The converse of a generalized Hölder inequality

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Abstract. Let \((\Omega, \Sigma, \mu)\) be a measure space with two sets \(A, B \in \Sigma\) such that \(0 < \mu(A) < 1 < \mu(B) < \infty\), and \(k\) a fixed positive integer. Suppose that \(\phi_1, \ldots, \phi_k\), are arbitrary bijections of \((0, \infty)\). The main result says that if
\[
\int_{\Omega} x_1 \cdots x_k d\mu \leq \phi_1^{-1} \left( \int_{\Omega(x_1)} \phi_1 \circ x_1 d\mu \right) \cdots \phi_k^{-1} \left( \int_{\Omega(x_k)} \phi_k \circ x_k d\mu \right)
\]
for all \(\mu\)-integrable nonnegative step functions \(x_1, \ldots, x_k\), then \(\phi_1, \ldots, \phi_k\) must be conjugate power functions (here \(\Omega(x) = \{\omega \in \Omega : x(\omega) \neq 0\}\)).

Introduction

For a measure space \((\Omega, \Sigma, \mu)\) denote by \(S = S(\Omega, \Sigma, \mu)\) the linear space of all \(\mu\)-integrable simple functions \(x : \Omega \to \mathbb{R}\), and by \(S_+ = S_+(\Omega, \Sigma, \mu)\) the set of all nonnegative \(x \in S(\Omega, \Sigma, \mu)\). For an arbitrary bijection \(\phi : (0, \infty) \to (0, \infty)\) the functional \(p_\phi : S \to \mathbb{R}_+ (\mathbb{R}_+ := [0, \infty))\) given by
\[
p_\phi(x) := \begin{cases} 
\phi^{-1} \left( \int_{\Omega(x)} \phi \circ |x| d\mu \right) & \text{if } \mu(\Omega(x)) > 0, \\
0 & \text{if } \mu(\Omega(x)) = 0, 
\end{cases} 
\]
x \in S(\Omega, \Sigma, \mu),

where \(\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}\), is well defined (cf. [3]).

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Note that for $\phi(t) := \phi(1)t^p, \ t > 0$, where $p \in \mathbb{R} \setminus \{0\}$ is arbitrary fixed, we have

$$p_\phi(x) = \left(\int_{\Omega(x)} |x|^p d\mu\right)^{\frac{1}{p}}, \ x \in S(\Omega, \Sigma, \mu), \ \mu(\Omega(x)) > 0.$$ and for $p \geq 1$ the functional $p_\phi$ becomes the $L^p$-norm. Let $k$ be a fixed positive integer, and $\phi_1, \ldots, \phi_k$ bijections of $(0, \infty)$. Suppose that the inequality

$$\int_{\Omega} x_1 \cdot \ldots \cdot x_k \ d\mu \leq p_{\phi_1}(x_1) \cdot \ldots \cdot p_{\phi_k}(x_k), \ x_1, \ldots, x_k \in S_+,$$

holds true. We prove that if $\phi_1, \ldots, \phi_k$ are multiplicatively conjugate, i.e. there is a constant $c > 0$ such that

$$\phi_1^{-1}(t) \phi_2^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t) = ct, \ t > 0,$$

and the measure space $(\Omega, \Sigma, \mu)$ is not trivial, then $\phi_1, \ldots, \phi_k$ must be power functions. The main purpose of this paper is to prove that if there are two sets $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty,$$

then $\phi_1, \ldots, \phi_k$ are multiplicatively conjugate power functions.

These results are the converses of a known generalized Hölder inequality (cf. Hardy–Littlewood–Pólya [1], p. 140, Theorem 188, also p. 21, Theorem 10). An analogous result for $k = 2$, under a little stronger assumptions, has been proved in [6].

The relevant results for the reversed Hölder inequality are also given.

1. Auxiliary results

A crucial role plays the following

**Lemma 1** ([5]). *Let $a$ and $b$ be real numbers such that*

$$0 < \min\{a, b\} < 1 < a + b.$$

*If a function $f : (0, \infty) \to \mathbb{R}_+$ satisfies the inequality*

$$af(s) + bf(t) \leq f(as + bt), \ s, t > 0,$$

*then $f(t) = f(1)t,$ (t > 0).*

For the reversed inequality we have the following
Lemma 2 ([4]). Let $a$ and $b$ be real numbers such that
\[ 0 < \min\{a, b\} < 1 < a + b. \]
If a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded in a neighbourhood of 0, $f(0) = 0$ and
\[ f(as + bt) \leq af(s) + bf(t), \quad s, t \geq 0, \]
then $f(t) = f(1)t$, $(t \geq 0)$.

We need also the following result on a simultaneous system of two functional equations.

Lemma 3 ([2]). Let $a, b, \alpha, \beta$ be positive real numbers and suppose that $h : (0, \infty) \to (0, \infty)$ is continuous at least at one point and satisfies the system of functional equations
\[ h(at) = \alpha h(t), \quad h(bt) = \beta h(t), \quad t > 0. \]
If $a \neq 1$ and $\frac{\log b}{\log a}$ is irrational then there exists a $q \in \mathbb{R}$ such that $h(t) = h(1)t^q$, for all $t > 0$.

2. The converse of generalized Hölder’s inequality for multiplicatively conjugate functions

We start this section with the following

Theorem 1. Let $(\Omega, \Sigma, \mu)$ be a measure space with two disjoint sets $A, B \in \Sigma$ of finite and positive measure, and $k$ a fixed positive integer. If $\phi_1, \ldots, \phi_k : (0, \infty) \to (0, \infty)$ are bijections such that for a positive $c$,
\[ \phi_1^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t) = ct, \quad t > 0, \]
and
\[ \int_{\Omega} x_1 \cdot \ldots \cdot x_k \, d\mu \leq p_{\phi_1}(x_1) \cdot \ldots \cdot p_{\phi_k}(x_k), \quad x_1, \ldots, x_k \text{ in } \mathbb{S}_+, \]
then $\phi_1, \ldots, \phi_k$ are conjugate power functions, i.e. there are $q_1, \ldots, q_k \in \mathbb{R}$, $q_1, \ldots, q_k \geq 1$, such that
\[ \phi_i(t) = \phi_i(1)t^{q_i}, \quad t > 0; \quad i = 1, \ldots, k, \]
and
\[ q_1^{-1} + \ldots + q_k^{-1} = 1. \]
Proof. For $k = 1$ we have $\phi_1^{-1}(t) = ct$, $t > 0$, and the theorem is obvious. A formal proof in the general case $k \geq 2$ requires quite complicated notation. Since the idea of the proof in the general case is exactly the same as in the case $k = 3$, we give the detailed argument for $k = 3$. For the simplicity of notations we put $\phi := \phi_1$, $\psi := \phi_2$, $\gamma := \phi_3$. By $\chi_A$ we denote the characteristic function of a set $A$. Put $a := \mu(A)$, $b := \mu(B)$.

Setting in inequality (2) arbitrary $x$, $y$, $z \in S_+$ of the form:

$$x := x_1\chi_A + x_2\chi_B, \quad y := y_1\chi_A + y_2\chi_B, \quad z := z_1\chi_A + z_2\chi_B,$$

$x_i$, $y_i$, $z_i > 0$,

and making use of the definition of $p_\phi$, we get the inequality

$$ax_1y_1z_1 + bx_2y_2z_2 \leq \phi^{-1}(a\phi(x_1) + b\phi(x_2))\psi^{-1}(a\psi(y_1) + b\psi(y_2))\gamma^{-1}(a\gamma(y_1) + b\gamma(y_2))$$

for all $x_i$, $y_i$, $z_i > 0$. Replacing here $x_i$, $y_i$, and $z_i$, respectively by $\phi^{-1}(x_i)$, $\psi^{-1}(y_i)$, and $\phi^{-1}(z_i)$, $i = 1, 2$, we obtain

$$a\phi^{-1}(x_1)\psi^{-1}(y_1)\gamma^{-1}(z_1) + b\phi^{-1}(x_2)\psi^{-1}(y_2)\gamma^{-1}(z_2) \leq \phi^{-1}(ax_1 + bx_2)\psi^{-1}(ay_1 + by_2)\gamma^{-1}(az_1 + bz_2)$$

(3)

for all $x_1$, $x_2$, $z_1$, $y_1$, $y_2$, $z_2 > 0$. Similarly, setting in (2)

$$x := x_1\chi_A, \quad y := y_1\chi_A, \quad z := z_1\chi_A, \quad x_1$, $y_1$, $z_1 > 0,$

we obtain

$$a\phi^{-1}(x_1)\psi^{-1}(y_1)\gamma^{-1}(z_1) \leq \phi^{-1}(ax_1)\psi^{-1}(ay_1)\gamma^{-1}(az_1),$$

$x_1$, $y_1$, $z_1 > 0$.

From (1) we have

(4) $$\psi^{-1}(t)\gamma^{-1}(t) = \frac{ct}{\phi^{-1}(t)}, \quad t > 0.$$ 

Hence, taking $z_1 := y_1$, we get

$$\frac{\phi^{-1}(ay_1)}{\phi^{-1}(y_1)} \leq \frac{\phi^{-1}(ax_1)}{\phi^{-1}(x_1)}, \quad x_1$, $y_1 > 0.$$

This implies that the function $t \to \frac{\phi^{-1}(t)}{\phi^{-1}(a^{-1}t)}$ is constant in $(0, \infty)$ and, consequently, we have

$$\frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)} = \frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)}, \quad x_1$, $y_1 > 0.$$
In the same way we show that
\[
\frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)} = \frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)}, \quad x_2, y_2 > 0.
\]

From (3) and (4) we obtain
\[
ay_1 \frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + by_2 \frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)} \leq (ay_1 + by_2) \frac{\phi^{-1}(ax_1 + bx_2)}{\phi^{-1}(ay_1 + by_2)}.
\]
Replacing here \(x_1, x_2, y_1, y_2\) resp. by \(a^{-1}x_1, b^{-1}x_2, a^{-1}y_1, b^{-1}y_2\) we get
\[
y_1 \frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)} + y_2 \frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)} \leq (y_1 + y_2) \frac{\phi^{-1}(x_1 + x_2)}{\phi^{-1}(y_1 + y_2)}.
\]
Now from (5) and (6) we obtain the inequality
\[
y_1 \frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + y_2 \frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)} \leq (y_1 + y_2) \frac{\phi^{-1}(x_1 + x_2)}{\phi^{-1}(y_1 + y_2)},
\]
valid for all \(x_1, x_2, y_1, y_2 > 0\). Setting
\[F(t) := \psi^{-1}(t)\gamma^{-1}(t), \quad t > 0,
\]
and making again use of (4) we can write this inequality in the form
\[
\phi^{-1}(x_1)F(y_1) + \phi^{-1}(x_2)F(y_2) \leq \phi^{-1}(x_1 + x_2)F(y_1 + y_2),
\]
\[x_1, x_2, y_1, y_2 > 0.
\]
Now we can prove that \(\phi\) and \(F\) are homeomorphisms in \((0, \infty)\). In view of (1) it is sufficient to show that either \(\phi^{-1}\) or \(F\) is increasing in \((0, \infty)\).
Suppose for instance that \(F\) is not increasing in \((0, \infty)\). Thus \(F(y_1) > F(y_1 + y_2)\) for some positive \(y_1, y_2\) and the last inequality implies that \(\phi^{-1}(x_1) < \phi^{-1}(x_1 + x_2)\) for all \(x_1, x_2 > 0\), i.e. that \(\phi^{-1}\) is increasing in \((0, \infty)\).

From (7), by induction, we obtain
\[
y_1 \frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + \ldots + n \frac{\phi^{-1}(x_n)}{\phi^{-1}(y_n)} \leq (y_1 + \ldots + y_n) \frac{\phi^{-1}(x_1 + \ldots + x_n)}{\phi^{-1}(y_1 + \ldots + y_n)},
\]
for all positive \(x_1, \ldots, x_n; y_1, \ldots, y_n\) and \(n \in \mathbb{N}\). Setting in this inequality \(x_1 = \ldots = x_n := s; y_1 = \ldots = y_n := t\), we get
\[
\frac{\phi^{-1}(nt)}{\phi^{-1}(t)} \leq \frac{\phi^{-1}(ns)}{\phi^{-1}(s)}, \quad s, t > 0; n \in \mathbb{N}.
\]
It follows that for every \( n \in \mathbb{N} \) the function \( t \rightarrow \frac{\phi^{-1}(nt)}{\phi^{-1}(t)} \), \( t > 0 \), is constant. Hence for every \( n \in \mathbb{N} \) there is \( \alpha_n > 0 \) such that

\[
\phi^{-1}(nt) = \alpha_n \phi^{-1}(t), \quad t > 0.
\]

Taking \( n = 2 \) and \( n = 3 \) we see that \( h := \phi^{-1} \) satisfies the system of functional equations

\[
h(2t) = \alpha h(t), \quad h(3t) = \beta h(t), \quad t > 0,
\]

where \( \alpha := \alpha_2, \beta := \alpha_3 \). Since \( h \) is continuous and \( \log 3/\log 2 \) is irrational, Lemma 3 implies that there is a \( q_1 \in \mathbb{R} \) such that

\[
\phi^{-1}(nt) = \alpha \phi^{-1}(t), \quad t > 0.
\]

By the monotonicity of \( \phi^{-1} \) we have \( q_1 > 0 \).

In the same way one can show that \( \phi^{-1}_i(t) = \phi^{-1}_i(1)t^{1/q_i} \) (\( t > 0 \)) for some \( q_i > 0 \), and \( i = 2, \ldots, k \). By (1) we have

\[
q_1^{-1} + \ldots + q_k^{-1} = 1,
\]

and consequently, \( q_i > 1, \ i = 1, \ldots, k \). This completes the proof. \( \square \)

**Remark 1.** Note that carrying out the argument for arbitrary \( k \in \mathbb{N}, \ k \geq 3 \), we can define the function \( F \) as follows

\[
F(t) := \phi^{-1}_2(t) \cdot \ldots \cdot \phi^{-1}_k(t), \quad t > 0.
\]

Similarly, applying Lemma 2, we can prove

**Theorem 2.** Let \((\Omega, \Sigma, \mu)\) be a measure space with two disjoint sets of finite and positive measure. If \( \phi_1, \ldots, \phi_k : (0, \infty) \rightarrow (0, \infty) \) are bijections such that for some positive \( c \):

\[
\phi^{-1}_1(t) \cdot \ldots \cdot \phi^{-1}_k(t) = ct, \quad t > 0,
\]

and

\[
P_{\phi_1}(x_1) \cdot \ldots \cdot P_{\phi_k}(x_k) \leq \int_{\Omega} x_1 \cdot \ldots \cdot x_k \, d\mu, \quad x_1, \ldots, x_k \in S_+,
\]

then \( \phi_1, \ldots, \phi_k \) are conjugate power functions i.e. there are \( q_1, \ldots, q_k \in \mathbb{R} \setminus \{0\} \) such that

\[
\phi_i(t) = \phi_i(1)t^{q_i}, \quad t > 0; \ i = 1, \ldots, k,
\]

and

\[
q_1^{-1} + \ldots + q_k^{-1} = 1.
\]
3. The main theorem

The main goal of this paper is to prove the following

**Theorem 3.** Suppose that \((\Omega, \Sigma, \mu)\) is a measure space with two sets \(A, B \in \Sigma\) such that

\[
0 < \mu(A) < 1 < \mu(B) < \infty.
\]

If \(\phi_1, \ldots, \phi_k : (0, \infty) \to (0, \infty)\) are arbitrary bijections such that

\[
\int_{\Omega} x_1 \cdot \ldots \cdot x_k \, d\mu \leq p_{\phi_1}(x_1) \cdot \ldots \cdot p_{\phi_k}(x_k), \quad x_1, \ldots, x_k \in S_+,
\]

then \(\phi_1, \ldots, \phi_k\) are conjugate power functions i.e. there are \(q_1, \ldots, q_k \in \mathbb{R}\), \(q_1, \ldots, q_k \geq 1\), such that

\[
\phi_i(t) = \phi_i(1)t^{q_i}, \quad t > 0; \ i = 1, \ldots, k,
\]

and

\[
q_1^{-1} + \ldots + q_k^{-1} = 1.
\]

**Proof.** Define \(f : (0, \infty) \to (0, \infty)\) by

\[
f(t) := \phi_1^{-1}(t)\phi_2^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t), \quad t > 0,
\]

and put \(a := \mu(A)\) and \(b := \mu(B \setminus A)\). Setting in the assumed inequality \(x_i := s_i \chi_A + t_i \chi_{B \setminus A} \in S_+\ (i = 1, \ldots, k)\), we obtain,

\[
a \phi_1^{-1}(s_1)\phi_2^{-1}(s_2) \cdot \ldots \cdot \phi_k^{-1}(s_k) + b \phi_1^{-1}(t_1)\phi_2^{-1}(t_2) \cdot \ldots \cdot \phi_k^{-1}(t_k) \leq \phi_1^{-1}(as_1 + bt_1)\phi_2^{-1}(as_2 + bt_2) \cdot \ldots \cdot \phi_k^{-1}(as_k + bt_k)
\]

for all positive \(s_1, \ldots, s_k; t_1, \ldots, t_k\). Taking here \(s_1 = s_2 = \ldots = s_k := s; t_1 = t_2 = \ldots = t_k := t\) gives

\[
a f(s) + b f(t) \leq f(as + bt), \quad s, \ t > 0.
\]

Since \(0 < a < 1 < a + b\) it follows by Lemma 1 that \(f(t) = f(1)t, (t > 0)\). Thus, by the definition of \(f\), the functions \(\phi_i, i = 1, \ldots, k\), are multiplicatively conjugate and our result is a consequence of Theorem 1.

**Remark 2.** In Theorem 3 (as well as in Theorem 1), if \(k \geq 2\) then \(q_i > 1\) for all \(i = 1, \ldots, k\). If \(k = 1\) then \(q_1 = 1\), and the basic Hölder inequality (2) becomes an equality.
Remark 3. Theorem 3 generalizes the main result of a paper [6] where the case $k = 2$ is considered, and the functions $\phi_1, \phi_2$ are assumed to be bijections of $\mathbb{R}_+ = [0, \infty)$.

Remark 4. If we assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, and $\phi(0) = 0$, then the definition of the functional $p_\phi : S_+ \to \mathbb{R}_+$ simplifies to the following formula

$$p_\phi(x) := \phi^{-1} \left( \int_{\Omega} \phi \circ x \, d\mu \right), \quad x \in S_+.$$  

Using this remark, and applying Lemma 2 and Theorem 2, we can prove

**Theorem 4.** Suppose that $(\Omega, \Sigma, \mu)$ is a measure space with two sets $A, B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$. If $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are bijections such that $\phi_i(0) = 0$, $i = 1, \ldots, k$, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$f(t) := \phi_1^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t), \quad t \geq 0,$$

is bounded in a neighbourhood of 0 and

$$p_{\phi_1}(x_1) \cdot \ldots \cdot p_{\phi_k}(x_k) \leq \int_{\Omega} x_1 \cdot \ldots \cdot x_k \, d\mu, \quad x_1, \ldots, x_k \in S_+,$$

then $\phi_1, \ldots, \phi_k$ are conjugate power functions, i.e. there are $q_1, \ldots, q_k \in \mathbb{R} \setminus \{0\}$, such that

$$\phi_i(t) = \phi_i(1)t^{q_i}, \quad t \geq 0; \quad i = 1, \ldots, k,$$

and

$$q_1^{-1} + \ldots + q_k^{-1} = 1.$$

Remark 5. In Theorem 4 (and Theorem 2), if $k \geq 2$ then at least one of the numbers $q_i$, $i = 1, \ldots, k$, is negative, and the relevant power function $\phi_i$ is decreasing in $(0, \infty)$. If $k = 1$ then $q_1 = 1$, and the assumed reversed Hölder inequality becomes an equality.
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References