Abstract. In this paper, we prove the following:
1. Let $G$ be a Moufang loop of order $p^\alpha m$, $(p, m) = 1$, $(p - 1, p^\alpha m) = 1$ and $p$ is a prime. Suppose $G$ has an element of order $p^\alpha$. Then $G = P \rtimes K$, a split extension of a normal subloop $K$ of order $m$ with a subloop $P$ order $p^\alpha$.

2. Let $G$ be a Moufang loop of odd order $p^2 m$, $(p, m) = 1$, and $p$ is the smallest prime dividing $|G|$. Then a similar result holds as in (1) with $\alpha = 2$.

I. Introduction

Let $G$ be a group of order $p^\alpha m$, $(p, m) = 1$, $(p - 1, p^\alpha m) = 1$ and suppose $G$ has an element $x$ of order $p^\alpha$. Then $G = \langle x \rangle \rtimes K$, a split extension of a normal subgroup $K$ of order $m$ with the subgroup $\langle x \rangle$ [4]. To prove an analogous result on a Moufang loop $G$, we need a normal subloop $K$ so that we can use induction on $G/K$. In other words, $G$ has to be nonsimple. By Liebeck [8], every simple nonassociative Moufang loop is isomorphic to one of the Paige’s loop $M^*(q)$. Considering the orders of elements in each of the conjugate classes of $M^*(q)$, as examined by Bannai and Song [2], we find that a Moufang loop $G$ with the given properties stated above cannot be simple. Then $G$ has a normal subloop $K$, and so induction will be possible.

Let $G$ be a group of order, $p^2 m$, $(p, m) = 1$, and $p$ is the smallest prime dividing $|G|$. Then $G = P \rtimes K$, a split extension of a normal subgroup $K$ of order $m$ with the subgroup $P$ of order $p^2$ [11]. We prove

Mathematics Subject Classification: 20N05.
Key words and phrases: Moufang loop, normal, extension, simple loop.
also an analogous result on a Moufang loop $G$ by using repeatedly several theorems of GLAUBERMAN [5].

There exist nonassociative Moufang loops of order $pq^3$ with $q = 1 \pmod{p}$ [10]. Moufang loops of odd order $p^2q^3$ can be similarly constructed. Hence our two splitting theorems are very useful in studying the structure of such finite Moufang loops.

II. Definitions

1. A binary system $\langle G, \cdot \rangle$, in which specification of any two of the elements $x, y, z$ in the equation $x \cdot y = z$ uniquely determines the third element, is called a quasigroup. If it further contains an identity element, then it is called a loop. Clearly, a group is a loop. But there are loops which are not associative.

2. A loop $\langle G, \cdot \rangle$ is a Moufang loop if $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z$ in $G$. From now on, $G$ is defined as a finite Moufang loop.

3. Define $zR(x, y) = (zx \cdot y)(xy)^{-1}$, $zL(x, y) = (yx)^{-1}(y \cdot xz)$ and $zT(x) = x^{-1} \cdot zx$. $I(G) = \langle R(x, y), L(x, y), T(x) \mid x, y \in G \rangle$ is called the inner mapping group of $G$.

4. Let $x$ and $y$ be elements of $G$. $x$ and $y$ are conjugate if there exists $\theta \in I(G)$ such that $x\theta = y$.

5. Let $H$ be a subloop of $G$ and $\pi$ a set of primes.

(i) $H$ is a normal subloop of $G$, $(H \triangleleft G)$, if $H\theta = H$ for all $\theta \in I(G)$ where $H\theta = \{h\theta \mid h \in H\}$.

(ii) $H$ is a $\pi$-loop if the order of every element of $H$ is a $\pi$-number. (A positive integer $n$ is a $\pi$-number if every prime divisor of $n$ lies in $\pi$).

(iii) $H$ is a Hall $\pi$-subloop of $G$ if $|H|$ is the largest $\pi$-number dividing $|L|$.

6. $G_a$, the associator subloop of $G$, is the subloop generated by all the associators $(x, y, z)$ where $(x, y, z) = (x \cdot yz)^{-1} \cdot (xy \cdot z)$. Write $G_a = (G, G, G)$ also.
7. $G_c$, the commutator subloop of $G$, is the subloop generated by all the commutators $[x, y]$ where $[x, y] = (yx)^{-1} \cdot (xy)$.

8. $N = N(L)$, the nucleus of $L$, is the subloops generated by all $n$ in $L$ where $(n, x, y) = (x, n, y) = (x, y, n) = 1$ for all $x, y$ in $L$.

9. $Z = Z(L)$, the centre of $L$, is the subloop generated by all $z$ in $N$ such that $[z, x] = 1$ for all $x$ in $L$.

III. Known results with Moufang loops

Let $G$ be a finite Moufang loop.

$R_1$. (a) $x \in G \Rightarrow |x| \mid |G|$. [1, p. 92, Thm. 1.2].

(b) $G$ is disassociative. [1, p. 117, Moufang’s Theorem].

(c) $x \in G$ and $\theta \in I(G) \Rightarrow x^n \theta = (x\theta)^n$ for any integer $n$. [1, p. 120, (4.1) and p. 117, Lemma 3.2].

$R_2$. $N$ and $Z$ are normal subloops of $G$. [1, p. 114, Thm. 2.1 and p. 60, Lemma 1.1]. In fact $N$ and $Z$ are associative by their definitions.

$R_3$. Let $H$ be a normal subloop of $G$ such that $H \subset N$. Then

(a) $G/C_G(H) < \text{Aut } H$ where $C_G(H) = \{g \mid g \in G, gh = hg \text{ for all } h \in H\}$.

(b) $C_G(H) \cap H = Z(H)$, the centre of $H$. If $H = N$, then $G_a \subset C_G(N)$. [7, p. 33, Thm. 3].

$R_4$. $G$ is a 2-loop if and only if $|G| = 2^m$ for some positive integer $m$. [6, p. 415, Thm.].

$R_5$. Suppose $|G|$ is odd and $K$ is a normal subloop of $G$.

(a) If $K$ is minimal normal in $G$, then $K$ is an elementary Abelian group and $(K, K, G) = 1$. [5, p. 402, Thm. 7].

(b) If $(K, K, G) = 1$ and $(|K|, |G/K|) = 1$, then $K \subset N$. [5, p. 405, Thm. 10].

(c) $G$ is solvable. [5, p. 413, Thm. 16].

(d) $G$ contains a Hall $\pi$-subloop, $\pi$ a set of primes. [5, p. 409, Thm. 12].

(e) If $H < G$ then $|H| \mid |G|$. [5, p. 359, Thm. 2].
\textbf{R}_6. If \( H \) is a subloop of \( G \), \( x \in G \) and \( d \) is the smallest positive integer such that \( x^d \in H \), then \(|\langle H, x \rangle| \geq |H|d| \text{ [3, p. 5, Lemma 0]}.\)

\textbf{R}_7. There exist simple nonassociative Moufang loops \( M^*(p^n) \) with \(|M^*(p^n)| = p^{3n}(p^{4n} - 1)/(d(p))\) where \( d(2) = 1 \) and \( d(p) = 2 \) if \( p \) is an odd prime. \text{ [9, p. 474, Thm. 4.1]}. \)

\textbf{R}_8. \( G \) is a nonassociative simple Moufang loop \( \iff \ G \) is isomorphic with \( M^*(p^n) \) for some prime \( p \). \text{ [8, p. 33, Theorem]}. \)

\textbf{R}_9. The conjugacy classes of \( M^*(p^n) \) contain elements whose orders are 1, \( p \), divisors of \( p^n - 1 \) or divisors of \( p^n + 1 \). \text{ [2, p. 224, Thm. 2.1.1 and p. 227, Thm. 2.1.2]}. 

\textbf{IV. Moufang loops of order} \( p^\alpha m \)

**Lemma 1.** Let \( G \) be a simple nonassociative Moufang loop of order \( 2^\alpha m, (2, m) = 1 \). Then \( G \) has no element of order \( 2^\alpha \).

**Proof.** By \( R_7 \) and \( R_8 \), \( G \) is isomorphic to \( M^*(q) \) for some \( q \) where \( q = p^n \) and \( p \) is a prime. Let \( x \) be any 2-element of \( M^*(q) \).

\text{Case 1:} \( p \geq 3 \). Let \( q - 1 = 2^{\beta_1}m_1 \) and \( q + 1 = 2^{\beta_2}m_2 \), where \( m_1 \) and \( m_2 \) are odd. Suppose \( \beta = \max\{\beta_1, \beta_2\} \). By \( R_9 \), \(|x| \leq 2^\beta \).

\[
|M^*(q)| = \frac{q^3(q^2 - 1)}{2} = \frac{q^3(q^2 + 1)}{2} (q - 1)(q + 1) = \frac{q^3(q^2 + 1)}{2} 2^{(\beta_1 + \beta_2)}m_1m_2 = 2^\alpha m.
\]

Since \( q \) is odd, \( 2 \not| (q^2 + 1) \). So \( \beta_1 + \beta_2 \leq \alpha \). Thus \(|x| \leq 2^\beta < 2^\beta + \beta \leq 2^\alpha \) as \( \beta_1 > 0 \) and \( \beta_2 > 0 \).

\text{Case 2: } p = 2. \text{ As } q - 1 \text{ and } q + 1 \text{ are odd, } 2 \not| (q - 1)(q + 1). \text{ So by } R_9, |x| = 2. \text{ Now } |M^*(2^n)| = 2^{3n}(2^{4n} - 1) = 2^\alpha m. \text{ So } 2^\alpha = 2^{3n} \geq 2^3 > 2 = |x|. \text{ Thus } G \text{ has no element of order } 2^\alpha.

**Lemma 2.** Let \( G \) be a Moufang loop and \( M \) a normal subloop of \( G \). Suppose \( H \) is a normal Hall \( \pi \)-subloop of \( M \). Then \( H \) is normal in \( G \) in each of the following two cases:

(a) \( G \) is of odd order;
(b) $|M| = 2^r m$ where $m$ is odd, $|H| = m$ and there exists an element of order $2^r$ in $M$.

**Proof.** Suppose $H \not< G$. Then there exists $\theta \in I(G)$ such that $H\theta \neq H$. Since any inner mapping $\theta$ is a permutation of $G$, $H\theta - H \neq \emptyset$. Let $h\theta \in H\theta - H$. Since $H \triangleleft M \triangleleft G$, $H\theta \subset M\theta = M$. Since $H$ and $\langle h\theta \rangle$ are both subloops of $M$ with $H \triangleleft M$, clearly $H\langle h\theta \rangle$ is a subloop of $M$. Now by $R_1(c)$, $(h\theta)^{|h\theta|} = (h)^{|h|}\theta = 1$. So $|h\theta| | |h|$. By $R_1(a)$, $|h| | |H|$.

So $|h\theta| | |H|$. Also $|H\langle h\theta \rangle| = \frac{|H| |(h\theta)|}{|H\langle h\theta \rangle|}$.

(a) Suppose $G$ is of odd order. Since $|h\theta| | |H|$, $H\langle h\theta \rangle$ is a $\pi$-loop in $M$ strictly containing the Hall $\pi$-subloop $H$. So $|H\langle h\theta \rangle| | |M|$. This is a contradiction by $R_5(e)$.

So $H \triangleleft G$ if $G$ is of odd order.

(b) Suppose $|M| = 2^r m$ where $m$ is odd, $|H| = m$ and there exists an element $x$ of order $2^r$ in $M$. Since $|h\theta| | |H|$, $|H\langle h\theta \rangle|$ is odd. Also $|H\langle h\theta \rangle| > m$ as $h\theta \notin H$. Now by $R_1(a)$, $x^d \notin H\langle h\theta \rangle$ for each $0 < d < 2^r$. But $x^{2^r} = 1 \in H\langle h\theta \rangle$. Thus $|H\langle h\theta \rangle, x| \geq |H\langle h\theta \rangle, 2^r| > R_6 > m2^r = |M|$.

This is a contradiction as $H\langle h\theta \rangle, x \subset M$. Hence $H \triangleleft G$ in this case also.

**Lemma 3.** Suppose $G$ is a Moufang loop of order $p^a m$, $(p, m) = 1$; $K$ is a normal subloop of $G$ such that $|G/K| = p^b m_0$, $m_0 | m$. Suppose there exists an element $x$ of order $p^a$ in $G$. Then $xK$ is an element of order $p^b$ in $G/K$.

**Proof.** $(xK)^{p^a} = x^{p^a} K = 1K \Rightarrow |xK| | p^a \Rightarrow xK$ is a $p$-element in $G/K \Rightarrow |xK| | p^b$ by $R_1(a)$. So $|xK| = p^\gamma$, $\gamma \leq \beta$. Then $(xK)^{p^\gamma} = x^{p^\gamma} K = 1K$ and $x^{p^\gamma} \in K$. Since $|K| = p^{\alpha - \beta} m/m_0$ and $x^{p^\gamma}$ is a $p$-element in $K$, $|x^{p^\gamma}| | p^{\alpha - \beta}$ by $R_1(a)$. Thus $(x^{p^\gamma})^{p^{\alpha - \beta}} = 1$ or $x^{p^{\alpha + \gamma - \beta}} = 1$. So $\alpha + \gamma - \beta \geq \alpha$. Then $\gamma \geq \beta$. Hence $\gamma = \beta$. So $|xK| = p^\beta$.

**Lemma 4.** Let $G$ be a Moufang loop of order $2^a m$, $(2, m) = 1$. Suppose $G$ has an element $x$ of order $2^a$. Then $G = \langle x \rangle \rtimes K$, i.e., $G$ is a split extension of a cyclic group $\langle x \rangle$ of order $2^a$ with a normal subloop $K$ of order $m$.

**Proof.** If $G$ is a group, we are through by [4, p. 14, Problem 2.16]. So we assume that $G$ is nonassociative. By Lemma 1, we know that $G$ is
nonsimple. Let $K$ be a maximal normal subloop of $G$. Let $|G/K| = 2^\beta m_0$, $0 \leq \beta \leq \alpha$, $m_0 \mid m$.

Case 1: $1 < m_0 < m : |G/K| = 2^\beta m_0$.

1(a) : $\beta = 0$. Then $|G/K| = m_0$ and $|K| = 2^\alpha (m/m_0)$. By Lemma 3, $|xK| = 1$. Hence $x \in K$. By induction, there exists a subloop $K_0$ of order $m/m_0$ normal in $K$. By Lemma 2, $K_0 \lhd G$. Now $|G/K_0| = 2^\alpha m_0$ and $xK_0$ is an element of order $2^\alpha$ in $G/K_0$ by Lemma 3. By induction, there exists a subloop $K_1/K_0$ of order $m_0$ normal in $G/K_0$. Then $K_1 \lhd G$ and $|K_1| = |K_0|m_0 = m$. So $G = \langle x \rangle \rtimes K_1$.

1(b) : $\beta \geq 1$. By Lemma 3, $xK$ is an element of order $2^\beta$ in $G/K$. By induction, there exists a subloop $K_1/K$ of order $m_0$ normal in $G/K$. Thus $K_1 \lhd G$ and $|K_1| = |K|m_0 > |K|$, contradicting the maximality of $K$.

Case 2: $m_0 = 1 : |G/K| = 2^\beta$.

2(a) : $\beta = 0$. $|G/K| = 1 \Rightarrow G = K$, a contradiction.

2(b) : $0 < \beta < \alpha$. $|K| = 2^{\alpha - \beta} m$. Since $xK \in G/K$, $(xK)^{2^\beta} = 1K$ and $x^{2^\beta} K$. Clearly $|x^{2^\beta}| = 2^{\alpha - \beta}$. By induction, $K$ has a normal subloop $K_0$ of order $m$. Thus $K_0 \lhd G$ by Lemma 2(b). So $G = \langle x \rangle \rtimes K_0$.

2(c) : $\beta = \alpha$. $|K| = m$ and $G = \langle x \rangle \rtimes K$.

Case 3: $m_0 = m : |G/K| = 2^\beta m$.

3(a) : $\beta = 0$. $|G/K| = m$. Suppose $m$ is not a prime. Then $G/K$ is solvable by $R_5(c)$. So it has proper normal subloop $K_1/K$. Then $K_1 \lhd G$ and $|K| < |K_1| < |G|$. This contradicts that $K$ is a maximal normal subloop of $L$. Now $m = p$, an odd prime. Now $|G| = 2^\alpha p$. By $R_4(a)$, there exists $w \in G$ such that $|w| = p$ as otherwise, $G$ would be a 2-loop, which is impossible by $R_4$. Now by $R_6$, $G = \langle x, w \rangle$ is a group by diassociativity, a contradiction.

3(b) : $0 < \beta < \alpha$. By Lemma 3, $xK$ is an element of order $2^\beta$ in $G/K$. By induction, there exists a subloop $K_1/K$ of order $m$ normal in $G/K$. Then $K_1 \lhd G$ and $|K_1| = m|K| > |K|$, a contradiction.

3(c) : $\beta = \alpha$. Then $|K| = 1$, a contradiction since $K$ is a maximal normal subloop of $G$. 


Theorem 1. Let $G$ be a finite Moufang loop of order $p^a$, $(p, m) = 1$, $(p - 1, p^a) = 1$. Suppose $G$ has an element $x$ of order $p^a$. Then $G = ⟨x⟩ \rtimes K$, i.e., $G$ is a split extension of a cyclic group $⟨x⟩$ of order $p^a$ and a normal subloop $K$ of order $m$.

**Proof.** By Lemma 4, we can assume that $p$ is an odd prime. Since $(p - 1, p^a) = 1$, $G$ is of odd order. By $R_5(3)$, $G$ is solvable. Let $K$ be a minimal normal subloop of $G$. By $R_5(4)$, $K$ is an elementary abelian $q$-group (where $q$ is a prime).

*Case 1: $q = p$. $K ≅ ⟨x⟩$. Otherwise, $K⟨x⟩$ is a $p$-subloop of $G$ whose order is bigger than $p^a$, contradicting $R_5(5)$. As $⟨x⟩$ is cyclic, $K$ is cyclic. So $K = C_p$ as it is an elementary abelian group.*

$1(a) : K \not\subset ⟨x⟩$. Then $\alpha ≥ 2$, $|G/K| = p^{a-1}m$ and $xK$ is an element of order $p^{a-1}$ by Lemma 3. By induction, there exists a subloop $K_1/K$ of order $m$ normal in $G/K$. Then $K_1 \triangleleft G$ and $|K_1| = pm$. Now $x^{p^{a-1}}$ is an element of order $p$ in $K_1$. By induction, there exists a subloop $K_2$ of order $m$ normal in $K_1$. Now $K_2$ is a normal Hall subloop in $K_1$ and $K_1 \triangleleft G$ implies that $K_2 \triangleleft G$ by Lemma 2(a). Thus $G = ⟨x⟩ \rtimes K_2$.

$1(b) : K = ⟨x⟩ = C_p$. Now $(K, K, G) = 1$ by $R_5(1)$ and $|K| = |G/K| = 1$ implies $K \subset N$, the nucleus of $G$, by $R_5(5)$. By $R_5(2)$, $G/C_G(K) ≤ \text{Aut}K$. As the order of the group of automorphisms of $C_p$ is $p - 1$, $|\text{Aut}K| ≤ p - 1$. As $(p - 1, |G|) = (p - 1, p^a) = 1$, $G = C_G(K)$. Thus $K \subset Z$, the centre of $G$. By $R_5(6)$, there exists a Hall subloop $H$ of order $m$ in $G$. Then $G = HZ$.

Now $G_a = (G, G, G) = (HZ, HZ, HZ) = (H, H, H) \subset H$; and $G_c = [G, G] = [HZ, HZ] = [H, H] \subset H$.

Let $h \in H$, $x, y \in G$.

Then $hT(x) = x^{-1}hx = hh^{-1}x^{-1}hx = h[h, x]$ and

$$hL(x, y) = hR(x^{-1}, y^{-1}), \quad \text{by } [1, \text{p.} \ 124, \ \text{Lemma} \ 5.4, \ (5.13)]$$

$$= h(h, y, x)^{-1}, \quad \text{by } [1, \text{p.} \ 124, \ \text{Lemma} \ 5.4, \ (5.16)].$$

Since $G_a \subset H$ and $G_c \subset H$, $h\theta \in H$ for all $\theta \in I(G)$. Thus $H \triangleleft G$ and $G = ⟨x⟩ \triangleleft H$. 


Case 2: $q \neq p$. Let $|K| = q^\gamma$. Then $|G/K| = p^\alpha \frac{m}{q^\gamma}$ where $q^\gamma \mid m$.

$2(a): m > q^\gamma$. By Lemma 3, $xK$ is an element of order $p^\alpha$ in $G/K$. By induction, there exists a normal subloop $K_1/K$ of order $m/q^\gamma$ in $G/K$. Therefore $K_1 \triangleleft G$ and $|K_1| = \frac{|K|m}{q^\gamma} = m$. Thus $G = \langle x \rangle \times K_1$.

$2(b): m = q^\gamma$. Then $G = \langle x \rangle \times K$ as required.

**Corollary 1.** Let $G$ be a Moufang loop of order $p^\alpha m$, $(p, m) = 1$, $(p - 1, p^\alpha m) = 1$ and suppose $G$ has an element of order $p^\alpha$. Then $G$ is solvable.

**Proof.** Case 1: $p = 2$. Then by Theorem 1, $G = C_{2^\alpha} \rtimes K$ with $|K| = m$ which is odd. So $G/K$ is isomorphic to $C_{2^\alpha}$ which is solvable. By $R_5(c)$, $K$ is solvable. Thus $G$ is solvable.

Case 2: $p \neq 2$. Then $|G|$ is odd as $(p - 1, p^\alpha m) = 1$. Thus $G$ is solvable by $R_5(c)$.

### V. Moufang loops of odd order $p^2 m$

**Theorem 2.** Let $G$ be a Moufang loop of odd order $p^2 m$, $(p, m) = 1$, $p$ the smallest prime dividing $|G|$. Then there exist subloops $M$ and $P$ in $G$ with $|P| = p^2$, $|M| = m$, $M \triangleleft G$ such that $G = P \rtimes M$.

**Proof.** If $G$ is a group, we are through by [10, p. 141, 6.3.16]. By $R_5(c)$, $G$ is solvable. Let $K$ be a minimal normal subloop of $G$. By $R_5(a)$, $K$ is elementary abelian. Let $|K| = q^\alpha$. Existence of $P$ is guaranteed by $R_5(d)$.

Case 1: $q \neq p$. If $|K| = m$, then $K = M$ and we are through. If $|K| < m$, then $|G/K| = p^2 (m/q^\alpha)$. By induction, there exists a normal subloop $M/K$ in $G/K$ with $|M/K| = \frac{m}{q^\alpha}$. Then $M \triangleleft G$ and $|M| = \frac{m}{q^\alpha} |K| = m$.

Case 2: $q = p$. Then by $R_5(e)$, $\alpha = 1$ or 2.

$2(a): \alpha = 1 : |K| = p$. By $R_5(d)$, we can get an element $xK$ of order $p$ in $G/K$. $|G/K| = pm$. So by Theorem 1, there exists a normal subloop $\hat{M}/K$ of order $m$ in $G/K$. Then $\hat{M} \triangleleft G$ and $|\hat{M}| = pm$. Similarly by $R_5(d)$ and by Theorem 1, there exists a subloop $M$ of order $m$ normal in $\hat{M}$. By Lemma 2(a), $M \triangleleft G$. 

2(b): \( \alpha = 2 : |K| = p^2 \). By \( R_5(a) \) and \( R_5(b) \), \( K \subset N \). Since \( K \) is an elementary abelian group, \( K = C_p \times C_p \).

Now by \( R_3(a) \), \( |G/C_G(K)| \mid \mid \text{Aut } K \mid = (p + 1)p(p - 1)^2 \) using [10, p. 141, 6.3.15]. Since \( K \subset C_G(K) \), and \( p \) is the smallest prime dividing \( |G| \), \( |G/C_G(K)| \mid \mid (p + 1) \). As \( p \) is odd and 2 does not divide the order of \( G, G = C_G(K) \). Thus \( K \subset Z \).

By \( R_5(d) \), there exists a subloop \( M \) of order \( m \) in \( G \). As \( G = KM = ZM \), it can be shown in a similar way as before (see the proof of Theorem 1, Case 1(b)) that \( M \triangleleft G \).

**Corollary 2.** Let \( G \) be a Moufang loop of odd order \( p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) where \( p_1 < p_2 < \cdots < p_m \) and \( 1 \leq \alpha_i \leq 2 \). Then there exists a subloop of order \( p_m^{\alpha_m} \) normal in \( G \).

**Proof.** For \( \alpha_1 = 1 \), \( R_5(d) \) guarantees the existence of an element of order \( p_1 \) in \( G \). So by Theorem 1 or Theorem 2, there exists \( M_1 \), a normal subloop in \( G \) with \( |M_1| = p_2^{\alpha_2} \cdots p_m^{\alpha_m} \). Again there exists a subloop \( M_2 \) of order \( p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) normal in \( M_1 \). By Lemma 2(a), \( M_2 \triangleleft G \). By this process, we get a subloop \( M_{m-1} \) of order \( p_m^{\alpha_m} \) normal in \( G \).

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(Received August 2, 1995; revised January 22, 1997)