On power values of binary forms over function fields

By JÁNOS VÉGSŐ (Debrecen)

Abstract. An inequality is given for the height to the “rational” solutions $x, y, z$ of the equation $F(x, y) = z^m$.

1. Introduction

Let $K$ be an algebraic function field and $F(X, Y) \in K[X, Y]$ be a binary form of degree $n$ having pairwise non-proportional linear factors in its splitting field. The equation

$F(x, y) = z^m$ in $x, y, z \in K$; $m \in \mathbb{Z}$

is a natural, common generalization of the superelliptic, Thue and Fermat equations. The first general theorem was obtained by SCHMIDT [S1], he gives an effective upper bound for the degree of the polynomial solutions. His result was extended by BRINDZA [B1], [B2] to the case of algebraic function fields when the unknowns are $S$-integers. By taking $z = 1$ or $y = 1$ we have Thue or superelliptic equations, respectively. For the related results we refer to [S2], [M1], [M2], [M3], [BM], [P] and [BPV].

The purpose of this note is to prove a bit surprisingly simple inequality which covers several previous results. Let $H_K(\alpha)$ denote the additive height of a non-zero $\alpha \in K$, that is

$H_K(\alpha) = \sum_v \max(0, v(\alpha))$,

where $v$ runs through the additive valuations of $K$ with value group $\mathbb{Z}$. 

Theorem. If the degree of $F$ is at least 5 then all the non-zero solutions of (1) in $x, y, z \in \mathbb{K}, m \in \mathbb{Z}$ satisfy

$$H_{\mathbb{K}}\left(\frac{x}{y}\right) \leq 10H_{\mathbb{K}}(z) + C,$$

where $C$ is an effectively computable constant depending only on $F$ and $\mathbb{K}$.

If $z = 1$ then the equation (1) can be written as

$$y^n = \frac{1}{F\left(\frac{x}{y}, 1\right)}$$

and the known properties of the additive height (see [M1] and [S2]) lead to a bound for $\max(H_{\mathbb{K}}(x), H_{\mathbb{K}}(y))$. Moreover, if $x$ and $y$ are relatively prime integral functions, that is the corresponding integral divisors represented by $x$ and $y$ are relatively prime, then

$$H_{\mathbb{K}}\left(\frac{x}{y}\right) \geq \max(H_{\mathbb{K}}(x), H_{\mathbb{K}}(y))$$

and one can derive an effective upper bound to the exponent $m$, provided that $z$ is not a constant.

Proof of the Theorem. In the sequel $c_1, \ldots, c_7$ will denote effectively computable constants depending only on $F$ and $\mathbb{K}$. In the splitting field $\mathbb{L}$ of $F$ we get the factorization

$$F(X, Y) = a \prod_{i=1}^{n}(X - \alpha_i Y), a \neq 0, \alpha_i \neq \alpha_j.$$  

Let $S$ denote the set of additive valuations of $\mathbb{L}$ including all the infinite valuations such that $v(z) \cdot v(a) \neq 0$ and $v(\alpha_i - \alpha_j) \neq 0$ for some $i, j$. Then

$$|S| \leq 2H_{\mathbb{L}}(z) + c_1 = 2[\mathbb{L} : \mathbb{K}]H_{\mathbb{K}}(z) + c_1.$$  

To show that

$$n|v\left(\frac{x - \alpha_i y}{x - \alpha_j y}\right); v \notin S$$

we suppose that there exists an $i$ for which $v(x - \alpha_i y) \neq 0$ and $v(z) = 0$. It implies the existence of an index $j$ with $v(x - \alpha_j y) \neq 0$ and $v(x - \alpha_i y) \cdot v(x - \alpha_j y) < 0$.  

Without loss of generality one can assume that
\[ v(x - \alpha_i y) < 0 \text{ and } v(x - \alpha_j y) > 0. \]
Combining the Siegel-identity with the property of valuations we obtain
\[ v(x - \alpha_k y) = v(x - \alpha_i y); k \neq j, \]
and it yields
\[ v(x - \alpha_j y) = (n - 1) \cdot |v(x - \alpha_i y)|, \]
\[ v \left( \frac{x - \alpha_i y}{x - \alpha_j y} \right) = n \cdot v(x - \alpha_i y). \]

Put \( u_1 = \frac{x - \alpha_1 y}{x - \alpha_2 y}, \) \( u_2 = \frac{x - \alpha_3 y}{x - \alpha_2 y}. \) Let \( k_i^+ \) and \( k_i^- \) denote the cardinality of the set of the valuations \( v \notin S \) satisfying \( v(u_i) > 0 \) and \( v(u_i) < 0, \) respectively, \( i = 1, 2. \) Applying the well-known inequality of Mason [M1] and the sum-formula we have
\[ k_i^+ n \leq H_L(u_1) < S + k_i^+ + k_i^- + k_2^+ + k_2^- + c_2, \]
\[ k_i^- n \leq H_L(u_1) < S + k_i^+ + k_i^- + k_2^+ + k_2^- + c_2, \]
\[ k_i^+ n \leq H_L(u_2) < S + k_i^+ + k_i^- + k_2^+ + k_2^- + c_3, \]
\[ k_i^- n \leq H_L(u_2) < S + k_i^+ + k_i^- + k_2^+ + k_2^- + c_3, \]
\[ n(k_i^+ + k_i^- + k_2^+ + k_2^-) \leq 4S + 4(k_i^+ + k_i^- + k_2^+ + k_2^-) + c_4. \]
A simple calculation leads to
\[ H_L(u_1) \leq 5S + c_5 = 10[\mathbb{L} : \mathbb{K}]H_\mathbb{K}(z) + c_6 \]
and
\[ [\mathbb{L} : \mathbb{K}]H_\mathbb{K} \left( \frac{x}{y} \right) = H_L \left( \frac{x}{y} \right) \leq 10[\mathbb{L} : \mathbb{K}]H_\mathbb{K}(z) + c_7. \]
and the theorem is proved.

References


JÁNOS VÉGSŐ
KOSSUTH LAJOS UNIVERSITY
H-4010 DEBRECEN, P.O.BOX 18
HUNGARY

(Received April 4, 1996)