On locally monomial functions

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Abstract. In the present paper the equation
\[ \Delta_n^nf(x) - n!f(y) = o(y^\alpha) \quad ((x, y) \to (0, 0), \ x \leq 0 \leq x + ny), \]
for real functions, where \( n \) is a natural number and \( \alpha \) a non-negative real number, is considered.

1. Introduction

The subject of this paper is related to the study of real polynomial and monomial functions with the aid of the Dinghas interval-derivative and the operator \( \tilde{D} \) defined below. In the sequel, in the Introduction we assume that \( f \) is a real function.

For real numbers \( x, y \) write
\[ \Delta_1^yf(x) = f(x + y) - f(x) \]
and, for \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \),
\[ \Delta_n^{n+1}f(x) = \Delta_1^y(\Delta_n^yf(x)). \]

For a non-negative integer \( n \) we say that \( f \) is a polynomial function of degree \( n \) if \( \Delta_n^{n+1}f(x) = 0 \) for all \( x, y \in \mathbb{R} \); \( f \) is called a monomial function.
of degree \( n \in \mathbb{N} \) if \( \Delta^nf(x) = n!f(y) \), \((x, y \in \mathbb{R})\). A monomial function of degree 1 is considered as an additive function, as well. (For polynomial and monomial functions we refer to [10].)

If, for a positive integer \( n \) and for a real number \( \xi \), the limit

\[
D^n f(\xi) := \lim_{(x,y)\to(\xi,0)} \frac{\Delta^nf(x)}{y^n}, \quad x \leq \xi \leq x + ny
\]

exists, then \( D^n f(\xi) \) is said to be the \( n \)th Dinghas interval-derivative of \( f \) at \( \xi \) (cf. [1]). We consider, furthermore, the operator

\[
\tilde{D}^n f(\xi) := \lim_{(x,y)\to(\xi,0)} \frac{\Delta^nf(x) - n!f(y)}{y^n},
\]

as far as it exists.

Polynomial and monomial functions can be characterized by the operators above: A. Simon and P. Volkmann proved in [6] that for a non-negative integer \( n \), a function is a polynomial function of degree \( n \) if and only if its \((n + 1)\)th Dinghas derivative is zero at all \( \xi \in \mathbb{R} \). It was shown in [2] that for a positive integer \( n \), a function \( f \) is a monomial function of degree \( n \) if and only if \( \tilde{D}^n f(\xi) = 0 \) for all \( \xi \in \mathbb{R} \). It was also proved in [2] that for \( n \in \mathbb{N} \), the property \( \tilde{D}^n f(0) = 0 \) implies \( f(ly) - l^n f(y) = o(y^n), \quad (y \searrow 0) \) for any integer \( l \).

The investigation of the local properties of the operators \( D \) and \( \tilde{D} \) are motivated by the result mentioned above. The following two problems in this field are due to P. Volkmann: given \( n \in \mathbb{N} \), does the property \( D^{n+1} f(0) = 0 \) imply that there exists a polynomial function \( p : \mathbb{R} \to \mathbb{R} \) of degree \( n \) such that \( f(z) - p(z) = o(z^n), \quad (z \to 0) \); and similarly does \( \tilde{D}^n f(0) = 0 \) imply that there exists a monomial function \( g : \mathbb{R} \to \mathbb{R} \) of degree \( n \) such that \( f(z) - g(z) = o(z^n), \quad (z \to 0) \)? A. Simon and P. Volkmann in [7] gave a positive answer to the first question in the case when \( n = 1 \). Furthermore, they proved the following more general theorem: for an arbitrary non-negative real number \( \alpha \neq 1 \) if

\[
\lim_{(x,y)\to(0,0)} \frac{\Delta^2 f(x)}{y^n}, \quad x \leq 0 \leq x + 2y
\]

is equal to 0.
then there exists a polynomial function \( p : \mathbb{R} \rightarrow \mathbb{R} \) of degree 1 such that \( f(z) - p(z) = o(|z|^{\alpha}) \), \((z \to 0)\).

Surprisingly, the answer to the question related to the operator \( \tilde{D}^nf(0) \) is negative. A counterexample is given by \( F : (-1,1) \rightarrow \mathbb{R}, F(x) = x\ln(\ln|x|) \) for \( x \neq 0, F(0) = 0 \). (See [3] and [7].) In the present paper the relation

\[
\lim_{(x,y) \to (0,0)} \frac{\Delta^nf(x) - n!f(y)}{y^\alpha} = 0,
\]

or in other words

\[
(1) \quad \Delta^nf(x) - n!f(y) = o(y^\alpha) \quad ((x,y) \to (0,0), x \leq 0 \leq x + ny)
\]

is studied (it is strongly related to some results in [7]), and a function \( f \) satisfying (1) is called a locally monomial function of degree \( n \) with order \( \alpha \), at 0.

In the second part of the paper we show that if, for \( n \in \mathbb{N}, \alpha \in \mathbb{R}, \alpha > n \), a function \( f \) is a locally monomial function of degree \( n \) with order \( \alpha \), at 0, then there exists a monomial function \( g : \mathbb{R} \rightarrow \mathbb{R} \) of degree \( n \) such that

\[
(2) \quad f(x) - g(x) = o(|x|^\alpha) \quad (x \to 0).
\]

For some similar results on monomial functions of degree 1 and 2 we refer to [8] and [9].

In the third part of the paper we prove that if \( f \) is a locally monomial function of degree 1 with order \( \alpha \) (i.e. a locally additive function with order \( \alpha \)), at 0, then even for \( 0 \leq \alpha < 1 \) there exists a monomial function \( g : \mathbb{R} \rightarrow \mathbb{R} \) of degree 1 (i.e. an additive function), such that (2) holds.

The results in the paper lead to the conjecture that for an arbitrary \( n \in \mathbb{N}, \alpha \geq 0, \alpha \neq n \) if the function \( f \) satisfies (1) then there exists a monomial function of degree \( n \) with property (2), but it may occur that exactly when \( \alpha = n \) (i.e. in the case of the operator \( \tilde{D} \)) there exists no such monomial function.
2. Locally monomial functions of degree $n$ with order $\alpha > n$

Lemma 1. For $n, \lambda \in \mathbb{N}$, $\lambda \geq 2$ put

$$A = \begin{pmatrix} \alpha_0^{(0)} & \ldots & \alpha_0^{(\lambda n)} \\
& \ddots & \vdots \\
\alpha_{(\lambda - 1)n}^{(0)} & \ldots & \alpha_{(\lambda - 1)n}^{(\lambda n)} \end{pmatrix},$$

where for $i = 0, \ldots, (\lambda - 1)n$ and $k = -i, \ldots, \lambda n - i$

$$\alpha_i^{(i+k)} = \begin{cases} (-1)^k \binom{n}{n-k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Let $a_i$ denote the $i$th row in $A$, $(i = 0, \ldots, (\lambda - 1)n)$. Furthermore, let $b = (\beta^{(0)} \ldots \beta^{(\lambda n)})$, where

$$\beta^{(k)} = \begin{cases} (-1)^{\frac{k}{\lambda}} \binom{n}{n-k}, & \text{if } \lambda | k \\ 0, & \text{if } \lambda \nmid k \end{cases}$$

for $k = 0, \ldots, \lambda n$.

There are positive integers $K_0, \ldots, K_{(\lambda - 1)n}$ such that

(3) \hspace{1cm} K_0 a_0 + \ldots + K_{(\lambda - 1)n} a_{(\lambda - 1)n} = b,

and

(4) \hspace{1cm} K_0 + \ldots + K_{(\lambda - 1)n} = \lambda^n.

PROOF. It is trivial that the lemma holds for $n = 1$, $\lambda \geq 2$, $\lambda \in \mathbb{N}$ with $K_0 = \ldots = K_{\lambda - 1} = 1$.

For $n, \lambda \geq 2$, $n, \lambda \in \mathbb{N}$ the existence of positive integers satisfying (3) was proved in Lemma 2 in [3]. The numbers $K_0, \ldots, K_{(\lambda - 1)n}$ satisfy

$$(1 + x + \ldots + x^{\lambda - 1})^n = K_0 + K_1 x + \ldots + K_{(\lambda - 1)n} x^{(\lambda - 1)n} \quad (x \in \mathbb{R}),$$

therefore, substituting $x = 1$ we get (4).
Theorem 1. Let $\alpha \geq 0$ be a real, $n$ be an arbitrary natural number and $f$ be a real function with property (1). Then we have

$$f(lz) - l^n f(z) = o(|z|^\alpha) \quad (z \to 0).$$

for any integer $l$.

Proof. In the special case $\alpha = n$ Theorem 1 was proved in [2]. The proof, given here, is similar, with some technical simplifications.

Let $\alpha \geq 0$ and $n \in \mathbb{N}$ be given numbers and let $f : \mathbb{R} \to \mathbb{R}$ satisfy (1). We show relation (5) in two steps.

I. At first we prove, by induction on $l$, that (1) implies

$$f(lz) = l^n f(z) + o(z^\alpha) \quad (z \searrow 0)$$

for any $l \in \mathbb{N}$.

The case $l = 1$ is trivial.

Let $l > 1$ be an arbitrary integer and suppose that

$$f(jy) - j^n f(y) = o(y^\alpha) \quad (y \searrow 0)$$

has already been proved for $j = 1, \ldots, l - 1$.

We define the real functions $\varepsilon_0, \ldots, \varepsilon_{(l-1)n}$ and $\varepsilon$ as follows:

$$\varepsilon_i(z) := \Delta_n z f(-iz) - n! f(z) \quad (i = 0, \ldots, (l - 1)n; z \in \mathbb{R})$$

and

$$\varepsilon(z) := \Delta_{lz} f(-(l - 1)nz) - n! f(lz) \quad (z \in \mathbb{R}).$$

Using the notation of Lemma 1 for $\lambda = l$ and by the well-known formula

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + ky) \quad (x, y \in \mathbb{R})$$

we get that these equations can be written as

$$\varepsilon_i(z) = \sum_{k=0}^{l} \alpha_i^{(k)} f((n - k)z) - n! f(z)$$

$$(i = 0, \ldots, (l - 1)n; z \in \mathbb{R})$$
\[ \varepsilon(z) = \sum_{k=0}^{n} \beta^k f((n-k)z) - n! f(lz) \quad (z \in \mathbb{R}). \]

By Lemma 1 there exist positive integers \( K_1, \ldots, K_{(l-1)n} \) for which

\[ K_0 a_0 + \ldots + K_{(l-1)n} a_{(l-1)n} - b = 0 \]

and

\[ K_0 + \ldots + K_{(l-1)n} = l^n. \]

Therefore, by the equations in (8) and (9) we obtain

\[ n! \left( f(lz) - l^n f(z) \right) = K_0 \varepsilon_0(z) + \ldots + K_{(l-1)n} \varepsilon_{(l-1)n}(z) - \varepsilon(z) \quad (z \in \mathbb{R}). \]

To prove (6), we show that for \( k = 0, \ldots, (l-1)n \)

\[ \varepsilon_k(z) = o(z^\alpha) \quad (z \downarrow 0) \quad (12) \]

and

\[ \varepsilon(z) = o(z^\alpha) \quad (z \downarrow 0). \]  \quad (13)

If we choose \( x = -(l-1)nz \) and \( y = lz \) for \( z > 0, z \in \mathbb{R} \), then \( x \leq 0 \leq x + ny \), so (1) and (9) imply (13).

If we replace \((x, y)\) by

\[ (0, z), (-z, z), \ldots, (-nz, z) \quad (z \in \mathbb{R}, z > 0), \]

then \( x \leq 0 \leq x + ny \), therefore, from (1) and (8) we have (12) for \( k = 0, \ldots, n \). In the case \( l = 2 \) property (12) is already proved. If \( l > 2 \) for \( k = 0, \ldots, (l-1)n \) we prove it by induction on \( k \). The proof is done for \( 0 \leq k \leq n \). Let \( n < k \leq (l-1)n \) be an arbitrary fixed integer and suppose that

\[ \varepsilon_r(z) = o(z^\alpha) \quad (z \downarrow 0) \]  \quad (14)

is true for \( r = 0, \ldots, k - 1 \). Set

\[ \tilde{l} = \left\lceil \frac{k-1}{n} \right\rceil + 1, \]
where $[\ ]$ denotes the integer part of a real number and we define $\tilde{\varepsilon} : \mathbb{R} \to \mathbb{R}$ as follows:

\begin{equation}
\tilde{\varepsilon}(z) := \Delta^nf(-kz) - n!f(\tilde{l}z).
\end{equation}

Since

$$k - n \leq n \left\lfloor \frac{k - 1}{n} \right\rfloor$$

for $x = -kz$ and $y = \tilde{l}z$, we have $x \leq 0$ and

$$x + ny = -kz + n \left( \left\lfloor \frac{k - 1}{n} \right\rfloor + 1 \right) z \geq 0,$$

hence (1) implies

\begin{equation}
\tilde{\varepsilon}(z) = o(z^\alpha) \ (z \searrow 0).
\end{equation}

Let $c = (\gamma_0, \ldots, \gamma_{n+k})$ be a vector with components $\gamma_0 = \ldots = \gamma_{n+k-\tilde{l}n-1} = 0$ and write

$$\gamma_{n+k-\tilde{l}n+j} = \begin{cases} (-1)^{\tilde{l}} \left( \frac{n}{n - j} \right), & \text{if } \tilde{l} | j \\ 0, & \text{otherwise} \end{cases}$$

for $j = 0, \ldots, \tilde{l}n$. The simple inequality

$$\left\lfloor \frac{k - 1}{n} \right\rfloor n \leq k - 1$$

yields

\begin{equation}
n + k - \tilde{l}n = n + k - \left\lfloor \frac{k - 1}{n} \right\rfloor n - n \\
\geq k - (k - 1) = 1,
\end{equation}

and then the components $\gamma_{n+k-\tilde{l}n}, \gamma_{n+k-\tilde{l}n+1}, \ldots, \gamma_{n+k}$ of the vector $c$, defined above, exist.
It is easy to see, like in (10) and (11), that (15) can be written in the following form:

\[(18) \tilde{\varepsilon}(z) = \sum_{j=0}^{n+k} \gamma_j f((n-j)z) - n! f(\tilde{l}z).\]

Let us omit the components \(\gamma_0, \ldots, \gamma_{n+k-\tilde{l}n-1}\) of the vector \(c\) and denote the resulting vector by \(\tilde{b} = (\tilde{\beta}^{(0)} \ldots \tilde{\beta}^{(\tilde{l}n)})\). It can be seen from the definition of \(c\) that \(\tilde{b}\) equals \(b = (\beta^{(0)} \ldots \beta^{(\tilde{l}n)})\) which was given for \(n\) and \(\lambda = \tilde{l}\) in Lemma 2.2. It is also easy to see, since we have cancelled only zeroes from \(c\), that (18) can be written as follows:

\[(19) \tilde{\varepsilon}(z) = \sum_{j=0}^{\tilde{l}n} \tilde{\beta}^{(j)} f(n - (n+k-\tilde{l}n+j)z) - n! f(\tilde{l}z) \quad (z \in \mathbb{R}).\]

Let us now consider the functions \(\varepsilon_{n+k-\tilde{l}n}, \varepsilon_{n+k-\tilde{l}n+1}, \ldots, \varepsilon_k\) and the corresponding coefficient vectors \(a_{n+k-\tilde{l}n}, a_{n+k-\tilde{l}n+1}, \ldots, a_k\) from (10). It follows from the definition of these vectors (see Lemma 1) that for their components \(i = n+k-\tilde{l}n, n+k-\tilde{l}n+1, \ldots, k\)

\[\alpha^{(0)}_i = \alpha^{(1)}_i = \ldots = a^{n+k-\tilde{l}n-2}_i = a^{n+k-\tilde{l}n-1}_i = 0.\]

If we omit these components from these vectors and denote them, in the order above, by

\[\tilde{a}_0 = (\tilde{\alpha}^{(0)}_0 \ldots \tilde{\alpha}^{(\tilde{l}n)}_0),\]

\[\vdots\]

\[\tilde{a}_{\tilde{l}-1n} = (\tilde{\alpha}^{(0)}_{(\tilde{l}-1)n} \ldots \tilde{\alpha}^{(\tilde{l}n)}_{(\tilde{l}-1)n}),\]

then we can write the functions \(\varepsilon_{n+k-\tilde{l}n}, \varepsilon_{n+k-\tilde{l}n+1}, \ldots, \varepsilon_k\) in the form

\[\varepsilon_{n+k-\tilde{l}n}(z) = \sum_{s=0}^{\tilde{l}n} \tilde{a}^{(s)}_0 f(n - (n+k-\tilde{l}n+s)z) - n! f(z)\]

\[(20) \vdots\]

\[\varepsilon_k(z) = \sum_{s=0}^{\tilde{l}n} \tilde{a}^{(s)}_{(\tilde{l}-1)n} f(n - (n+k-\tilde{l}n+s)z) - n! f(z).\]
One can see that \( \tilde{a}_0, \ldots, \tilde{a}_{(\tilde{l}-1)n} \) are equal to the vectors

\[
\begin{align*}
a_0 &= (\alpha_0^{(0)} \ldots \alpha_0^{(n)}) \\
&\quad \vdots \\
a_{(\tilde{l}-1)n} &= (\alpha_{(\tilde{l}-1)n}^{(0)} \ldots \alpha_{(\tilde{l}-1)n}^{(n)}),
\end{align*}
\]

defined for \( n \) and \( \lambda = \tilde{l} \) in Lemma 1. So by this lemma, there exist positive integers \( \tilde{K}_0, \ldots, \tilde{K}_{(\tilde{l}-1)n} \) such that

\[
\tilde{K}_0 \tilde{a}_0 + \ldots + \tilde{K}_{(\tilde{l}-1)n} \tilde{a}_{(\tilde{l}-1)n} - \tilde{b} = 0
\]

and

\[
\tilde{K}_0 + \ldots + \tilde{K}_{(\tilde{l}-1)n} = \tilde{l}n.
\]

Thus (19) and (20) imply

\[
\tilde{K}_0 \varepsilon_{n+k-\tilde{l}n}(z) + \tilde{K}_1 \varepsilon_{n+k-\tilde{l}n+1}(z) + \ldots + \tilde{K}_{(\tilde{l}-1)n} \varepsilon_k(z) = \tilde{\varepsilon}(z) + n! f(\tilde{l}z) - n! \tilde{l}^n f(z) \quad (z \in \mathbb{R}),
\]

that is

\[
\varepsilon_k(z) = -\frac{1}{\tilde{K}_{(\tilde{l}-1)n}} \left( \tilde{K}_0 \varepsilon_{n+k-\tilde{l}n}(z) + \tilde{K}_1 \varepsilon_{n+k-\tilde{l}n+1}(z) + \ldots + \tilde{K}_{(\tilde{l}-1)n-1} \varepsilon_{k-1}(z) + \tilde{\varepsilon}(z) + n! \left( f(\tilde{l}z) - \tilde{l}^n f(z) \right) \right) \quad (z \in \mathbb{R}).
\]

From

\[
\tilde{l} = \left\lfloor \frac{k-1}{n} \right\rfloor + 1 \leq \frac{k-1}{n} + 1 \leq \frac{(l-1)n-1+n}{n} < l
\]

together with the inductive hypothesis (7) we get:

\[
f(\tilde{l}z) - \tilde{l}^n f(z) = o(z^\alpha) \quad (z \searrow 0).
\]

By (14) and (17)

\[
\varepsilon_r(z) = o(z^\alpha) \quad (z \searrow 0)
\]

for \( r = n + k - \tilde{l}n, \ldots, k-1 \). Combining (21), (16) and the previous two formulae we get

\[
\varepsilon_k(z) = o(z^\alpha) \quad (z \searrow 0).
\]
II. Now we prove that under our assumptions \( f(0) = 0 \) and

\[
(22) \quad f(-z) - (-1)^n f(z) = o(|z|^\alpha) \quad (z \to 0).
\]

We consider the functions

\[
\varepsilon_0(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(kz) - n! f(z)
\]

and

\[
\varepsilon_1(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f((k-1)z) - n! f(z),
\]

defined in (8). By the well-known formulae

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n - n! = 0
\]

and

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k-1)^n - n! = 0
\]

we can write the functions \( \varepsilon_0 \) and \( \varepsilon_1 \) in the form

\[
(23) \quad \varepsilon_0(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (f(kz) - k^n f(z))
\]

and

\[
(24) \quad \varepsilon_1(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (f((k-1)z) - (k-1)^n f(z)).
\]

In the first part of the proof we have shown that \( \varepsilon_0(z) = o(z^\alpha) \), \( \varepsilon_1(z) = o(z^\alpha) \) and \( f(lz) - l^n f(z) = o(z^\alpha), \ (z \searrow 0, \ l = 1, \ldots, n) \). This relation together with (23) implies \( f(0) = 0 \), therefore, applying (24) we get

\[
f(-z) - (-1)^n f(z) = o(z^\alpha) \quad (z \searrow 0)
\]

which yields (22).

Finally, (6) and (22) prove Theorem 1.
Theorem 2. Let $\delta > 0$ be a real number and $n \in \mathbb{N}$. If the function $f : [-\delta, \delta] \to \mathbb{R}$ satisfies the property

$(25) \quad \Delta_y^n f(x) - n! f(y) = 0 \quad (x \in [-\delta, 0], \; y, x + ny \in [0, \delta]),$

then for any integer $l$ there exists a real number $\delta_l > 0$ such that

$(26) \quad f(lz) - l^n f(z) = 0 \quad (z \in [-\delta_l, \delta_l]).$

Proof. Let $\delta > 0$ and $n \in \mathbb{N}$ be given and let $f : [-\delta, \delta] \to \mathbb{R}$ be a function satisfying (25). We prove that for an arbitrary integer $l$ with $\delta_l = \frac{\delta}{|ln|}$ equation (26) holds.

The proof can be done in a similar way as in the proof of Theorem 1, therefore, we give the outline of the argument, only.

At first, we show by induction on $l$ that for any $l \in \mathbb{N}$

$(27) \quad f(lz) - l^n f(z) = 0 \quad (z \in \left[-\frac{\delta}{ln}, \frac{\delta}{ln}\right]).$

For $l > 1$ we define the functions

$\varepsilon_0, \ldots, \varepsilon_{(l-1)n}$ and $\varepsilon : \left[-\frac{\delta}{ln}, \frac{\delta}{ln}\right] \to \mathbb{R}$

by the same formula as in (8) and (9) and we use a similar method as in the proof of Theorem 1, to show that

$\varepsilon_k(z) = 0 \quad (z \in \left[-\frac{\delta}{ln}, \frac{\delta}{ln}\right])$

for $k = 0, \ldots, (l - 1)n$ and

$\varepsilon(z) = 0 \quad (z \in \left[-\frac{\delta}{ln}, \frac{\delta}{ln}\right]).$

By Lemma 1

$n! \left(f(lz) - l^n f(z)\right) = K_0 \varepsilon_0(z) + \ldots + K_{(l-1)n} \varepsilon_{(l-1)n}(z) - \varepsilon(z)$

$\quad \left(z \in \left[-\frac{\delta}{ln}, \frac{\delta}{ln}\right]\right),$

which proves (27).
To prove

\begin{equation}
(28) \quad f(-z) - (-1)^n f(z) = 0 \quad (z \in \left[ -\frac{\delta}{n}, \frac{\delta}{n} \right])
\end{equation}

we consider the functions \( \varepsilon_0 \) and \( \varepsilon_1 \) on the interval \( \left[ -\frac{\delta}{n^2}, \frac{\delta}{n^2} \right] \). Here we have

\[ \varepsilon_0(z) = 0 \quad (z \in \left[ -\frac{\delta}{n^2}, \frac{\delta}{n^2} \right]) \]

and

\[ \varepsilon_1(z) = 0 \quad (z \in \left[ -\frac{\delta}{n^2}, \frac{\delta}{n^2} \right]), \]

therefore, we get, by the method used in the second part of the proof of Theorem 1, that

\begin{equation}
(29) \quad \Delta^n_y f(x) - n! f(y) = 0
\end{equation}

for \( x, y, x + ny \in [-\bar{\delta}, \bar{\delta}] \).

**Proof.** Let \( \bar{\delta} > 0 \) and \( n \in \mathbb{N} \) be given numbers and let \( f : [-\delta, \delta] \to \mathbb{R} \) be a function with property (25). Let, furthermore, \( \bar{\delta} = \frac{\delta}{2n} \) and \( \bar{x} \) and \( \bar{y} \) be fixed numbers for which \( \bar{x}, \bar{y}, \bar{x} + n\bar{y} \in [-\delta, \delta] \).

It is trivial, that in the case when \( \bar{y} = 0 \) equation (29) holds. For an arbitrary function \( \varphi : \mathbb{R} \to \mathbb{R} \) we have the simple formula

\[ \Delta^n_y \varphi(x) = (-1)^n \Delta^n_{-y} \varphi(x + ny) \quad (x, y \in \mathbb{R}), \]

so by

\[ f(-y) - (-1)^n f(y) = 0 \quad (y \in [-\delta, \bar{\delta}]), \]
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which was proved in Theorem 2, we can write

$$
\Delta^n_y f(x) - n! f(y) = (-1)^n (\Delta^n_y f(x + ny) - n! f(-y))
$$

$$(x, y, x + ny \in [-\delta, \delta]).$$

Therefore, we may suppose that $\bar{y} \in (0, \bar{\delta}].$

In the case when $\bar{x} \in [-\delta, 0]$ and $\bar{x} + n\bar{y} \in [0, \delta]$, (29) comes from (1).

If $\bar{x}, \bar{y}$ have these properties, then there exist natural numbers $m$ such that $\bar{x} - m\bar{y} \leq 0$. Let $m_0$ be the smallest natural number with this property and we define $x_\mu^* = \bar{x} - (m_0 - \mu)\bar{y}$ for $\mu = 0, \ldots, m_0$.

We prove by induction on $\mu$ that by

$$
(30) \quad c_\mu := \Delta^n_y f(x_\mu^*) - n! f(\bar{y})
$$

$c_\mu = 0$ for $\mu = 0, \ldots, m_0$, which with $\mu = m_0$ implies

$$
\Delta^n_y f(\bar{x}) - n! f(\bar{y}) = 0,
$$

which is our statement.

By (25), obviously, $c_0 = 0$.

Let $\mu \in \{1, \ldots, m_0\}$ and suppose that $c_\nu = 0$ is already proved for $\nu = 0, \ldots, \mu - 1$. Taking

$$
x = x_\mu^* - iy, \quad y = \bar{y} \quad (i = 1, \ldots, n)
$$

and

$$
x = x_\mu^* - ny, \quad y = 2\bar{y},
$$

respectively, the inductive hypothesis and (25) lead to

$$
\Delta^n_y f(x_\mu^* - iy) - n! f(y) = 0 \quad (i = 1, \ldots, n)
$$

and

$$
\Delta^n_{2\bar{y}} f(x_\mu^* - n\bar{y}) - n! f(2\bar{y}) = 0.
$$
It is easy to see that with the notation of Lemma 1 (for $$\lambda = 2$$) we can write these equations as follows

$$\sum_{k=0}^{2n} \alpha_i^{(k)} f(x^*_{\mu} + (n-k)\bar{y}) - n! f(\bar{y}) = 0 \quad (i = 1, \ldots, n)$$

and

$$\sum_{k=0}^{2n} \beta^{(k)} f(x^*_{\mu} + (n-k)\bar{y}) - n! f(2\bar{y}) = 0.$$ 

Furthermore, (30) has the form

$$\sum_{k=0}^{2n} \alpha_0^{(k)} f(x^*_{\mu} + (n-k)\bar{y}) - n! f(\bar{y}) = c_\mu.$$ 

By Lemma 1 for $$a_i = (\alpha_i^{(0)}, \ldots, \alpha_i^{(2n)})$$, $$(i = 0, \ldots, n)$$ and $$b = (\beta^{(0)}, \ldots, \beta^{(2n)})$$ there exist positive integers $$K_0, \ldots, K_n$$ such that 

$$K_0 a_0 + \ldots + K_n a_n - b = 0$$ and 

$$K_0 + \cdots + K_n = 2^n.$$ 

Therefore, by the equations in (31), (32) and (33) we get

$$-(K_0 + \cdots + K_n) n! f(\bar{y}) + n! f(2\bar{y}) = K_0 c_\mu,$$

that is

$$-2^n n! f(\bar{y}) + n! f(2\bar{y}) = K_0 c_\mu.$$ 

By Theorem 2 we have $$f(2\bar{y}) - 2^n f(\bar{y}) = 0$$, which implies $$c_\mu = 0$$.

**Theorem 4.** Let $$n$$ be a natural number and $$\alpha > n$$ be a real number. If a function $$f : \mathbb{R} \to \mathbb{R}$$ satisfies

(1) \(\Delta_n^y f(x) - n! f(y) = o(y^\alpha) \quad ((x, y) \to (0, 0), \ x \leq 0 \leq x + ny)\)

then there exists a monomial function $$g : \mathbb{R} \to \mathbb{R}$$ of degree $$n$$ such that

(2) \(f(x) - g(x) = o(|x|^\alpha) \quad (x \to 0).\)

**Proof.** Let $$n \in \mathbb{N}$$ and $$\alpha > n$$, $$\alpha \in \mathbb{R}$$ be given. For a function $$f : \mathbb{R} \to \mathbb{R}$$ satisfying (1) Theorem 1 implies

(34) \(f(lz) - l^n f(z) = o(|z|^\alpha) \quad (z \to 0)\).
for any integer $l$. Let now $l \in \mathbb{N}$, $l > 1$ be fixed. It is easy to see, that (34) is equivalent to the following statement: there exist a real number $\delta > 0$ and a continuous, increasing function $h : [0, \delta] \to \mathbb{R}$ with the property $\lim_{z \searrow 0} h(z) = 0$ such that

$$|f(lz) - l^n f(z)| \leq |z|^\alpha h(|z|) \quad (z \in [-\delta, \delta]).$$

Therefore, for an arbitrary $z_0 \in [-\delta, \delta]$ and $k \in \mathbb{N}$ we have

$$\left| f\left(\frac{z_0}{l^{k-1}}\right) - l^n f\left(\frac{z_0}{l^k}\right) \right| \leq \frac{|z_0|^\alpha}{l^{k\alpha}} h\left(\frac{|z_0|}{l^k}\right).$$

With

$$\varepsilon_k(z_0) := l^{(k-1)n} f\left(\frac{z_0}{l^{k-1}}\right) - l^{kn} f\left(\frac{z_0}{l^k}\right)$$

we get

$$|\varepsilon_k(z_0)| \leq l^{(k-1)n} \frac{|z_0|^\alpha}{l^{k\alpha}} h\left(\frac{|z_0|}{l^k}\right)$$

and the monotony of $h$ yields

(35) \quad $|\varepsilon_k(z_0)| \leq \frac{1}{l^{k(\alpha-n)}} \frac{|z_0|^\alpha}{l^n} h(|z_0|).$

For an arbitrary $N \in \mathbb{N}$ we obtain

(36) \quad $\varepsilon_1(z_0) + \cdots + \varepsilon_N(z_0) = f(z_0) - l^N f\left(\frac{z_0}{l^N}\right).$

Since $\alpha > n$

$$\sum_{k=1}^{\infty} \frac{1}{l^{k(\alpha-n)}} = \frac{1}{l^{\alpha-n-1}},$$

therefore,

$$\sum_{k=1}^{\infty} \varepsilon_k(z_0)$$

is convergent, so the limit

(37) \quad $g(z_0) = \lim_{k \to \infty} l^{kn} f\left(\frac{z_0}{l^k}\right)$
exists, and (35) and (36) yield

$$|f(z_0) - g(z_0)| \leq \frac{1}{l^{\alpha-n}-1} \frac{|z_0|^\alpha}{l^n} h(|z_0|),$$

which implies (2).

For $x \in [-\delta, 0], x + ny \in [0, \delta]$ by (1) we have

$$\lim_{k \to \infty} \Delta_{\frac{x}{k}}^n f \left( \frac{x}{k^n} \right) - n!f \left( \frac{y}{k^n} \right) = 0,$$

and (37) gives

$$\Delta_{\frac{y}{k}}^n g(x) - n!g(y) = \lim_{k \to \infty} l_{k^n} \left( \Delta_{\frac{x}{k}}^n f \left( \frac{x}{k^n} \right) - n!f \left( \frac{y}{k^n} \right) \right) = 0,$$

which together with Theorem 3 show that there exists a real number $\bar{\delta} > 0$ such that $g$ is a monomial function of degree $n$ on the interval $[-\bar{\delta}, \bar{\delta}]$. This result and the known extension theorem for monomial functions (cf. [5], for instance) imply our statement.

3. Locally additive functions with order $\alpha \neq 1$

**Lemma 2.** Let $\delta$ be a positive real number and $f : [-\delta, \delta] \to \mathbb{R}$. If there exists a real number $K \geq 0$ such that

$$|f(x + y) - f(x) - f(y)| \leq K \quad (x \in [-\delta, 0], y, x + y \in [0, \delta]),$$

then we have

$$|f(x + y) - f(x) - f(y)| \leq 3K$$

for all $x, y, x + y \in [-\delta, \delta]$.

**Proof.** Let $\bar{x}$ and $\bar{y}$ be fixed real numbers such that $\bar{x}, \bar{y}, \bar{x} + \bar{y} \in [-\delta, \delta]$. Then we have one of the following relations:

(A) $\bar{x} \in [-\delta, 0], \bar{y} \in [0, \delta], \bar{x} + \bar{y} \in [0, \delta]$;

(B) $\bar{x} \in [-\delta, 0], \bar{y} \in [0, \delta], \bar{x} + \bar{y} \in [-\delta, 0]$;

(C) $\bar{x} \in [0, \delta], \bar{y} \in [0, \delta], \bar{x} + \bar{y} \in [0, \delta]$;
Case (A) is trivial.
In case (B) we get the following inequalities from (38):
- \(|f(\bar{y}) - f(\bar{x} + \bar{y}) - f(-\bar{x})| \leq K\), with \(x = \bar{x} + \bar{y}\) and \(y = -\bar{x}\);
- \(|-f(0) + f(\bar{x}) + f(-\bar{x})| \leq K\), with \(x = \bar{x}\) and \(y = -\bar{x}\);
- \(|f(\bar{y}) - f(0) - f(\bar{y})| \leq K\), with \(x = 0\) and \(y = \bar{y}\);
and the addition of these inequalities implies (39).
In case (F) we get (39) by case (B) and with \(x = \bar{y}\) and \(y = -\bar{y}\).

The remaining cases can be treated by the substitutions
- \(x = -\bar{y}\) and \(y = \bar{y}; x = -\bar{y}\) and \(y = \bar{x} + \bar{y}; x = 0\) and \(y = \bar{y}\) in case (C);
- \(x = \bar{y}\) and \(y = -\bar{y}; x = \bar{x}\) and \(y = -\bar{x} + \bar{y}; x = \bar{x} + \bar{y}\) and \(y = -\bar{x} - \bar{y}\) in case (D); \(x = \bar{y}\) and \(y = \bar{x}\) in case (E), respectively.

**Theorem 5.** Let \(\alpha \geq 0 \alpha \neq 1\) be a real number and let \(f : \mathbb{R} \to \mathbb{R}\) be a function with the property

\[f(x + y) - f(x) - f(y) = o(y^\alpha) \quad (x \leq 0 \leq x + y, \, y \searrow 0).\] (40)

Then there exists an additive function \(a : \mathbb{R} \to \mathbb{R}\) such that

\[f(x) - a(x) = o(|x|^\alpha) \quad (x \to 0).\]

**Proof.** For \(\alpha > 1\) the statement is proved in Theorem 4.
In the sequel, \(\alpha \in [0,1]\). In this case the proof is similar to some reasoning in [7].
By (40) there exist real numbers \(\delta > 0\) and \(K > 0\) such that

\[|f(x + y) - f(x) - f(y)| \leq K \quad (x \in [-\delta, 0], \, y, x + y \in [0, \delta]),\]

hence from Lemma 2 we have

\[|f(x + y) - f(x) - f(y)| \leq 3K \quad (x, y, x + y \in [-\delta, \delta]).\]

Z. Kominek proved ([4], Lemma 1) that this property implies the existence of an additive function \(a : \mathbb{R} \to \mathbb{R}\) such that

\[|f(x) - a(x)| \leq 12K \quad (x \in [-\delta, \delta]).\]
For the function $\varepsilon : [-\delta, \delta] \to \mathbb{R}, \varepsilon(x) = f(x) - a(x)$ we have $\varepsilon(0) = 0$ and by Theorem 1

$$\varepsilon(2z) - 2\varepsilon(z) = o(|z|^\alpha) \quad (z \to 0).$$

It is easy to see, that this property is equivalent to the following: there exist a real number $\delta_1 > 0$ and a continuous, increasing function $h : [0, \delta_1] \to \mathbb{R}$ such that $\lim_{z \to 0} h(z) = 0$ and

$$|\varepsilon(2z) - 2\varepsilon(z)| \leq |z|^\alpha h(|z|) \quad (z \in [-\delta_1, \delta_1]).$$

Introducing the function

$$\bar{\varepsilon}(z) = \begin{cases} \frac{\varepsilon(z)}{|z|^\alpha}, & \text{if } z \in [-\delta_1, \delta_1], \ z \neq 0, \\ 0, & \text{if } z = 0 \end{cases}$$

we have

$$\left| |z|^\alpha \bar{\varepsilon}(z) - \frac{1}{2} 2^\alpha |z|^\alpha \bar{\varepsilon}(2z) \right| \leq \frac{1}{2} |z|^\alpha h(|z|) \quad (z \in [-\delta_1, \delta_1])$$

and

$$|\bar{\varepsilon}(z) - 2^{\alpha-1}\bar{\varepsilon}(2z)| \leq \frac{1}{2} h(|z|) \quad (z \in [-\delta_1, \delta_1]).$$

Write

$$s_k = \sup \left\{ |\bar{\varepsilon}(z)| \mid \frac{\delta_1}{2^k} \leq |z| \leq \frac{\delta_1}{2^{k-1}} \right\} \quad (k \in \mathbb{N}).$$

Then

$$s_{k+1} \leq 2^{\alpha-1} s_k + \frac{1}{2} h \left( \frac{\delta_1}{2^k} \right), \quad (k \in \mathbb{N})$$

therefore, $\lim_{k \to \infty} s_k = 0$ and

$$\varepsilon(z) = o(|z|^\alpha) \quad (z \to 0).$$
References


