A characterization of the local structure of static stars

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Abstract. It is shown that the existence of a “spherical coweakly-affine static reference frame” in a spacetime gives rise to a 2 by 2 warped product metric tensor describing a static star locally.

1. Introduction

Intuitively, a “static star” refers to a gravitational field generated by a time independent nonrotating source. Throughout this paper, we only will consider the gravitational field exterior to the celestial body of a “static star” with no matter present, yet still call that gravitation a “static star”. In the literature, formal descriptions of “static stars” are made by using asymptotic considerations (see [4, Ch. 9]). Yet the physical observations are made locally and hence, these observations are suited to build a “static star” model.

A local characterization of Schwarzschild and Reissner metrics is studied in [3] by using the (local) concepts of infinitesimal isotropy (or equivalently, null anisotropy) and weak-affinity. From the physical point of view, infinitesimal isotropy refers to “spherical symmetry” and weak-affinity, together with infinitesimal isotropy, refers to nonrotation. Indeed this is the case in Schwarzschild and Reissner metrics.

In the current theory of “static stars”, staticity is expressed by the existence of a static reference frame on some open subset of the spacetime. In this paper, we will make it our starting point in order to reach

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Partially supported by a project DGICYT PB94-0633-C02-01 (Spain).
Supported by a fellowship from TUBITAK-BAYG (Turkey).
back to infinitesimal isotropy. We will formulate a “static star” by only imposing conditions on the static reference frame, an important one is the coweak-affinity of the static reference frame. Then with some “sphericality” assumptions, we will obtain a 2 by 2 warped product decomposition of the metric locally. Yet to reach infinitesimal isotropy, we will make in addition the assumption of symmetry with respect to the stress-energy tensor. Then the spacetime will become the one described in [3], which has a 2 by 2 warped product decomposition locally.

2. Preliminaries

Let \( M \) be a 4-dimensional spacetime. A future directed unit timelike vector field \( Z_1 \) is called a reference frame. Let \( \omega_1(\cdot) = \langle \cdot, Z_1 \rangle \) be the 1-form associated to \( Z_1 \). We call \( Z_1 \) an irrotational (or locally synchronizable) reference frame if \( \omega_1 \wedge d\omega_1 = 0 \). Also it can be shown that if \( Z_1 \) is irrotational, then locally there exist functions \( t \) and \( h > 0 \) such that \( \omega_1 = -h dt \), i.e., \( Z_1 = -g \nabla t \) locally (see [11, pag. 52–59]). A reference frame \( Z_1 \) is called stationary if there exists a function \( f > 0 \) such that \( Z = f Z_1 \) is a Killing vector field. A stationary reference frame \( Z_1 \) is called static if \( Z_1 \) is irrotational (see [11 pag. 219]). A spacetime \( M \) is called static (resp., stationary) if there exists a static (resp., stationary) reference frame on \( M \). In fact, if \( M \) is a static spacetime with static reference frame \( Z_1 \), then \( M \) is locally a warped product \( I_g^2 \times N \), where \( I \subset \mathbb{R} \), \( N \) is a Riemannian manifold, and \( g > 0 \) is a smooth function. Then \( Z_1 = -g \nabla t \) and the Killing field \( Z = f Z_1 = -g^2 \nabla t = \frac{\partial}{\partial t} \) (cf. [8 pag. 360–361]). It follows that the orthogonal vector bundle \( Z_1^\perp \) to \( Z_1 \) is integrable with totally geodesic leaves and \( f = g \).

3. Static spacetimes

Lemma 3.1. If \( Z \) is a timelike Killing vector field on \( M \), then \( \nabla_Z Z = f \nabla f \perp Z \), where \( \langle Z, Z \rangle = -f^2 \).

Proof. First note that, by the Killing identity, \( \langle \nabla_Z Z, Z \rangle = 0 \) and hence \( \nabla_Z Z \perp Z \). Also, for every \( X \in \Gamma TM \),

\[
0 = \langle \nabla_Z Z, X \rangle + \langle Z, \nabla_X Z \rangle = \langle \nabla_Z Z, X \rangle + \frac{1}{2} X \langle Z, Z \rangle
\]

\[
= \langle \nabla_Z Z, X \rangle - \frac{1}{2} X (f^2) = \langle \nabla_Z Z - f \nabla f, X \rangle.
\]

Hence \( \nabla_Z Z = f \nabla f \). \qed
**Definition 3.2.** A stationary reference frame $Z_1$ on $M$ is called proper if $\nabla f \neq 0$ at each $p \in M$, where $Z = f Z_1$ ($f > 0$) is a Killing vector field.

**Remark 3.3.** Note that the properness of $Z_1$ is also used for the global characterization of the static parts of the Schwarzschild and Reissner metrics (see [5] and [6]).

**Definition 3.4.** Let $Z_1$ be a proper stationary reference frame on $M$. The unit acceleration $P_1$ of $Z$ is defined by $P_1 = \frac{\nabla f}{\|\nabla f\|}$. Also the canonical distributions $W_1$ and $W_2$ in $TM$ are defined by $W_1 = \text{span}\{Z_1, P_1\}$ and $W_2 = W_1^\perp$. Furthermore, if $Z_1$ is static then $P_1$ is called the fundamental gyroscope of $Z_1$.

**Remark 3.5.** Indeed, if $Z_1$ is static, then it can be easily shown that $F_{Z_1} P_1 = 0$ by using the fact that $Z_1^\perp$ has totally geodesic leaves, where $F_{Z_1}$ is the Fermi–Walker connection over $Z_1$ (see [11, pag. 50–52]). Thus $P_1$, as being the unit acceleration of $Z_1$, can be named as the fundamental gyroscope of $Z_1$.

**Lemma 3.6.** Let $Z_1$ be a proper stationary reference frame on $M$. Then

1. $[Z, \nabla f] = 0$ and hence $W_1$ is integrable.
2. $[Z, P_1] = 0$ and hence $Z\|\nabla f\| = 0$.

**Proof.** (1) Let $X \perp Z$ be a vector field. Then

$$\langle \nabla_X \nabla f, Z \rangle = -\langle \nabla f, \nabla_X Z \rangle = \langle \nabla \nabla f, X \rangle$$

$$= \langle \nabla_Z \nabla f, X \rangle + \langle [\nabla f, Z], X \rangle = \langle \nabla_X \nabla f, Z \rangle + \langle [\nabla f, Z], X \rangle$$

and hence $[\nabla f, Z] \perp X$. Also,

$$\langle \nabla_Z \nabla f, Z \rangle = -\langle \nabla f, \nabla_Z Z \rangle = \langle \nabla \nabla f, Z \rangle$$

and hence $[Z, \nabla f] \perp Z$. Thus $[Z, \nabla f] = 0$, and it also follows that $W_1$ is integrable.

(2) By (1), $[Z, P_1] \in \Gamma W_1$ and, since

$$\langle \nabla_Z P_1, Z \rangle = -\langle P_1, \nabla_Z Z \rangle = \langle \nabla P_1, Z, Z \rangle$$

and

$$\langle [Z, P_1], P_1 \rangle = \langle \nabla_Z P_1, P_1 \rangle - \langle \nabla P_1, Z, P_1 \rangle = 0,$$

it follows that $[Z, P_1] = 0$ and hence $Z\|\nabla f\| = 0$. □
Lemma 3.7. If $Z_1$ is a proper static reference frame on $M$, then $W_2$ is also integrable.

**Proof.** Recall from the preliminaries that locally $Z = f Z_1 = -g^2 \nabla t$. Hence $\nabla t$ and $\nabla f$ are orthogonal to $W_2$. Then for any $X, Y \in \Gamma W_2$,

$$\langle \nabla t, [X, Y] \rangle = \langle \nabla t, \nabla_X Y \rangle - \langle \nabla t, \nabla_Y X \rangle$$

$$= - \langle \nabla_X \nabla t, Y \rangle + \langle \nabla_Y \nabla t, X \rangle = 0$$

and similarly, $\langle \nabla f, [X, Y] \rangle = 0$. Thus it follows that $[X, Y] \in \Gamma W_2$. □

**Definition 3.8.** Let $X$ be a vector field on $M$. The affinity tensor field of $X$ is defined by

$$(\mathcal{L}_X \nabla)(U, V) = \mathcal{L}_X \nabla_U V - \nabla_U \mathcal{L}_X V - \nabla_{\mathcal{L}_X U} V,$$

where $\mathcal{L}$ is the Lie derivative.

**Remark 3.9.** Note that every Killing vector field is affine.

**Definition 3.10.** A proper static reference frame $Z_1$ on $M$ is called coweakly-affine if $P_1$ is a geodesic vector field (i.e., $\nabla P_1 P_1 = 0$) and

$$\langle (\mathcal{L}_{P_1} \nabla)(U, V), V \rangle = 0$$

for every $U, V \perp P_1$.

**Proposition 3.11.** Let $Z_1$ be a proper static reference frame on $M$. If $P_1$ is a geodesic vector field and $\langle (\mathcal{L}_{P_1} \nabla)(X, Y), Y \rangle = 0$ for every $X, Y \in \Gamma W_2$ then $Z_1$ is a coweakly-affine static reference frame.

**Proof.** First note that, since $\mathcal{L}_{P_1} Z = 0$ and $\nabla_Z Z = f \|\nabla f\| P_1$,

$$\langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Z \rangle = \langle \mathcal{L}_{P_1} \nabla_Z Z, Z \rangle - \langle \nabla_Z \mathcal{L}_{P_1} Z, Z \rangle$$

$$- \langle \nabla_{\mathcal{L}_{P_1} Z} Z, Z \rangle = 0.$$

Also note that since $\mathcal{L}_{P_1} \nabla$ is tensorial, it suffices to check

$$\langle (\mathcal{L}_{P_1} \nabla)(U, V), V \rangle = 0$$

only for $U = Z + X$ and $V = Z + Y$, where $X, Y \in \Gamma W_2$ are Lie parallel vector fields along $P_1$. (Note that since $P_1$ is a geodesic vector field and $W_2$ is integrable, such $X, Y \in W_2$ can always be constructed locally). Then

$$\langle (\mathcal{L}_{P_1} \nabla)(Z + X, Z + Y), Z + Y \rangle$$

$$= \langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Z \rangle + \langle (\mathcal{L}_{P_1} \nabla)(Z, Y), Z \rangle$$

$$+ \langle (\mathcal{L}_{P_1} \nabla)(X, Z), Z \rangle + \langle (\mathcal{L}_{P_1} \nabla)(X, Y), Z \rangle$$

$$+ \langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Y \rangle + \langle (\mathcal{L}_{P_1} \nabla)(Z, Y), Y \rangle$$

$$+ \langle (\mathcal{L}_{P_1} \nabla)(X, Z), Y \rangle + \langle (\mathcal{L}_{P_1} \nabla)(X, Y), Y \rangle.$$
By the assumption, the first and the last terms vanish. Others can easily be shown to vanish. For example, since \( \mathcal{L}_{P_1} Z = \mathcal{L}_{P_1} X = \mathcal{L}_{P_1} Y = 0 \),

\[
\langle (\mathcal{L}_{P_1} \nabla)(X, Z), Y \rangle = \langle \mathcal{L}_{P_1} \nabla_X Z - \nabla_X \mathcal{L}_{P_1} Z - \nabla \mathcal{L}_{P_1} X Z, Y \rangle = \langle \mathcal{L}_{P_1} \nabla_X Z, Y \rangle = 0;
\]

since \( \nabla_X Z = 0 \) by the fact that \( \langle \nabla_X Z, Z \rangle = -\langle \nabla Z, X \rangle = 0 \), \( \langle \nabla_X Z, P_1 \rangle = -\langle Z, \nabla_X P_1 \rangle = 0 \) (since \( Z^+ \) is totally geodesic) and \( \langle \nabla_X Z, Y \rangle = -\langle Z, \nabla_X Y \rangle = \langle \nabla_Y Z, X \rangle = -\langle \nabla_X Z, Y \rangle = 0 \) for every \( Y \in \Gamma W_2 \).

Also,

\[
\langle (\mathcal{L}_{P_1} \nabla)(Z, Y), Y \rangle = \langle \mathcal{L}_{P_1} \nabla_Z Y - \nabla_Z \mathcal{L}_{P_1} Y - \nabla \mathcal{L}_{P_1} Z Y, Y \rangle = \langle \mathcal{L}_{P_1} \nabla_Z Y, Y \rangle + \langle \mathcal{L}_{P_1} \mathcal{L}_Z Y, Y \rangle = \langle \mathcal{L}_{P_1} \mathcal{L}_Z Y, Y \rangle = 0
\]

since \( \mathcal{L}_{P_1} \mathcal{L}_Z Y = [P_1, [Z, Y]] = -[Z, [Y, P_1]] - [Y, [P_1, Z]] = 0 \) by the Jacobi identity. \( \square \)

**Lemma 3.12.** Let \( Z_1 \) be a coweakly-affine static reference frame. Then

\[
\langle \nabla_U \nabla V P_1, V \rangle = \langle \nabla \nabla_U V P_1, V \rangle
\]

for every \( U, V \perp P_1 \).

**Proof.** Let \( R \) be the curvature tensor. Then

\[
R(U, P_1)V = \nabla_U \nabla V P_1 - \nabla V \nabla_U P_1 - \nabla_{[U, P_1]} V
= \nabla_U \nabla V P_1 + \nabla_U \mathcal{L}_{P_1} V - \nabla \nabla_U V P_1 - \mathcal{L}_{P_1} \nabla_U V + \nabla \mathcal{L}_{P_1} U V
= \nabla_U \nabla V P_1 - \nabla \nabla_U V P_1 - (\mathcal{L}_{P_1} \nabla)(U, V).
\]

Hence

\[
0 = \langle R(U, P_1)V, V \rangle = \langle \nabla_U \nabla V P_1, V \rangle - \langle \nabla \nabla_U V P_1, V \rangle - \langle (\mathcal{L}_{P_1} \nabla)(U, V), V \rangle.
\]

Thus \( \langle \nabla_U \nabla V P_1, V \rangle = \langle \nabla \nabla_U V P_1, V \rangle \). \( \square \)

**Definition 3.13.** Let \( Z_1 \) be a coweakly-affine static reference frame on \( M \). Then the shape operator \( L_{P_1} : W_2 \ni W_2 \) of \( W_2 \) is defined by \( L_{P_1} X = -\nabla_X P_1 \). Also the extrinsic curvature function \( \kappa_i : M \mapsto \mathbb{R} \) of \( W_i, (i = 1, 2) \), is defined by

\[
\kappa_i(p) = \frac{\langle R(x_i, y_i)y_i, x_i \rangle}{\langle x_i, x_i \rangle \langle y_i, y_i \rangle - \langle x_i, y_i \rangle^2},
\]

where \( x_i, y_i \in W_{i_p} \) with \( \langle x_i, x_i \rangle \langle y_i, y_i \rangle - \langle x_i, y_i \rangle^2 \neq 0 \).
Remark 3.14. Note that $L_{P_1}$ is well-defined since $\langle \nabla_X P_1, Z \rangle = -\langle P_1, \nabla_X Z \rangle = 0$ and $\langle \nabla_X P_1, P_1 \rangle = \frac{1}{2} X \langle P_1, P_1 \rangle = 0$ for every $X \in W_2$. Also $\kappa_i$ is well-defined since it corresponds to the curvatures of the planes in $W_i$, $i = 1, 2$.

Definition 3.15. Let $Z_1$ be a coweakly-affine static reference frame on $M$. $Z_1$ is called spherical if $\det L_{P_1} \geq 0$ and $\kappa_i > 0$, $i = 1, 2$.

Remark 3.16. Let $Z_1$ be a spherical coweakly affine static reference frame. The sphericality of $Z_1$ can physically be interpreted as the shape of a star must be “extrinsically and intrinsically” spherical and radially attractive rather than repulsive. The “nonrotationality” of the star is essentially described by that $Z_1$ is a coweakly-affine static reference frame. Here, note that every Killing field is an affine vector field. Thus, since $P_1 = \nabla f \| \nabla f \|$ and is weakly-affine, $P_1$ can also be considered as a “weakly-static” vector field. Hence by assuming that $Z_1$ is coweakly-affine static, actually we are imposing some staticity on the distribution $W_1 = \text{span}\{Z_1, P_1\}$. But the “staticity” of $W_1$ should be considered as an “affine staticity”, that is, which does not involve the curvature tensor.

Now we can give a local characterization of a “spherical static star”.

Theorem 3.17. Let $Z_1$ be a spherical coweakly-affine static reference frame on $M$. Then $M$ is locally a warped product $(M_1 \times_{\psi^2} M_2, \langle \cdot, \cdot \rangle_1 \oplus \psi^2 \langle \cdot, \cdot \rangle_2)$, where $(M_1, \langle \cdot, \cdot \rangle_1)$ and $(M_2, \langle \cdot, \cdot \rangle_2)$ are respectively, Lorentzian and Riemannian surfaces with positive curvature. Furthermore if $\nabla \kappa_2 \perp W_2$, then $(M_2, \langle \cdot, \cdot \rangle_2)$ is of constant positive curvature.

Proof. Note that by Lemmas 3.6 and 3.7, $W_1$ and $W_2$ are integrable. First we will show that the integral manifolds of $W_1$ are totally geodesic. Indeed, it suffices to show that $\nabla Z Z \in \Gamma W_2$, $\nabla P_1 P_1 \in \Gamma W_2$ and $\nabla P_1 Z \in \Gamma W_2$. The first two statements follow immediately since $\nabla Z Z = f \| \nabla f \| P_1$ and $\nabla P_1 P_1 = 0$. For the last one, let $X \in \Gamma W_2$, then since $\langle \nabla_{P_1} Z, X \rangle = -\langle \nabla_X Z, P_1 \rangle = \langle Z, \nabla_X P_1 \rangle = 0$ by the Remark 3.14, it follows that $\nabla P_1 Z \in \Gamma W_1$. Next we will show that the integral manifolds of $W_2$ are totally umbilic. Let $X, Y \in \Gamma W_2$ and $\mathcal{I}$ be the 2nd fundamental form tensor of the integral manifolds of $W_2$.

Note that since $\langle \nabla_X Y, Z \rangle = -\langle \nabla_X Z, Y \rangle = 0$ by the Remark 3.14,

$\mathcal{I}(X, Y) = \langle \nabla_X Y, P_1 \rangle P_1 = -\langle Y, \nabla_X P_1 \rangle P_1$. 
Hence, if $T$ and $\perp$ denote the components tangent and orthogonal to $W_2$ respectively, and $\nabla^\perp$ is the normal connection,

\[
(\nabla_X \parallel)(Y, Y) = \nabla^\perp_X (Y, Y) - 2(\langle (\nabla_X Y)^T, Y \rangle \\
= -X\langle Y, \nabla_Y P_1 \rangle + \langle Y, \nabla_Y P_1 \rangle (\nabla_X P_1)^\perp + 2\langle Y, \nabla_2 (\nabla_X Y)^T \rangle P_1 \\
= \langle \nabla_X Y, \nabla_X P_1 \rangle P_1 - \langle Y, \nabla_X \nabla_Y P_1 \rangle P_1 \\
= \langle Y, \nabla_X \nabla_Y P_1 \rangle P_1 - \langle Y, \nabla_X \nabla_Y P_1 \rangle P_1 \\
= 0 \quad \text{by Lemma 3.12.}
\]

Thus, since $(\nabla_X \parallel)$ is symmetric, it follows that $(\nabla_X \parallel) = 0$.

This has two implications: First, from the Codazzi equation (cf. [8, pag. 115]), $R(X, Y)V \in \Gamma W_2$ for every $X, Y, V \in \Gamma W_2$. Second, if $c_i$, $i = 1, 2$ are the eigenvalues of $L_{P_1}$ then $\nabla c_i \perp W_2$, that is, $c_i$ is constant along the integral manifolds of $W_2$. Now we will show that $c_1 = c_2$. Indeed, let $X_1, X_2$ be orthonormal eigenvectors of $L_{P_1}$ corresponding to $c_1$ and $c_2$ respectively. Then, since

\[
\langle \nabla_X \nabla_Y P_1, Y \rangle = \langle \nabla_Y P_1, \nabla_X Y \rangle
\]

and

\[
\langle \nabla_X \nabla Y P_1, Y \rangle = X\langle \nabla_Y P_1, Y \rangle - \langle \nabla_Y P_1, \nabla_X Y \rangle,
\]

Lemma 3.12 can also be written as

\[
X\langle \nabla_Y P_1, Y \rangle = 2\langle \nabla_Y P_1, \nabla_X Y \rangle
\]

for $X, Y \in \Gamma W_2$. Hence, by setting $Y = X_1 - X_2$, since $X\langle \nabla_Y P_1, Y \rangle = X(c_1 + c_2) = 0$, we obtain

\[
0 = 2\langle \nabla_Y P_1, \nabla_Y Y \rangle = 2(c_2 - c_1)\langle X_1, \nabla_X X_2 \rangle
\]

for every $X \in \Gamma W_2$. But if $c_2 \neq c_1$, then $(\nabla_X X_i)^T = 0$ ($i = 1, 2$) for every $X \in \Gamma W_2$ and hence the curvature tensor $R_2$ of the induced Riemannian structure of the integral manifolds of $W_2$ is identically zero. But then from the Gauss equations

\[
0 = \langle R_2(X_1, X_2)X_2, X_1 \rangle = \langle R(X_1, X_2)X_2, X_1 \rangle \\
+ \langle L_{P_1} X_1, X_1 \rangle \langle L_{P_1} X_2, X_2 \rangle = \kappa_2 + c_1 c_2 = \kappa_2 + \text{det } L_{P_1}
\]

in contradiction with the assumption that $Z_1$ is spherical. Thus $c_1 = c_2 = c$ and it follows that $\parallel(X, Y) = \langle X, Y \rangle P$, where $P = cP_1$. Hence the integral
manifolds of $W_2$ are totally umbilic with normal parallel normal curvature vector field $P$. Then it follows from [9, Prop. 3(c)] that $M$ is locally a warped product $M_1 \times \psi^2 M_2$ with $M_1$ of curvature $\kappa_1 > 0$ and $M_2$ is of constant curvature $\kappa_2 + c^2 > 0$. Furthermore, if $\kappa_2$ is constant along the integral submanifolds of $W_2$ then $M_2$ is of constant curvature. □

Remark 3.18. Note that the local warping function $g^2$ in $I_{g^2} \times N$ may be quite different from the warping function $\psi^2$ in $M_1 \times \psi^2 M_2$. For example in the Schwarzschild metric in usual coordinates, $g^2 = (1 - 2m/r)$ and $\psi^2 = r^2$. Also we can introduce Schwarzschild type coordinates for $M_1$. Note that since $[P_1, Z] = 0$, there exists a chart $(t, r)$ such that $Z = \partial/\partial t$ and $P_1 = \partial/\partial r$. Also since $Z f = 0$, where $\langle Z, Z \rangle = -f^2$, $f$ is only a function of $r$. Hence the metric is locally of the form $-f^2(r) dt \otimes dt + dr \otimes dr$ on $M_1$. Furthermore, by [8, Prop. 35, pag. 206], since $\Pi(X, Y) = \langle X, Y \rangle P = -\langle X, Y \rangle (\nabla \psi \nabla \psi)$, $\nabla \psi$ is proportional to $P$ and hence $\langle \nabla \psi, Z \rangle = 0$. Thus $\psi$ is only a function of $r$. That is, the 2 by 2 warped metric above is locally a spherically symmetric static metric [7, pag. 594]

Remark 3.19. Static parts of the Schwarzschild and Reissner metrics are examples to the above theorem which describes the metric of the static part of a “static star”. Yet Schwarzschild and Reissner metrics are in fact, special cases of the above theorem as being infinitesimally isotropic.

Definition 3.20. Let $Z_1$ be a spherical coweakly-affine static reference frame on a spacetime $M$ obeying the Einstein equation for a stress-energy tensor $T$ with $\text{tr} T = 0$. $Z_1$ is called symmetric with respect to $T$ if $T(Z_1, Z_1) = -T(P_1, P_1)$.

Theorem 3.21. Let $M$ be a spacetime obeying the Einstein equation for a stress-energy tensor $T$ with $\text{tr} T = 0$. If $Z_1$ is a spherical coweakly-affine static reference frame on $M$ and is symmetric with respect to $T$, then $M$ is locally a warped product given in Theorem 3.17, and furthermore:

(a) $R(z, x)y = \mu(x, y)z$ for every $z \in W_1$, $x, y \in W_2$ and viceversa, where $\mu < 0$ is a smooth function.

(b) $R(x, y)z = \kappa_i R_0(x, y)z$ for every $x, y, z \in W_i$, $i = 1, 2$, where $R_0(x, y)z = \langle z, y \rangle x - \langle x, z \rangle y$.

(c) $T = \eta(-\langle \cdot , \rangle_1 \oplus \langle \cdot , \rangle_2)$, where $\eta = 2\mu + \kappa_2 = -2\mu - \kappa_1 \geq 0$.

Proof. By Theorem 3.17, $M$ is locally a warped product. Now by using [8, Prop. 7.35, 7.42 and 7.43],
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(1) $R(x, y)z = K_1 R_0(x, y)z$ for every $x, y, z \in W_1$, since $W_1$ is totally geodesic.

(2) $R(x, y)z = (K_2 + (P, P)) R_0(x, y)z - (P, P) R_0(x, y)z = K_2 R_0(x, y)z$
   by the Gauss equations.

(3) Note that since $P_1 = -\nabla \frac{\psi}{c \psi}$ and is a geodesic vector field,
   $\langle \nabla P_1 \nabla \psi, Z_1 \rangle = 0$. Hence $Z_1$ and $P_1$ are the eigenvectors
   of the Hessian tensor $\nabla \nabla \psi$, corresponding to the eigenvalues, say $a$ and $b$, respectively. Then, since

   \[
   Ric(U, V) = \kappa_1 \langle U, V \rangle - \frac{2}{\psi} \langle \nabla_U \nabla \psi, V \rangle
   \]

   for every $U, V \in \Gamma W_1$ and $T = Ric$, it follows that

   \[
   Ric(Z_1, Z_1) = -\kappa_1 + \frac{2}{\psi} a = -Ric(P_1, P_1) = -\kappa_1 + \frac{2}{\psi} b.
   \]

   Thus $a = b = -\psi \mu$ for some function $\mu$ and hence $\nabla_U \nabla \psi = -\psi \mu U$ for every $U \in \Gamma W_1$.

   Then, for any $U \in \Gamma W_1$ and $X, Y \in \Gamma W_2$,

   \[
   R(U, X)Y = -\langle X, Y \rangle \frac{\nabla_U \nabla \psi}{\psi} = \mu \langle X, Y \rangle U
   \]

   and for any $X \in \Gamma W_2$ and $U, V \in \Gamma W_1$

   \[
   R(X, U)V = -\langle \nabla_U \nabla \psi, V \rangle \psi X = \mu \langle U, V \rangle X.
   \]

   Also, since $0 \leq T(Z_1, Z_1) = Ric(Z_1, Z_1) = -\kappa_1 - 2 \mu$ and $\kappa_1 > 0$, it follows that $\mu < 0$. Hence we showed (a) and (b), that is $M$ is infinitesimally isotropic with respect to $TM = W_1 \oplus W_2$. Then it follows from [2, Prop. 4.4] that

   \[
   T = Ric = (\kappa_1 + 2 \mu) \langle , \rangle_1 \oplus (\kappa_2 + 2 \mu) \langle , \rangle_2.
   \]

   But $\text{tr} T = 0$, $\kappa_1 + \kappa_2 + 4 \mu = 0$, and hence $T = \eta(-\langle , \rangle_1 \oplus \langle , \rangle_2)$, where

   $\eta = \kappa_2 + 2 \mu = -\kappa_1 - 2 \mu \geq 0$. \hfill \Box

Remark 3.22. Note that the stress-energy tensor $T$ in the above theorem corresponds to the electromagnetic stress-energy tensor in the Reissner solution, where $\eta$ corresponds to negative Faraday stresses (cf. [1, pag. 124]). Indeed, with no matter present but a radial electric field in the “affinely static”, $W_1 \oplus W_2$ possesses such a stress-energy tensor, (cf. [7, pag. 840]). Also see [5], [6] for a global version of the above theorem by making asymptotic considerations.
Remark 3.23. A spacetime with a curvature tensor as in Theorem 3.21 is called infinitesimally isotropic (equivalently, null anisotropic) with respect to the decomposition $TM = W_1 \oplus W_2$ (see [2] and [3]). In fact, Theorem 3.21 together with Theorem 3.17 gives the same conclusion as [3, Th. 3.11]. But we note that [3, Th. 3.11] is also applicable to non-static parts of a “spherically symmetric static star” to give a warped product $M_1 \times \varphi^2 M_2$ locally. Indeed, although there exists no static reference frame in the “black hole” regions of Scharzschild and Reissner spacetimes, the metric is still a warped product $M_1 \times \varphi^2 M_2$. In other words, [3, Th. 3.11, 3.16] may be considered as a local characterization of “spherically symmetric static stars”.

Acknowledgement. The second author (DNK) is grateful to the Department of Geometry and Topology of the University of Santiago de Compostela for their support and hospitality during the preparation of this paper.

References