurrence form $\omega$ is necessarily closed, i.e., $d\omega = 0$. With the help of (3.1) and (3.2) one also derives from $I^\perp = \eta^r \otimes \xi_r$ that $I^\perp$ is exterior covariant closed, i.e., $d\nabla(I^\perp) = 0$ and this is matching the fact that $I^\perp$ is the soldering form of the leaf $M^\perp$. By reference to Proposition 2.5 it is seen by (3.2) that the structure covectors $\eta^r$ are of class 2.

Let now denote by $\varphi = \eta^{2m+1} \wedge \ldots \wedge \eta^{2m+4}$ the simple form which corresponds to $D^\top_p$ (or equivalently the volume element of $M^\perp$). By (3.2) one has at once $d\varphi = 0$ and therefore since one may write $D^\top_p \subset \ker(\varphi) \cap \ker(d\varphi)$ we conclude that the horizontal distribution $D^\top_p$ is also involutive. Then setting $M^\top$ for the $2m$ leaf of $D^\top_p$, it is seen that $\xi_r$ are geodesic normal section for the immersion $\kappa : M^\top \to M$, which is totally geodesic. It follows from the above discussion that the manifold $M$ under consideration is the local product $M = M^\top \times M^\perp$, where $M^\top$ and $M^\perp$ are totally geodesic submanifolds of $M$, tangent to the horizontal distribution $D^\top$ and the vertical distribution $D^\perp$ of $M$ respectively.

Further since the dual form $\omega$ of $T$ is expressed by $\omega = t_r \eta^r$ then by virtue of (3.2) one may set

\[(3.3) \quad dt_r = \lambda \eta^r \quad \Rightarrow \quad d\lambda - \lambda \omega = 0\]

which shows that $\omega$ is an exact form. In consequence of this fact, equations (3.2) may be expressed, using the notation introduced in Definition 2.1(1),
as $d^{-\omega} \eta^r = 0$, thus proving that the structure covectors of $M(\xi^r, \eta^r, g)$ are cohomologically exact.

Taking now the covariant differential of the generator vector field $T$, one derives on behalf of (3.1) and (3.3)

\[(3.4)\] \[\nabla T = (\lambda + 2t) \mathfrak{I}^\perp - \nu \otimes T; \quad 2t = \|T\|^2\]

which shows the significative fact that $T$ is a closed torse forming (def. 2.1(3)). Since this quality implies that $T$ is a gradient vector field, this fact is in accordance with equation (3.3). We also derive from (3.4)

\[(3.5)\] \[dt = \lambda \omega \implies t + \lambda = a = \text{const.}\]

Next operating on (3.1) by the exterior covariant derivative operator $d\nabla$ one quickly derives by (3.2) and (3.4) that one has $d\nabla (\nabla \xi^r) = \nabla^2 \xi^r = 2a \eta^r \wedge \mathfrak{I}^\perp$. The above equations reveal the important fact that all the vectors $\{\xi^r\}$ on $M^\perp$ are exterior concurrent vector fields (see [20]). Then since the conformal scalar $2a$ is constant, we conclude by reference to [16] that the vertical submanifold $M^\perp$ is a space form of curvature $-2a$.

Next by (3.2), (3.3) and (3.5) one derives succesively $\mathcal{L}_T \eta^r = (a + t) \eta^r - t_r \omega$ and $d(\mathcal{L}_T \eta^r) = (2a + \lambda) \omega \wedge \eta^r$. In consequence of the last equation and by reference to [14] we agree to say that the generator vector $T$ defines a relative infinitesimal conformal transformation of the considered almost contact 4-structure, thus finishing the proof.

4. Conformal-type structures induced by a $(T.P.A.C.)$ 4-structure

In the present section we consider on $M^\perp$ the 2-form $\psi$ of rank 2 (if $\Omega \in \Lambda^2 M$, rank $r$ is the smallest integer such that $\Omega^{r+1} = 0$), defined by $\psi = \eta^{2m+1} \wedge \eta^{2m+2} + \eta^{2m+3} \wedge \eta^{2m+4}$. On behalf of (3.2) one quickly derives by exterior differentiation of $\psi$ that $d\psi = 2\omega \wedge \psi \Leftrightarrow d^{-2\omega} \psi = 0$ (the last equality obtained on behalf of Definition 2.1(1)). Therefore following a known definition it is seen that $\psi$ is a conformal symplectic form on $M^\perp$ having $\omega$ (resp. $T$) as covector of Lee (resp. vector field of Lee). In addition in the case under discussion one may say that $\psi$ is a $d^{-2\omega}$-exact form.

It should be noticed that this property is in accordance with the general properties of $T$-parallel connections (see also [14]). If $Y \in \Gamma TM^\perp$ is any vertical vector field, then by reference to [12] we set $^bY = -i_Y \psi$. Do not confuse with the the musical isomorphism $b : \Gamma TM \rightarrow \Gamma TM^*$, which is denoted by $X \rightarrow X^b$. For instance, $\omega = T^\flat$. 


In the case under discussion and in order to simplify we write
\[
\beta = -bT = t_{2m+1}^2t_{3m+2} + t_{2m+3}^2t_{3m+4} - t_{2m+2}^2t_{3m+1} - t_{2m+4}t_{3m+3}\]
and by (3.3) and (3.2) one gets \(d\beta = 2\lambda \psi + \omega \wedge \beta\) by which after a standard calculation one derives \(\mathcal{L}_T \psi = 2(a + t)\psi - \omega \wedge \beta\). Since \(\omega\) is an exact form, then following [1] the above equation shows that \(T\) defines a weak infinitesimal conformal transformation of \(\psi\). Then we obtain \(d(\mathcal{L}_T \psi) = 8a \omega \wedge \psi\). Therefore we may also say that \(T\) defines a relative infinitesimal conformal transformation of \(\psi\).

Consider now the vertical vector field \(C = C^r \xi_r\) and set \(\varrho = bC\).

Then in order that \(C\) be an infinitesimal conformal transformation of \(\psi\), one finds making use of (3.2)

\[
dC^r = C^r \omega. \tag{4.1}\]

This implies \(d\varrho = 2\omega \wedge \varrho \iff d^{-2}\varrho = 0\) and setting \(s = g(C, T)\) one may write \(\mathcal{L}_T \psi = 2s \psi\). In the light of this problem, and making use of (3.1) and (4.1) one derives

\[
\nabla C = sI^\perp + C \wedge T \tag{4.2}\]

which reveals the important fact that \(C\) is a structure conformal vector field having \(2s = \rho\) as conformal scalar (see Definition 2.3). Setting \(\alpha = C^\flat\) one finds by (3.4) and (4.2)

\[
ds = \lambda \alpha + s \omega \tag{4.3}\]

and on the other hand by (3.2) one has

\[
d\alpha = 2\omega \wedge \alpha \iff d^{-2}\omega \alpha = 0. \tag{4.4}\]

Hence one may say that as \(\psi\) the dual form \(\alpha\) of \(C\) is \(d^{-2}\omega\)-exact. It should be noticed that equation (4.4) is in accordance with the general properties of structure conformal vector fields [19] (see also [14], [15]).

By (3.3), (4.3) and (4.4) it is seen that the existence of the structure conformal vector field \(C\) is determined by the exterior differential system \(\Sigma_e\) whose characteristic numbers are \(r = 3, s_0 = 2, s_1 = 1\). Since \(r = s_0 + s_1\) it follows by E. Cartan’s test [5] that \(\Sigma_e\) is involutive and \(C\) is determined by 1 arbitrary function of 2 arguments.

Next since \(\rho = 2s\), it follows at once from (4.3), by duality: grad \(\rho = 2\lambda C + \rho T\). But as it is known div \(Z = tr[\nabla Z]\), \(Z \in \Gamma TM\), and so one gets from (3.4) div \(T = 4a + 2t\) and \(C\) being a conformal vector field one has
\[ \text{div } C = 4 \rho. \] Therefore by the general formulas \( \Delta f = -\text{div}(\text{grad } f), \ f \in C^\infty M, \) a short calculation gives

\[
\Delta \rho = -8a \rho
\]

which shows that \( \rho \) is an \textit{eigenfunction} of \( \Delta \) and has \(-8a \) associated \textit{eigenvalue}. Following a known theorem, it follows that if \( M^\perp \) is compact, then necessarily \( a = -\mu^2 \) (\( \mu = \text{const.} \)), that is, \( M^\perp \) is an elliptic submanifold of \( M \).

On the other hand taking the covariant differential of \( \text{grad } \rho \), then by a standard calculation one infers

\[
\nabla \text{grad } \rho = 4a \rho I^\perp
\]

which reveals that \( \text{grad } \rho \) is \textit{concurrent} vector field on \( M^\perp \) [6] (we recall that concurrency is of conformal nature). Accordingly on behalf of the definition given in [14], we may say in the case under consideration \( C \) has the \textit{divergence conformal property}. It is worth to point out that if \( M^\perp \) is an elliptic submanifold of \( M \) (i.e., \( a = -\mu^2 \)), then following Obata’s theorem [24], \( M^\perp \) is \textit{isometric} to a sphere of radius \( \frac{1}{2} \mu \).

Further since \( M^\perp \) is a space form, then we recall [16] that any vector field on \( M^\perp \) is E.C., with the same conformal scalar \( 2a \). Consequently, if \( \mathcal{R} \) denotes the Ricci tensor of \( \nabla \), one has

\[
\mathcal{R}(C, Z) = -6a g(C, Z), \ Z \in \Gamma TM^\perp.
\]

Then by (4.5), (4.6), (4.7) and making use of Proposition 2.2(3) and carrying out the calculations one derives \( \mathcal{L}_C g(C, Z) = \frac{4}{3} \rho g(C, Z) \). Therefore one may state that the (S.C)-vector field \( C \) defines an infinitesimal conformal transformation of all the functions \( g(C, Z) \) where \( Z \in \Gamma TM^\perp \). It should be noticed that this situation is similar to that of [14]. In addition by (3.1) and (4.2) one finds

\[
[C, \xi_r] = -\frac{\rho}{2} \xi_r
\]

which shows that the structure vector fields \( \xi_r \) admit \textit{infinitesimal transformations} of generator \( C \). Next making use of Orsted’s lemma (Proposition 2.2(1)) it follows

\[
\mathcal{L}_C \eta^r = \rho \eta^r.
\]

Hence making use of a known terminology, it follows that \( C \) defines an \textit{almost contact transformation} of the structure covectors \( \eta^r \).
Finally we denote by $\mathcal{P} = \xi_{2m+1} \wedge \xi_{2m+2} + \xi_{2m+3} \wedge \xi_{2m+4}$ the Poisson bivector \cite{12} associated with the conformal symplectic form $\psi$. Since $\mathcal{P}$ may be expressed as

$$
\mathcal{P} = \eta^{2m+2} \otimes \xi_{2m+1} - \eta^{2m+1} \otimes \xi_{2m+2} \\
+ \eta^{2m+4} \otimes \xi_{2m+3} - \eta^{2m+3} \otimes \xi_{2m+4}
$$

then since the Lie derivative is additive, one gets by (4.8) and (4.9) that $\mathcal{L}_C \mathcal{P} = 0$ which shows that $C$ defines an infinitesimal automorphism of $\mathcal{P}$.

Next operating on the vector valued 1-form $\mathcal{P}$ by the operator $d \nabla$ one derives after two successive computations $d \nabla \mathcal{P} = \omega \wedge \mathcal{P} - 2 \psi \otimes \mathcal{I} - \beta \wedge \mathcal{I} \perp \in \Lambda^2(M, TM)$ ($\beta = -b \mathcal{T}$) and $d \nabla^2 \mathcal{P} = 4a \psi \wedge \mathcal{I} \perp$. Therefore (see Proposition 2.5) the last equality shows that $\mathcal{P}$ is a 2-exterior vector valued 1-form. Moreover, taking into account $\mathcal{L}_T \psi = 2s \psi$ a short calculation gives $\mathcal{L}_C (d \nabla^2 \mathcal{P}) = \frac{4}{3} d \nabla^2 \mathcal{P}$ that is $C$ defines an infinitesimal conformal transformation of $d \nabla^2 \mathcal{P}$.

Then one has the

**Theorem 4.1.** Let $M(\xi_\tau, \eta_\tau, g)$ be a $(2m+4)$-dimensional Riemannian manifold endowed with a $(T,P,A,C)$ 4-structure discussed in Section 2 and having $T$ as generator vector field. Let $M^\perp$ be the space form submanifold of $M$, tangent to the vertical distribution $D^\perp = \{\xi_\tau\}$ of $M$. One has the following properties:

1. $M^\perp$ is equipped with a conformal symplectic structure $\text{CSp}(4, \mathbb{R})$ defined by the form $\psi \in \Lambda^2 M^\perp$ (of rank 2) and such that the covector of Lee corresponding to $\text{CSp}(4, \mathbb{R})$ is the dual form $\omega$ of $T$, that is, $d \psi = 2 \omega \wedge \psi$ and $T$ defines a relative infinitesimal conformal transformation of $\psi$, that is, $d(\mathcal{L}_T \psi) = 8a \omega \wedge \psi$, $(a = \text{const.})$

2. Any vector field $C$ which defines an infinitesimal conformal transformation of $\psi$ is a structure conformal vector field, i.e., $\nabla C = g(T, C) \mathcal{I}^\perp + C \wedge T$ and one has $\mathcal{L}_C \psi = \rho \psi$; $\rho = 2g(T, C)$ and $\mathcal{L}_C g(C, Z) = \frac{4}{3} \rho g(C, Z)$, $Z \in \Gamma TM^\perp$.

3. The conformal scalar $\rho$ ($\mathcal{L}_C g = \rho g$) is an eigenfunction of $\Delta$ and if $M^\perp$ is compact, then $a = -\mu^2$ and $M^\perp$ is isometric to a sphere of radius $\frac{1}{2} \mu$.

4. The Poisson bivector $\mathcal{P}$ associated with $\psi$ is a 2-exterior vector valued 1-form, i.e., $d \nabla^2 \mathcal{P} = 4a \psi \wedge \mathcal{I} \perp$ and $C$ defines an infinitesimal automorphism of $\mathcal{P}$. 
5. Framed $f$-structures

In the present section we assume that the manifold $M(\xi^r, \eta^r, g)$ under consideration is endowed with a framed $f$-structure $\phi$ [27], that is $\phi$ is a tensor field of type $(1,1)$ and rank $2m$ which satisfies:

1. $\phi^3 + \phi = 0$
2. $\phi^2 = -I + \sum \eta^r \otimes \xi^r; \phi \xi^r = 0; \eta^r \circ \phi = 0$
3. $g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z) \eta^r(Z'); Z, Z' \in \Gamma TM$ and the fundamental 2-form $\Omega$ associated with the $f$-structure satisfies:
   (4) $\Omega(Z, Z') = g(\phi Z, Z'); \Omega^m \wedge \varphi \neq 0,$ $\varphi$ being the volume element of $M^\perp,$ i.e., $\varphi = \eta^2m+1 \wedge \eta^2m+2 \wedge \eta^2m+3 \wedge \eta^2m+4.$

Such a manifold $M(\phi, \Omega, \xi^r, \eta^r, g)$ is, as known, defined as framed $f$-manifold.

With respect to the cobasis $O^* = \text{covect}\{\omega^A, \eta^r\}$ the form $\Omega$ is expressed by $\Omega = \sum \omega^a \wedge \omega^{a*}; a \in \{1, \ldots, m\}; a* = a+m$ and the horizontal connection forms $\vartheta^A_B$ satisfies the known Kählerian conditions
   (5.1) $\vartheta^a_B = \vartheta^{a*}_{a*}; \vartheta^a_{b*} = \vartheta^b_{a*}.$

Since on the other hand by (3.1) it is seen that the transversal connection forms $\vartheta^r_A$ vanish, one gets by exterior differentiation $d\Omega = 0.$ Since $\Omega$ is of constant rank and closed it follows that it is a presymplectic form on $M$ and a symplectic form on $M^\top.$ We notice that in this case ker($\Omega$) coincides with the vertical distribution $D^\perp_\perp$ of $M$ which may be also called characteristic distribution of $\Omega.$ In addition by condition (3) of a framed $f$-structure and $\vartheta^r_A = 0$ one has $(\nabla \phi)Z = 0,$ $Z \in \Gamma TM,$ that is $\nabla$ and $\phi$ commute.

Recall now that the torsion tensor field $S$ of an $f$-structure is the vector valued 2-form defined by $S = N_\phi + S^\perp$ where $N_\phi(Z, Z') = [\phi Z, \phi Z'] + \phi^3[Z, Z'] - \phi[Z, \phi Z'] - \phi[\phi Z, Z']$ is the Nijenhuis tensor field, and $S^\perp = 2\sum \omega^a \wedge \omega^{a*}$ is the vertical component of $S.$ By (3.10), (5.6) and $(\nabla \phi)Z = 0$ it is easily seen that $S$ vanishes on $D^\perp.$ In this case, the $f$-structure $(\phi, \xi^r, \eta^r)$ is said to be horizontal-normal (or $D^\perp$-normal) [2].

Consequently, following a definition of A. Bejancu [2] the framed $f$-manifold $M(\phi, \Omega, \xi^r, \eta^r, g)$ under consideration is a framed-CR manifold. On the other hand, taking into account that $\Omega$ is closed, the horizontal submanifold $M^\top$ of $M$ moves to a symplectic submanifold.

It also should be noticed that by (3.2) one may write $S^\perp$ as $S^\perp = 2\omega \wedge I^\perp \Rightarrow d^\perp S^\perp = 0$ that is, $S^\perp$ is a closed vector valued 2-form. We agree with the following
**Definition 5.1.** Let $M$ be a framed $f$-manifold and let $S^\perp$ be the vertical component of its associated torsion tensor. If the covariant differential of $S^\perp$ is closed, i.e., $d^\nabla S^\perp = 0$, we say that $M$ is a *vertical closed framed f-manifold*.

Now since one finds $\mathcal{L}_T \xi_r = [T, \xi_r] = t_r T - (t + a) \xi_r$ then one get at once $\mathcal{L}_T S^\perp = 2\lambda S^\perp$. Accordingly the Lee vector field $T$ defines an infinitesimal conformal transformation of $S^\perp$.

Then we can state the following

**Theorem 5.2.** Let $M(\phi, \Omega, \xi_r, \eta^r, g)$ be a framed $f$-manifold endowed with a $T$-parallel almost contact 4-structure, and let $S^\perp$ be the vertical component of the torsion tensor field $S$ associated with the $f$-structure defined by $\phi$.

Any such $M$ is a framed $f$-CR manifold which is vertical torsion closed, i.e., $d^\nabla S^\perp = 0$, and may be viewed as the local Riemannian product $M = M^\dagger \times M^\perp$ such that:

(i) $M^\dagger$ is a totally geodesic Kählerian submanifold of $M$, tangent to $\{\xi_r\}^\perp$;

(ii) $M^\perp$ is a totally geodesic space form submanifold of $M$, tangent to $\{\xi_r\}$;

(iii) the Lee vector field $T$ of the (T.P.A.C.) 4-structure defines an infinitesimal conformal transformation of $S^\perp$.

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**References**


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