On ideals and extensions of near-rings

By STEFAN VELDSMAN (Port Elizabeth)

Abstract. Given a chain of ideals $J \triangleleft I \triangleleft N$ in a near-ring $N$, we consider necessary and sufficient conditions on $J$, $I$, $N$, $N/I$ and $I/J$ respectively to ensure that $J \triangleleft N$.

§1. Introduction

Near-rings considered will be right near-rings; the variety of all near-rings will be denoted by $\mathcal{V}$ and the subvariety of 0-symmetric near-rings will be denoted by $\mathcal{V}_0$. To facilitate discussions, the conditions mentioned in the Abstract will be formulated in terms of a subclass $\mathcal{M}$ of $\mathcal{V}$.

The class $\mathcal{M}$ satisfies condition
(F) If $J \triangleleft I \triangleleft N$ and $I/J \in \mathcal{M}$, then $J \triangleleft N$
(G) If $J \triangleleft I \triangleleft N$ and $J \in \mathcal{M}$, then $J \triangleleft N$
(H) If $J \triangleleft I \triangleleft N$ and $I \in \mathcal{M}$, then $J \triangleleft N$
(K) If $J \triangleleft I \triangleleft N$ and $N \in \mathcal{M}$, then $J \triangleleft N$
(L) If $J \triangleleft I \triangleleft N$ and $N/I \in \mathcal{M}$, then $J \triangleleft N$.

It is our purpose here to describe the near-rings in $\mathcal{M}$ for each of the above five cases. For rings, this has been done, see [8] or SANDS [3]. For the purpose of comparison, we recall:

In the variety of rings, a subclass $\mathcal{M}$ satisfies condition:
(F) if and only if the rings in $\mathcal{M}$ are quasi semiprime, i.e. if $R \in \mathcal{M}$ and $xR = 0$ or $Rx = 0$ ($x \in R$), then $x = 0$ (or equivalently, $R$ has zero middle annihilator, i.e. $RxR = 0$ ($x \in R$) implies $x = 0$).
(G) if and only if $R^2 = R$ for all $R \in \mathcal{M}$.
(H) if and only if every ideal $I$ of $R \in \mathcal{M}$ is invariant under all double homothetisms of $R$ (cf RÉDEI [2]).
(K) if and only if for all $R \in \mathcal{M}$ and for all $a \in R$, $(a) = (a)^2 + Za$ where $(a)$ is the ideal in $R$ generated by $a$ and $Z$ is the integers.

Mathematics Subject Classification: 16A76.
If $N$ is a near-ring, then $N^+$ will denote the underlying group. For nearrings $N_i$, $i = 1, 2, \ldots, k$ and subsets $U_i \subseteq N_i$, $(U_1, U_2, \ldots, U_k)$ denotes the subset $\{(u_1, u_2, \ldots, u_k) \mid u_i \in U_i\}$ of $N_1 \times N_2 \times \ldots \times N_k$.

§2. On condition (F)

In the variety of 0-symmetric near-rings, this problem has been settled in [9]: A class of near-rings $M$ in $\mathcal{V}_0$ satisfies condition (F) if and only if every near-ring in $M$ is quasi semi-equiprime. A near-ring $N$ is quasi semi-equiprime if $xN = 0$ ($x \in N$) implies $x = 0$ and whenever $\theta : I \to N$ is a surjective homomorphism with $I \triangleleft A$ then $x - y \in \ker\theta (x, y \in I)$, implies $ax - ay \in \ker\theta$ for all $a \in A$.

In the variety of all near-rings, a complete description is still outstanding. The construction in [7] shows that $M$ does not contain any non zero constant near-rings. In fact, it is conjectured that $M = \{0\}$; the strongest motivation for this coming from the example in [6] which shows that any class which contains the two element field cannot satisfy condition (F).

§3. On condition (G)

This case can quickly be disposed of using a construction given in [5] which resembles one given by BETSCH and KAARLI [1]. Let $N$ be a near-ring and let $K$ be the near-ring with $K^+ = N^+ \oplus N^+ \oplus N^+$ and with multiplication

$$(a, b, c)(x, y, z) = \begin{cases} (b, 0, cz) & \text{if } y \text{ and } z \text{ are non-zero} \\ (0, 0, cz) & \text{otherwise.} \end{cases}$$

Apart from verifying the associativity of the multiplication as well as the right distributivity over the addition, it is straightforward to see that $N \cong (0, 0, N) \triangleleft (N, 0, N) \triangleleft K$, $(0, 0, N) \triangleleft K$ if and only if $N = 0$ and $K$ is 0-symmetric if and only if $N$ is 0-symmetric.

**Theorem 3.1.** In either one of $\mathcal{V}$ or $\mathcal{V}_0$, a subclass $M$ satisfies condition (G) if and only if $M = \{0\}$.

**Proof.** Suppose $\mathcal{V}$ satisfies condition $G$ and let $N \in \mathcal{V}$. Then $N \cong (0, 0, N) \triangleleft (N, 0, N) \triangleleft K$; hence $(0, 0, N) \triangleleft K$, which yields $N = 0$. 

\(\text{(L) if and only if } M = \{0\}\)
§4. On condition (L)

As in the variety of rings, a subclass \( \mathcal{M} \) of \( \mathcal{V} \) or \( \mathcal{V}_0 \) satisfies condition (L) if and only if \( \mathcal{M} = \{0\} \). To verify this for near-rings, we need two constructions:

4.1 Let \( N \) be a near-ring and let \( G \) be the group \( G = N^+ \oplus U \) where \( U \) is any non-zero group. As is well-known, \( N \) can be identified with a subnear-ring of \( M(G) = \{ f \mid f : G \rightarrow G \text{ a function} \} \) via \( \theta : N \rightarrow M(G) \),

\[
\theta(n) = \theta_n : G \rightarrow G, \quad \theta_n(g) = \begin{cases} ng & \text{if } g \in N \\ n & \text{if } g \in G \setminus N. \end{cases}
\]

Let \( K_1 \) be the near-ring with \( K_1^+ = N^+ \oplus M(G)^+ \) and multiplication

\[
(n, g)(m, h) = (nm, nh).
\]

Let \( X = \{ f \in M(G) \mid f(U) \subseteq U \} \). Then \((0, X) \triangleleft (0, M(G)) \triangleleft K_1 \) and \( K_1/(0, M(G)) \cong N \). Note that \((0, X)\) is a right ideal in \( K_1 \). It is a left ideal of \( K_1 \) if and only if \( N \) is constant. Indeed, \((0, X)\) is a left ideal of \( K_1 \) if and only if \( n(g + h) - ng \in X \) for all \( n \in N \), \( g \in M(G) \) and \( h \in X \). Let \( g \) and \( h \) be the functions defined by

\[
g(x) = \begin{cases} x & \text{if } x \in N \\ 0 & \text{if } x \in G \setminus N \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 0 & \text{if } x \in N \\ x & \text{if } x \in G \setminus N. \end{cases}
\]

Clearly \( h \in X \); thus if \((0, X) \triangleleft K_1 \), then \( n(g + h) - ng \in X \). Thus, for

\[
0 \neq u \in U, \quad n(g(u) + h(u)) - ng(u) \in U,
\]

i.e. \( n(u) - n(0) \in U \)

i.e. \( n - n0 \in U \cap N = \{0\} \); hence \( n = n0 \).

Conversely, if \( N \) is constant, then \((0, X) \triangleleft K_1 \). If \( N \) is 0-symmetric, and
we replace \( M(G) \) above with \( M_0(G) \), then everything above stays valid
except in this case \((0, X) \triangleleft K_1 \) if and only if \( N = 0 \).

4.2 Let \( N \) be a near-ring and let \( G \) be any group which properly contains \( N^+ \). We regard \( N \) as a subnear-ring of \( M(G) \). Let \( K_2 \) be the near-ring with \( K_2^+ = N^+ \oplus M(G)^+ \) and multiplication \((n, f)(m, g) = (nm, fm)\).

Then \((0, M_0(G)) \triangleleft (0, M(G)) \triangleleft K_2 \) and \( K_2/(0, M(G)) \cong N \). Furthermore, \((0, M_0(G))\) is a left ideal in \( K_2 \), but it is an ideal if and only if \( N \) is 0-symmetric. Indeed, if it is a right ideal, then \((0, 1_G)(n, 0) \in (0, M_0(G))\) which gives \( n0 = 0 \). The converse is obvious.

**Theorem 4.3.** In either one of \( \mathcal{V} \) or \( \mathcal{V}_0 \), a subclass \( \mathcal{M} \) satisfies condition (L) if and only if \( \mathcal{M} = \{0\} \).

**Proof.** Firstly, if \( \mathcal{M} \) is a subclass in \( \mathcal{V}_0 \) which satisfies condition (L) and \( N \in \mathcal{M} \), we have from the construction in 4.1 that \( N = 0 \). If \( \mathcal{M} \) is a
subclass in \( \mathcal{V} \) which satisfies condition \((L)\) and \( N \in \mathcal{M} \), the construction in 4.1 shows that \( N \) is constant and the construction in 4.2 shows that \( N \) is 0-symmetric. Hence \( N = 0 \).

§5. On condition \((K)\)

Principal ideals in a near-ring, contrary to the ring case, have no nice finite description. This seems to be the most serious obstacle in describing the near-rings in a class \( \mathcal{M} \) which satisfies condition \((K)\). For a near-ring \( N \) and \( a \in S \subseteq N \), \( S \) a subnear-ring of \( N \), the ideal in \( S \) generated by \( a \) will be denoted by \( \langle a, S \rangle \).

**Theorem 5.1.** Let \( \mathcal{M} \) be a class of near-rings. Then \( \mathcal{M} \) satisfies condition \((K)\) if and only if \( \langle a, N \rangle = \langle a, \langle a, N \rangle \rangle \) for all \( a \in N \), \( N \in \mathcal{V} \).

**Proof.** Firstly, if \( a \in N \in \mathcal{M} \) and \( \mathcal{M} \) satisfies condition \((K)\), then \( \langle a, \langle a, N \rangle \rangle \triangleleft N \). Hence \( \langle a, N \rangle = \langle a, \langle a, N \rangle \rangle \). Conversely, if the condition is satisfied, choose \( J \triangleleft I \triangleleft N \in \mathcal{M} \) and let \( j \in J \), \( n, m \in N \). Then \( \langle j, N \rangle \cap J \subseteq \langle j, N \rangle \subseteq I \); hence

\[
\langle j, N \rangle = \langle j, \langle j, N \rangle \rangle \subseteq \langle j, N \rangle \cap J \subseteq J.
\]

Thus \( n + j - n, jn \) and \( n(m + j) - nm \in J \) which yields \( J \triangleleft N \).

§6. On condition \((H)\)

We start with finding the near-ring analogue of the Schreier group extensions or the Everett ring extensions (cf Rédei [2]): Given near-rings \( A \) and \( B \) determine all near-rings \( N \) such that \( N \) is an extension of \( A \) by \( B \), i.e. \( A \triangleleft N \) and \( N/A = B \). This problem has earlier been settled for composition rings and 0–symmetric near-rings by Steinegger [4]. Strictly speaking, it involves finding all triples \((\zeta, N, \eta)\) where

\[
0 \rightarrow A \xrightarrow{\zeta} N \xrightarrow{\eta} B \rightarrow 0
\]

is a short exact sequence. Two extensions \((\zeta, N, \eta)\) and \((\zeta', N', \eta')\) of \( A \) by \( B \) are equivalent if there exists an isomorphism \( \chi : N \rightarrow N' \) such that the diagram
commutes. $\chi$ is called an *equivalence isomorphism*. In order to simplify notation and discussions, our exposition will not be as rigorous as required above; instead, we identify $A$ with $\zeta(A)$ and $N/\zeta(A)$ with $B$. In such a case, it means $\chi(a) = a$ for all $a \in A$ and $\chi(n) + A = n + A$ for all $n \in N$.

The elements of $A$ and $B$ will always be taken as $A = \{a, b, c, \ldots\}$ and $B = \{\alpha, \beta, \gamma, \ldots\}$ respectively with 0 denoting the additive identity of both $A$ and $B$.

Consider a quintuple of functions $(F, [-,-], G, H, \langle -,- \rangle)$ with $F : B \to M(A)$, $[-,-] : B \times B \to A$, $G : A \times B \to M(A)$, $H : B \times B \to M(A)$ and $\langle -,- \rangle : B \times B \to A$ which satisfy the *initial conditions*

$$F(0) = 1_A$$

and

$$G(b, \beta) \in M_0(A), G(b, 0)(a) = ab$$

and

$$H(0, \beta) = 0; H(\alpha, \beta) \in M_0(A)$$

$$\langle 0, \beta \rangle = 0.$$ 

On the cartesian product $A \times B$ define two operations by

$$(a, \alpha) + (b, \beta) = (a + F(\alpha)(b) + [\alpha, \beta], \alpha + \beta)$$

and

$$(a, \alpha)(b, \beta) = (G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle, \alpha\beta).$$

We say these two operations are the operations induced by the function quintuple. $A \times B$ together with these two operations is called an *$E$-sum of $A$ and $B$ w.r.t. the quintuple $(F, [-,-], G, H, \langle -,- \rangle)$* and is denoted by $A\sharp B$. The two functions $[-,-]$ and $\langle -,- \rangle$ are called the *additive and multiplicative factor systems* of the $E$-sum respectively.

Although $A$ is (as a near-ring) isomorphic to $(A, 0)$ (via $a \to (a, 0)$), in general $A\sharp B$ may have no particular structure w.r.t. the induced operations. A function quintuple $(F, [-,-], G, H, \langle -,- \rangle)$ is called an *amicable system for $A$ w.r.t. $B$ if it satisfies the following conditions for all $a, b, c \in A$ and $\alpha, \beta, \gamma \in B$:

(E1) $F(\alpha) \in \text{End}(A^+)$

(E2) $[\alpha, \beta] + F(\alpha + \beta)(c) = F(\alpha)(F(\beta)(c)) + [\alpha, \beta]$

(E3) $[\alpha, \beta] + [\alpha + \beta, \gamma] = F(\alpha)([\alpha, \beta]) + [\alpha, \beta + \gamma]$

(E4) $G(\alpha, \gamma) \in \text{End}(A^+)$

(E5) $G(c, \gamma)(F(\alpha)(b)) + G(c, \gamma)([\alpha, \beta]) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle$

$= H(\alpha, \gamma)(c) + \langle \alpha, \gamma \rangle + F(\alpha \gamma)(G(c, \gamma)(b)) + F(\alpha \gamma)(H(\beta, \gamma)(c))$

$+ F(\alpha \gamma)(\langle \beta, \gamma \rangle) + [\alpha \gamma, \beta \gamma]$. 
(E6) \[ G(c, \gamma) \left( H(\alpha, \beta)(b) \right) + G(c, \gamma) \left( \langle \alpha, \beta \rangle \right) + H(\alpha \beta, \gamma)(c) + \langle \alpha \beta, \gamma \rangle \]
\[ = H(\alpha, \beta \gamma)(G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta \gamma \rangle \]

(E7) \[ G(c, \gamma) \circ G(b, \beta) = G \left( G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta \gamma \right). \]

**Theorem 6.1.** Let \( A^\sharp B \) be an E-sum of \( A \) and \( B \) w.r.t. a function quintuple \((F, [-, -], G, H, \langle - , - \rangle)\). Then \( A^\sharp B \) is a near-ring if and only if the function quintuple is an amicable system for \( A \) w.r.t. \( B \). In such a case, \( A^\sharp B \) is an extension of \( A \) by \( B \).

**Proof.** Assume \( A^\sharp B \) is a near-ring. Since the addition is associative, it follows that
\[ F(\alpha)(\beta) + [\alpha, \beta] + F(\alpha + \beta)(c) + [\alpha + \beta, \gamma] \]
\[ = F(\alpha) \left( b + F(\beta)(c) + [\beta, \gamma] \right) + [\alpha, \beta + \gamma] \]
From this equality and the initial conditions, (E1), (E2) and (E3) are obtained by putting \( \beta = \gamma = 0 \), \( b = \gamma = 0 \) and \( b = c = 0 \) respectively. The right distributivity of the multiplication over the addition, using (E1), gives
\[ G(c, \gamma) \left( a + F(\alpha)(b) + [\alpha, \beta] \right) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle \]
\[ = G(c, \gamma)(a) + H(\alpha, \gamma)(c) + \langle \alpha, \gamma \rangle + F(\alpha \gamma) \left( G(c, \gamma)(b) \right) \]
\[ + F(\alpha \gamma) \left( H(\beta, \gamma)(c) \right) + F(\alpha \gamma)(\langle \beta, \gamma \rangle) + [\alpha \gamma, \beta \gamma] \]
Substituting \( \alpha = \beta = 0 \) yields (E4). Then using this in (2) gives (E5). Using (E4), the associativity of the multiplication gives
\[ G(c, \gamma) \left( G(b, \beta)(a) \right) + G(c, \gamma) \left( H(\alpha, \beta)(b) \right) + G(c, \gamma) \left( \langle \alpha, \beta \rangle \right) \]
\[ + H(\alpha \beta, \gamma)(c) + \langle \alpha \beta, \gamma \rangle \]
\[ = G \left( G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta \gamma \right) (a) \]
\[ + H(\alpha, \beta \gamma)(G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta \gamma \rangle \]
Substituting \( a = 0 \) and \( \alpha = 0 \) in this equality, (E6) and (E7) respectively are obtained.

Conversely, if the function quintuple is an amicable system, we verify that \( A^\sharp B \) is a near-ring. The associativity of the addition will follow if the equality in (1) holds. Using (E1), (E3) and then (E2), the right hand side of (1) becomes:
\[ F(\alpha) \left( b + F(\beta)(c) + [\beta, \gamma] \right) + [\alpha, \beta + \gamma] \]
\[ = F(\alpha)(b) + F(\alpha) \left( F(\beta)(c) \right) + F(\alpha) \left( [\beta, \gamma] \right) + [\alpha, \beta + \gamma] \]
\[ = F(\alpha)(b) + F(\alpha) \left( F(\beta)(c) \right) + [\alpha, \beta] + [\alpha + \beta, \gamma] \]
\[ = F(\alpha)(b) + [\alpha, \beta] + F(\alpha + \beta)(c) + [\alpha + \beta, \gamma]. \]
It can be verified that \((0, 0)\) is the additive identity and every element \((a, \alpha)\) has an additive inverse \(-\alpha = (\alpha, -\alpha)\) for all \(\alpha \in B\). Hence \(A \# B\) is a group. For the right distributivity, we need the equality in (2). Using (E4), the left hand side becomes
\[
G(c, \gamma)(a) + G(c, \gamma)(F(a)(b)) + G(c, \gamma)([\alpha, \beta]) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle
\]
which equals the right hand side in view of (E5).

Finally, for the associativity, we require the equality in (3). Using (E7) and (E6), the right hand side becomes
\[
G(c, \gamma)(G(b, \beta)(a)) + G(c, \gamma)(H(\alpha, \beta)(b)) + G(c, \gamma)(\langle \alpha, \beta \rangle)
+ H(\alpha \beta, \gamma)(c) + \langle \alpha, \beta, \gamma \rangle = G(G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta \gamma)(a)
+ H(\alpha, \beta \gamma)(G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta \gamma \rangle.
\]
Thus \(A \# B\) is a near-ring. If we identify \(A\) with \((A, 0)\) and \(B\) with \{(0, \alpha) + (A, 0) \mid \alpha \in B\}\), we have \(A \# A \# B \# A = B\).

**Remark.** Conditions (E1), (E2) and (E3), which are equivalent to \(A \# B\) being a group under the induced addition, implies \(F(\alpha) \in \text{Aut}(A^+)\) for all \(\alpha \in B\). Indeed, by (E1) we only have to verify that \(F(\alpha)\) is bijective. If \(F(\alpha)(a) = F(\alpha)(b)\), then
\[
(0, \alpha) + (a, 0) = (F(\alpha)(a), \alpha) = (F(\alpha)(b), \alpha) = (0, \alpha) + (b, 0).
\]
Hence \((a, 0) = (b, 0)\) which yields the injectivity. Substituting \(\beta = 0\) in (E2) gives \(F(\alpha)(c) = F(\alpha)(F(0)(c))\); hence \(c = F(0)(c)\) which yields \(F(0) = 1_A\). Using (E3) with \(\beta = -\alpha\) and \(\gamma = \alpha\), gives \([\alpha, -\alpha] = F(\alpha)([-\alpha, \alpha])\). Thus, for any \(c \in A\), using (E2) with \(\beta = -\alpha\) and (E1) yield \(F(\alpha)([-\alpha, \alpha]) + c = F(\alpha)(F(-\alpha)(c)) + F(\alpha)([-\alpha, \alpha])\), i.e.
\[
c = F(\alpha)(-[\alpha, \alpha] + F(-\alpha)(c) + [-\alpha, \alpha])
\]
which shows that \(F(\alpha)\) is surjective.
All extensions of \(A\) by \(B\) are, up to equivalence, an \(E\)-sum of \(A\) and \(B\) for a suitable amicable system. This is our next result.

**Theorem 6.2.** Let \(A\) and \(B\) be near-rings and let \(N\) be an extension of \(A\) by \(B\). Then \(N\) is equivalent to an \(E\)-sum \(A \# B\) for some amicable system \((F, [-, -], G, H, \langle -, - \rangle)\).

**Proof.** Each \(\alpha \in B = N/A\) is a subset; let \(f\) be a choice function with \(f(\alpha) \in \alpha\) and \(f(0) = 0\). Every element \(n \in N\) can uniquely be expressed as \(n = a + f(\alpha)\) for some \(a \in A, \alpha \in B\). Define a function
\( \phi : N \to A \times B \) by \( \phi \left( a + f(\alpha) \right) = (a, \alpha) \). Then \( \phi \) is an injection. Define a quintuple of functions by:

\[
F : B \to M(A), \quad F(\alpha)(a) = f(\alpha) + a - f(\alpha) \\
[-, -] : B \times B \to A, \quad [\alpha, \beta] = f(\alpha) + f(\beta) - f(\alpha + \beta) \\
G : A \times B \to A, \quad G(b, \beta)(a) = a \left( b + f(\beta) \right) \\
H : B \times B \to M(A) \quad \text{by} \quad H(\alpha, \beta)(a) = f(\alpha)[a + f(\beta)] - f(\alpha)f(\beta) \\
\langle -, - \rangle : B \times B \to A \quad \text{by} \quad \langle \alpha, \beta \rangle = f(\alpha)f(\beta) - f(\alpha\beta).
\]

These functions are all well-defined; for example, we verify it for \( H \): Since \( f(\alpha) \in \alpha \) and \( f(\beta) \in \beta \), \( f(\alpha) = n_1 + a_1 \) and \( f(\beta) = n_2 + a_2 \) for suitable \( n_1, n_2 \in N, a_1, a_2 \in A \). Then

\[
f(\alpha) \left[ a + f(\beta) \right] - f(\alpha)f(\beta) = (n_1 + a_2) \left[ a + (n_1 + a_2) \right] - (n_1 + a_1)(n_2 + a_2)
\]

which is in \( A \) since \( A \triangleleft N \). The quintuple \( (F, [-, -], G, H, \langle -, - \rangle) \) satisfies the initial conditions since \( f(0) = 0 \). In addition, we will show that they form an amicable system for \( A \) w.r.t. \( B \). But this will follow if we can show that \( A \triangleright B \) is a near-ring respect to the addition and multiplication induced by this function quintuple. To this effect, it is sufficient to show that \( \phi \) preserves addition and multiplication. Firstly we note that

\[
( a + f(\alpha) ) + ( b + f(\beta) ) = a + f(\alpha) + b - f(\alpha) + f(\alpha) + f(\beta) \\
- f(\alpha + \beta) + f(\alpha + \beta) = a + F(\alpha)(b) + [\alpha, \beta] + f(\alpha + \beta).
\]

The first three terms are in \( A \); hence it is the unique expression of \( ( a + f(\alpha) ) + ( b + f(\beta) ) \in N \) in the form \( c + f(\gamma) \). Thus

\[
\phi \left( ( a + f(\alpha) ) + ( b + f(\beta) ) \right) = ( a + F(\alpha)(b) + [\alpha, \beta], \alpha + \beta ) \\
= (a, \alpha) + (b, \beta).
\]

Likewise,

\[
( a + f(\alpha) )\left( b + f(\beta) \right) \\
= a \left( b + f(\beta) \right) + f(\alpha) \left( b + f(\beta) \right) - f(\alpha)f(\beta) \\
+ f(\alpha)f(\beta) - f(\alpha\beta) + f(\alpha\beta) \\
= G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle + f(\alpha\beta)
\]

and the first three are terms in \( A \). Hence

\[
\phi \left( ( a + f(\alpha) )\left( b + f(\beta) \right) \right) = ( G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle, \alpha\beta ) \\
= (a, \alpha)(b, \beta).
\]

Hence \( \phi \) is a near-ring isomorphism. In fact, it is an equivalence isomorphism: If \( a \in A \), then \( \phi(a) = \phi(a + 0) = (a, 0) \). As usual, we identify
{n + A | n ∈ N} with \( A^*_B/(A, 0) = \{ (0, \alpha) + (A, 0) | \alpha \in B \} \) via \( n = a + f(\alpha) \) for some unique \( a \in A, \alpha \in B \). Then

\[
\phi(n) + (A, 0) = (a, \alpha) + (A, 0) = (0, \alpha) + (A, 0) = n + A.
\]

Remark. For any two near-rings \( A \) and \( B \) there always exists at least one \( E \)-sum \( A^*_B \) with an amicable system of functions, namely \( F(\alpha) = 1_A, [\alpha, \beta] = 0 = (\alpha, \beta), G(c, \gamma)(a) = ac \) and \( H(\alpha, \beta) = 0 \). This is nothing but the direct sum \( A \oplus B \) of the near-rings \( A \) and \( B \).

An amicable system \((F, [−, −], G, H, (−, −))\) for \( A \) w.r.t. \( B \) is called a factor-free amicable system if \([\alpha, \beta] = 0 = (\alpha, \beta)\). In such a case, it will be denoted by \((F, G, H)\) and the initial conditions and the conditions \((E1)\) to \((E7)\) simplify to:

\[
F: B \to \text{Aut}(A^+) \text{ is a group homomorphism (i.e. } F(\alpha + \beta) = F(\alpha) \circ F(\beta)).
\]

\[
G: A \times B \to \text{End}(A^+) \text{ and } H: B \times B \to M_0(A) \text{ are functions with } H(0, \alpha) = 0 \text{ and}
\]

\[
(F1) \quad G(c, \gamma) \left( F(\alpha)(b) \right) + H(\alpha + \beta, \gamma)(c) = H(\alpha, \gamma)(c) + F(\alpha \gamma) \left( G(c, \gamma)(b) \right) + F(\alpha \gamma) \left( H(\beta, \gamma)(c) \right)
\]

\[
(F2) \quad G(c, \gamma) \left( H(\alpha, \beta)(b) \right) + H(\alpha \beta, \gamma)(c) = H(\alpha, \beta \gamma) \left( G(c, \gamma)(b) + H(\beta, \gamma)(c) \right)
\]

\[
(F3) \quad G(c, \gamma) \circ G(b, \beta) = G \left( G(c, \gamma)(b) + H(\beta, \gamma)(c) \right) \beta \gamma.
\]

((F1), (F2) and (F3) follows from \((E5)\), \((E6)\) and \((E7)\) respectively.) A triple \((f, g, h)\), where \( f, g, h \in M_0(A) \), is called a triple homothetism of \( A \) if \( f = F(\alpha) \), \( g = G(b, \beta) \) and \( h = H(\beta, \alpha) \) for some \( b \in A, \alpha, \beta \in B \) where \((F, G, H)\) is a factorfree amicable system for \( A \) with respect to some \( B \). If \( I \triangleleft A \), then \( I \) is invariant under the triple homothetism \((f, g, h)\) if \( f(I) = I, g(I) \subseteq I \text{ and } h(I) \subseteq I \).

Theorem 6.3. Let \( \mathcal{M} \) be a class of near-rings. Then \( \mathcal{M} \) satisfies condition \((H)\) if and only every ideal \( I \triangleleft A \) for \( A \in \mathcal{M} \) is invariant under every triple homothetism of \( A \).

Proof. Let \( I \triangleleft A \in \mathcal{M} \), \( \mathcal{M} \) satisfies condition \((H)\), and let \((f, g, h)\) be a triple homothetism of \( A \). By definition, there is a near-ring \( B \) and a factorfree amicable system \((F, G, H)\) for \( A \) w.r.t. \( B \) such that \( f = F(\alpha_0), g = G(b_0, \beta_0) \) and \( h = H(\beta_0, \alpha_0) \). The \( E \)-sum \( A^*_B \) is a near-ring, w.r.t. the operations induced by \( F, G \) and \( H \), and \( (I, 0) \triangleleft (A, 0) \triangleleft A^*_B \). By condition \((H)\), \((I, 0) \triangleleft A^*_B \); hence \((0, \alpha) + (i, 0) = (0, \alpha) \in (I, 0)\). This means \( F(\alpha)(i) \in I \) and in particular \( f(i) = F(\alpha_0)(i) \in I \). Thus \( f(I) \subseteq I \).
For the reverse inclusion, since $F(\alpha)(i) \in I$ for all $\alpha$, we have $i = F(\alpha_0)(F(-\alpha_0)(i)) \in F(\alpha_0)(I) = f(I)$; hence $f(I) = I$. Furthermore, $(i, 0)(b, \beta) \in (I, 0)$; hence $g(b, \beta)(i) \in I$ and in particular, $g(i) = G(b_0, \beta_0)(i) \in I$. Thus $g(I) \subseteq I$. Lastly, from $(0, \beta)[(0, \alpha) + (i, 0)] - (0, \beta)(0, \alpha) \in (I, 0)$ we have $H(\beta, \alpha)(F(\alpha)(i)) \in I$. In particular for $\beta = \beta_0$ and $\alpha = \alpha_0$ and since the restriction of $F(\alpha_0)$ to $I$ is an automorphism of $I$, $h(i) = H(\beta_0, \alpha_0)(F(\alpha_0)(j)) \in I$ for a suitable $j \in I$. Thus $h(I) \subseteq I$.

Conversely, suppose the ideals of $A \in \mathcal{M}$ are invariant under triple homothetism of $A$. Consider $I \triangleleft A \triangleleft B$. Define functions $F : B \to \text{Aut}(A^+)$, $G : A \times B \to \text{End}(A^+)$ and $H : B \times B \to M_0(A)$ by $F(\alpha)(a) = \alpha + a - \alpha$, $G(a, \beta)(c) = c(a + \beta)$ and $H(\alpha, \beta)(a) = \alpha(a + \beta) - \alpha\beta$. Then $(F, G, H)$ constitutes a factorfree amicable system for $A$ w.r.t. $B$. Since $I$ is invariant under triple homothetisms and $(F(\alpha), G(b, \beta), H(\beta, \alpha))$ is a triple homothetism for all $b \in A$, $\alpha, \beta \in B$, it follows that $I \triangleleft B$.

In conclusion we may mention that in the ring case, it is possible to express a double homothetism (in this case, $F(\alpha) = 1_A$ for all $\alpha$ and is thus omitted in the triple $(f, g, h)$) only in terms of $A$ and the conditions (E1) to (E7) simplify considerably.

References


Stefan Veldsman
Dept. of Mathematics
University of Port Elizabeth
Po Box 1600
Port Elizabeth
6000 South Africa

(Received June 11, 1990)