The mean values of multiplicative functions IV

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Dedicated to Professors Imre Kátai and Zoltán Daróczy
on their 60th birthday

Abstract. The mean value theorem for the product of multiplicative functions with arguments from arithmetic progression when the variable of these progressions runs over primes is proved. This theorem is used for the investigation of the limit behaviour of a sum of additive functions.

1. Results

Let \( g_l : \mathbb{N} \rightarrow \mathbb{C}, l = 1, \ldots, s \) be multiplicative functions. Throughout the paper \( p \) and \( q \) denote primes; \( c, c_1, \ldots \) are positive constants; \( m, n, k \) are positive integers; \( a_1, \ldots, a_s \) are positive integers, also; and \( b_1, \ldots, b_s \) are integers.

In this paper we continue the investigations of publications [6], [7], [8], [9]. Let

\[ G(n) = G(n; g_1, \ldots, g_s) = g_1(a_1n + b_1) \cdots g_s(a_s n + b_s). \]

We consider the asymptotic behaviour of the sum

\[ M_x(G) = \frac{1}{\pi(x)} \sum_{p \leq x} G(p). \]

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as $x \to \infty$. This sum was earlier analyzed by I. KátaI [4].

Define the multiplicative functions $g_l$ and $g_l^{*}$, $l = 1, \ldots, s$, by

$$g_l(p^m) = \begin{cases} g_l(p^m) & \text{if } p \leq r, \\ 1 & \text{if } p > r, \end{cases}$$

and the multiplicative functions $h_l$, $h_l^{*}$, $l = 1, \ldots, s$, by means of the convolution $h_l = g_l \ast \mu$, $h_l^{*} = g_l^{*} \ast \mu$, where $\mu$ denotes the Möbius function.

Let us introduce some notations we shall use below. Let

$$(d_1, \ldots, d_k), [d_1, \ldots, d_k]$$

mean the greatest common divisor and the least common multiple of the integers $d_1, \ldots, d_k$, respectively;

$\varphi$ be the Euler function; $a = \max(a_1, \ldots, a_s)$, $b = \max(b_1, \ldots, b_s)$;

$$\Delta_{kl} = a_k b_l - a_l b_k, \quad 1 \leq k < l \leq s, \quad \Delta = \max_{1 \leq k \leq l \leq s} |\Delta_{kl}|;$$

$$w_p = \sum h_1(p^{m_1}) \cdots h_s(p^{m_s}) \varphi([p^{m_1}, \ldots, p^{m_s}]),$$

where the prime $'$ means that the summation is taken over all collections $(p^{m_1}, \ldots, p^{m_s})$ with non-negative integer exponents $m_l$, $l = 1, \ldots, s$, for which

$$(p^{m_l}, a_l) = 1, \quad (p^{m_l}, b_l) = 1, \quad l = 1, \ldots, s,$$

and $(p^{m_k}, p^{m_l}) \mid \Delta_{kl}, \quad 1 \leq l < k \leq s$;

$$P(x) = \prod_{p \leq x} w_p, \quad P(r, x) = \prod_{r < p \leq x} w_p,$$

$$S(r, x) = \sum_{l=1}^{s} \sum_{r < p \leq x} \frac{|g_l(p) - 1|^2}{p}.$$

We shall use the conditions:

$$(A) \quad \sum_{r < p \leq x} \frac{\text{Re} \left( g_1(p) + \cdots + g_s(p) \right) - s}{p} \leq C;$$

$$(B) \quad |g_l(n)| \leq \psi_l(n) \ll (\log n)^{A_l}, \quad l = 1, \ldots, s,$$
for $n \geq 2$, where $A_l$ are non-negative constants and the functions $\psi_l$ do not decrease,

$$
\Psi(n) = \psi_1(a_1n + b_1) \cdots \psi_s(a_sn + b_s), \quad A = \max(A_1, \ldots, A_s);
$$

(C)

$$
S(r, x) \leq \frac{1}{4}, \quad s \leq r, \quad s(\log r)^{A-1} \leq r, \quad \Delta \leq r, \quad \Delta_{kl} \neq 0, \quad a \leq r, \quad b_l \leq r,
$$

$$
a_l + b_l > 0, \quad (a_l, b_l) = 1, \quad b_l \neq 0, \quad a_l x + b_l \leq x^{3/2},
$$

$$
l = 1, \ldots, s, \quad 1 \leq k < l.
$$

We can formulate now our main result.

**Theorem.** Let the multiplicative functions $g_1, \ldots, g_s$ satisfy the conditions (A), (B), and (C) with some collection of required constants. Assume further that $B > 0$, $\alpha \geq \alpha_0 > 0$, $1 - 1/s < \alpha < 1$. Then for $2 \leq r \leq \sqrt{\log x}$

$$
M_x(G) - P(x)
$$

$$
\ll \left( \frac{1}{(\log x)^B} + \frac{(ax + b)^{s(1-\alpha)}}{x} \log x \right) \exp \left( \frac{csr^\alpha(\log r)^{A-1}}{(1-\alpha)^{A+1}} \right)
$$

$$
+ \Psi(x) s \left( \frac{(\log r)^{A-1}}{r} + \log \log(a|b|+2) \left( (S(r, x))^{1/4} + \frac{\log (a + b)}{\log x} \right) \right),
$$

where the constant $c$ and the one in the symbol $\ll$ may depend on $B$, on $\alpha_0$, and on the constants from (A) and (B), only.

The conditions (C) are not essential. They could be weakened, but the proof of Theorem would be more difficult.

It is easy to apply our Theorem to the functions from a set $G$ (for the definition of $G$ see [8]). For example, let

$$
A_k = \{ n \mid p^{\nu_n} \parallel n \Rightarrow m < k \}
$$

denote a set of $k$-free positive integers.
Corollary 1. For $x \geq 2$ and $s \ll (\log \log x)^{1/3}$

$$\frac{1}{\pi(x)} \sum_{p \leq x} \frac{\varphi(p+1) \ldots \varphi(p+s)}{(p+1) \ldots (p+s)} = \prod_p w_p + R(x),$$

$$\frac{1}{\pi(x)} \sum_{p \leq x} \frac{1}{p^{1+\ldots+p+s} \in A_k} 1 = \prod_p v_p + Q(x).$$

The values of $w_p$ and $v_p$ are defined by (1) and can be evaluated as follows. Let $s = \xi p + \eta$, where $\xi, \eta$ are integers for which $\xi \geq 0, 0 \leq \eta < p$. Then

$$w_p = \begin{cases} 
1 + \frac{1}{p-1} \left( \eta \left( \left(1 - \frac{1}{p}\right)^{\xi+1} - 1 \right) 
+ (p - \eta - 1) \left( \left(1 - \frac{1}{p}\right)^{\xi} - 1 \right) \right) & \text{if } p \leq s, \\
1 - \frac{s}{p(p-1)} & \text{if } p > s,
\end{cases}$$

$$v_p = \begin{cases} 
0 & \text{if } p^k \leq s, \\
1 - \frac{s - \xi}{p^{k-1}(p-1)} & \text{if } p^k > s.
\end{cases}$$

For the remainder terms we have

$$R(x) \ll \frac{s \log \log s}{(\log \log x)^{1/3}(\log \log \log x)^{3/2}},$$

$$Q(x) \ll \frac{s \log \log s}{(\log \log x)^{1/3}(\log \log \log x)^{2/3}}.$$
Let further the values \( s, a_l, b_l, l = 1, \ldots, s \), not depend on \( x \). Denote

\[
(3) \quad \sum_{|f_l(p)| \leq 1} \frac{f_l^2(p)}{p}, \quad l = 1, \ldots, s, \\
(4) \quad \sum_{|f_l(p)| > 1} \frac{1}{p}, \quad l = 1, \ldots, s, \\
(5) \quad \sum_{|f_l(p)| \leq 1, |f_s(p)| \leq 1} \frac{f_1(p) + \cdots + f_s(p)}{p}.
\]

Let also be

\[
(6) \quad a_l + b_l > 0, \ (a_l, b_l) = 1, \ b_l \neq 0, \ l = 1, \ldots, s; \\
\Delta_{jk} \neq 0, \ 1 \leq j < k \leq s.
\]

**Corollary 2.** Let \( f_l, l = 1, \ldots, s \), be real-valued additive functions, let the series (3), (4), (5) converge, and let the conditions (6) be fulfilled. Then the distribution functions

\[
\frac{1}{\pi(x)} \# \left\{ p \mid p \leq x, \ f_1(a_1 p + b_1) + \cdots + f_s(a_s p + b_s) \leq z \right\}
\]

converge weakly towards a limit distribution as \( x \to \infty \), and the characteristic function of this limit distribution is equal to

\[
\prod_p w_p,
\]

where \( w_p \) is defined by (1) with \( g_l = e^{itf_l}, l = 1, \ldots, s \).

Corollaries 1 and 2 clearly imply

**Corollary 3.** Let \( s \) not depend on \( x \). Then the distribution functions

\[
\frac{1}{\pi(x)} \# \left\{ p \mid p \leq x, \ \frac{\varphi(p+1) \cdots \varphi(p+s)}{(p+1) \cdots (p+s)} \leq e^z \right\}
\]

converge weakly towards a limit distribution, as \( x \to \infty \). The characteristic function of this limit distribution is

\[
\prod_p w_p,
\]
with

\[
    w_p = \begin{cases}
        1 + \frac{1}{p-1} \left( \eta \left( \left( 1 - \frac{1}{p} \right)^{it(x+1)} - 1 \right) \right) & \text{if } p \leq s, \\
        +(p - \eta - 1) \left( \left( 1 - \frac{1}{p} \right)^{it} - 1 \right) & \text{if } p > s,
    \end{cases}
\]

where \( \xi \) and \( \eta \) are defined in Corollary 1.

In the following corollary we give an example of a sum of additive arithmetical functions with shifted arguments which is uniformly distributed mod 1 on the set of primes. The additive functions may depend on \( x \).

Let

\[
    f_1(a_1n + b_1) + \cdots + f_s(a_s n + b_s) = \mathcal{F}(n).
\]

We say that \( \mathcal{F}(n) \) is asymptotically uniformly distributed mod 1 on the set of primes if

\[
    \frac{1}{\pi(x)} \# \{ p \mid p \leq x, \text{ fractional part of } \mathcal{F}(p) \in [\alpha, \beta) \} \rightarrow \beta - \alpha
\]

as \( x \to \infty \), for every \( \alpha, \beta, 0 \leq \alpha < \beta \leq 1 \).

**Corollary 4.** Let real-valued additive arithmetical functions \( f_1, \ldots, f_s \) be such, that the following conditions are satisfied:

(a) \( f_l(p) \to 0 \) as \( p \to \infty \), for \( l = 1, \ldots, s \);

(b) there exists \( r, 2 \leq r \leq \log \log x \), such that

\[
    \sum_{r < p \leq x} f_l^2(p) \to 0
\]

as \( x \to \infty \), for \( l = 1, \ldots, s \);

(c) at least one of the functions \( f_k \) satisfies

\[
    \sum_{p \leq x} \frac{f_k^2(p)}{p} \to \infty, \quad x \to \infty.
\]

Let the conditions (6) also be fulfilled. Then the sum \( \mathcal{F}(n) \) is asymptotically uniformly distributed mod 1 on the set of primes.
For example, let
\[ f_l(p) = \frac{1}{\log p}, \quad l = 1, \ldots, s - 1, \]
\[ f_s(p) = \begin{cases} 
\frac{1}{\log p} & \text{if } p > \log \log x, \\
\frac{1}{(\log \log p)^{1/2}} & \text{if } p \leq \log \log x.
\end{cases} \]

Then the sum \( F(n) \) is asymptotically uniformly distributed mod 1 on the set of primes.

Finally we formulate a local limit law for the sum of additive integer-valued functions.

Let \( \lambda_k, k \in \mathbb{Z} \), be defined by means of the equality
\[ \sum_{k=-\infty}^{+\infty} \lambda_k e^{itk} = \prod_p w_p \quad \text{for all } t \in \mathbb{R}, \]
where \( w_p = w_p(t) \) are determined by (1) with \( g_l = e^{itf_l} \). Let us put also
\[ \varepsilon(x) = \left( \sum_{l=1}^{s} \sum_{p > \log \log x \atop f_l(p) \neq 0} \frac{1}{p} \right)^{1/4}. \]

**Corollary 5.** Let the integer-valued additive arithmetical functions \( f_1, \ldots, f_s \) be such, that the series
\[ \sum_{f_l(p) \neq 0} \frac{1}{p}, \quad l = 1, \ldots, s, \]
converge and the conditions (6) are fulfilled. Then
\[ \frac{1}{\pi(x)} \# \{ p \leq x \mid F(p) = k \} = \lambda_k + O \left( \varepsilon(x) + \frac{1}{\log \log x} \right) \]
uniformly for all \( k \in \mathbb{Z} \) and \( x \geq 2 \).
2. Auxiliary lemmas

We shall use some estimates and statements which we formulate as lemmas.

Lemma 1 (Turán–Kubilius). Let \( f(p) \) be complex numbers, \( \xi \in \mathbb{N} \), \( \eta \in \mathbb{Z} \), \( \eta \neq 0 \), \( (\xi, \eta) = 1 \), \( \xi + \eta > 0 \), \( \xi x + \eta \leq x^K \), \( 0 < \beta < 1 \). Then for \( x \geq 2 \)
\[
\frac{1}{\pi(x)} \sum_{p \leq x} \left| \sum_{q \leq x^\beta \atop q \nmid \xi} f(q) \right|^2 \leq \sum_{q \leq x^\beta \atop q \nmid \xi} \frac{|f(q)|^2}{q},
\]
where the constant in the symbol \( \ll \) depends only on \( \beta \) and \( K \).

For the proof of Lemma 1, it is enough to repeat the proof of Lemma 4.12 from [2] and to satisfy ourselves that the conditions of Lemma 1 guarantee the uniformity of our estimate with respect to \( f \), \( x \), \( \xi \), and \( \eta \).

Consider further a system of congruences
\[
(7) \quad a_l n + b_l \equiv 0 \pmod{d_l}, \quad (a_l, b_l) = 1, \quad l = 1, \ldots, s.
\]

The Chinese residue theorem includes as a special case the following

Lemma 2. The system of congruences (7) has a solution if and only if \( (a_l, d_l) = 1 \), \( l = 1, \ldots, s \), and
\[
(d_k, d_l) \mid (a_k b_l - a_l b_k), \quad 1 \leq k < l \leq s.
\]
If the solution exists, it is exactly one residue class modulo \([d_1, \ldots, d_s]\).

Lemma 3 ([9, Lemma 3]). Assume \( \gamma \geq 0 \). Then uniformly in \( u \), \( 0 \leq u < 1 \),
\[
\sum_{n=1}^{\infty} u^n n^\gamma \ll \frac{u}{(1 - u)^{\gamma + 1}}.
\]

Lemma 4 ([9, Lemma 4]). Assume \( c_1 \leq \gamma \leq 1 \). Then uniformly in \( \gamma \) and \( u \geq 2 \)
\[
\sum_{p \leq u} \frac{1}{p^{1-\gamma}} \ll \frac{u^\gamma}{\log u}.
\]
Lemma 5. Assume $\gamma > 0$, $\beta \in \mathbb{R}$. Then uniformly for all $u \geq 2$

$$
\sum_{p > u} \frac{(\log p)^\beta}{p^{1+\gamma}} \ll \frac{(\log u)^\beta}{u^\gamma \log u}.
$$

Proof. The assertion of the lemma follows from the following inequality. Let $\delta > 0$. Then

$$
\sum_{p > u} \frac{1}{p^{1+\delta}} = \sum_{k=1}^\infty \sum_{2^{k-1}u < p \leq 2^k u} \frac{1}{p^{1+\delta}}
\ll \sum_{k=1}^\infty \frac{1}{(2^{k-1}u)^{1+\delta}} \log(2^{k-1}u) \sum_{2^{k-1}u < p \leq 2^k u} \frac{\log p}{p}
\ll \frac{1}{u^\delta \log u} \sum_{k=1}^\infty \frac{1}{2^{(k-1)\delta}} (\log 2 + O(1)) \ll \frac{1}{u^\delta \log u}.
$$

Lemma 6 (Brun–Titchmarsh, [5, Ch. 5, Theorem 2.1]). Let $\gamma$ be a real number, $0 < \gamma < 1$. Then the inequality

$$
\pi(x, d, v) \ll \frac{x}{\varphi(d) \log x}
$$

holds uniformly for all $x \geq 2$ and integers $v$ and $d$ with $1 \leq d \leq x^\gamma$.

Lemma 7 ([3, Theorem 3.7]). The inequality

$$
\pi(x, d, v) \ll \frac{x}{\varphi(d) \log \frac{x}{d}}
$$

holds uniformly for all $x \geq 2$ and integers $d$ and $v$ with $1 \leq d < x$, $(v, d) = 1$.

Lemma 8 ([5, Ch. 4, Theorem 7.5]). There is a positive constant $c_2$, such that uniformly for all $x \geq 2$

$$
\pi(x) - \text{li} x \ll x e^{-c_2 \sqrt{\log x}}.
$$
Lemma 9 (Bombieri [1]). Let $K$ be a positive real number. Then there exists a further real number $L$, such that uniformly for all $x \geq 2$

$$
\sum_{d \leq x^{1/2} (\log x)^{-1}} \max_{(d,v) = 1} \left| \pi(x, d, v) - \frac{\text{li} x}{\varphi(d)} \right| \ll \frac{x}{(\log x)^K}.
$$

Lemma 10 ([5, Ch. 2, Theorem 4.2]). Let $k_1, k_2$ be positive integers, $l_1, l_2$ be integers, $(k_j, l_j) = 1$ for $j = 1, 2$, $k_1 l_2 - k_2 l_1 \neq 0$. Then

$$
\# \{ n \mid n \leq x, k_j n + l_j \text{ are primes for } j = 1, 2 \} \ll \frac{x}{\log^2 x} \prod_{p \mid k_1 k_2 (k_1 l_2 - k_2 l_1)} \left( 1 - \frac{1}{p} \right)^{-1}
$$

uniformly in $x \geq 2$, $k_j, l_j, j = 1, 2$.

Lemma 11. Let $k$ and $d$ be positive integers, $(k, d) = 1$, $v$ be an integer, $v \neq 0$. Then

$$
\# \{ (p, q) \mid p \leq x, kq - dp = v \} \ll \frac{x}{\varphi(k) \log^2 x} \prod_{p \mid d} \left( 1 - \frac{1}{p} \right)^{-1}
$$

uniformly in $x \geq 2$, $k$, $d$, and $v$.

**Proof.** It follows from the condition $(k, d) = 1$ that the equation

(8)

$$
km - dn = v
$$

has solutions in positive integers $m$ and $n$. All these solutions have the form

$$
m = m_0 + dt, \quad n = n_0 + kt, \quad t = t_0, t_0 + 1, \ldots,
$$

where $m_0, n_0$ is the integer solution of the equation (8),

$$
km_0 - dn_0 = v, \quad 0 \leq n_0 < k,
$$

and $t_0$ is the least positive integer for which $m_0 + dt_0 > 0$. The number of such pairs $(m, n)$, where both components are primes $(p, q)$, $p \leq x$, can be
estimated by Lemma 10. Thus

\[ \# \{ (p, q) \mid p \leq x, kq - ap = v \} \]

\[ = \# \left\{ t \mid t \leq \frac{x - n_0}{k}, m_0 + dt \text{ and } n_0 + kt \text{ are primes} \right\} \]

\[ \ll \frac{x}{\varphi(k) \log^2 \frac{x}{k}} \prod_{p \mid d} \left( 1 - \frac{1}{p} \right)^{-1} \]

and Lemma 11 is proved.

3. Proofs

Proof of the Theorem. Let \( l \) belong to the set \( \{1, \ldots, s\} \). The following estimates will be used later. We obtain from (B), Lemma 3 and Lemma 4, that

\[ \sum_{d > x} \frac{|h_{lr}(d)|}{\varphi(d)} \leq \frac{1}{z^\alpha} \sum_{d=1}^{\infty} \frac{|h_{lr}(d)|d^\alpha}{\varphi(d)} = \frac{1}{z^\alpha} \prod_{p \leq r} \left( 1 + \sum_{m=1}^{\infty} \frac{|h_{lr}(p^m)|p^{\alpha m}}{\varphi(p^m)} \right) \]

\[ \leq \frac{1}{z^\alpha} \prod_{p \leq r} \left( 1 + \frac{c_3 (\log p)^{A_l}}{p^{1-\alpha}} \left( 1 - \frac{1}{p^{1-\alpha}} \right)^{-A_l-1} \right) \]

\[ \leq \frac{1}{z^\alpha} \prod_{p \leq r} \left( 1 + \frac{c_4 (\log p)^{A_l}}{(1-\alpha)^{A_l+1}p^{1-\alpha}} \right) \]

\[ \leq \frac{1}{z^\alpha} \exp \left( \frac{c_5 (\log p)^{A_l}}{(1-\alpha)^{A_l+1}} \sum_{p \leq r} \frac{(\log p)^{A_l}}{p^{1-\alpha}} \right) \]

\[ \leq \frac{1}{z^\alpha} \exp \left( \frac{c_6 r^\alpha (\log r)^{A_l-1}}{(1-\alpha)^{A_l+1}} \right), \]

where the constant \( c_6 \) does not depend on \( \alpha \). Analogously

\[ \sum_{d=1}^{\infty} \frac{|h_{lr}(d)|}{\varphi(d)} \leq \exp \left( c_7 (\log r)^{A_l} \log \log r \right) \]
and

\[ \sum_{d=1}^{\infty} \frac{|h_{ir}(d)|}{(\varphi(d))^{1-\alpha}} \leq \exp \left( c_8 r^\alpha (\log r)^{A_1 - 1} \right). \]  

Split the left-hand side of (2) in the following way:

\[ M_x(G) - P(x) = P(r, x)(M_x(G_r) - P(r)) \]
\[ + \frac{1}{\pi(x)} \sum_{p \leq x} G_r(p)(G_r^*(p) - P(r, x)), \]

where

\[ G_r(n) = G(n; g_{1r}, \ldots, g_{sr}) \quad \text{and} \quad G_r^*(n) = G(n; g_{1r}^*, \ldots, g_{sr}^*). \]

Let \( \omega(n) \) mean the number of distinct prime divisors of the number \( n \) and

\[ M_{x0}(G_r) = \frac{1}{\pi(x)} \sum_{p \leq x, (p, b_l) = 1, \forall l} G_r(p). \]

The value of \( M_x(G_r) \) can be written in the form

\[ M_x(G_r) = M_{x0}(G_r) + O \left( \frac{\Psi(x)}{\pi(x)} (\omega(|b_1|) + \cdots + \omega(|b_s|)) \right) \]
\[ = M_{x0}(G_r) + O \left( \frac{\Psi(x)}{\pi(x)} (|b_1| + \cdots + |b_s|) \right) = M_{x0}(G_r) + O \left( \frac{1}{\sqrt{x}} \right). \]

Put

\[ R_1 = |M_{x0}(G_r) - P(r)|, \]
\[ R_2 = \left| \frac{1}{\pi(x)} \sum_{p \leq x} G_r(p)(G_r^*(p) - P(r, x)) \right|. \]

Then evidently

\[ (12) \quad M_x(G) - P(x) \ll \left| P(r, x) \right| \left( R_1 + \frac{1}{\sqrt{x}} \right) + R_2. \]
From the definition of $h_{lr}$ and then from Lemma 2, we obtain that

$$M_{x0}(G_r) = \frac{1}{\pi(x)} \sum_{p \leq x} \sum_{d_1 \mid a_1p + b_1} \cdots \sum_{d_s \mid a_sp + b_s} h_{1r}(d_1) \cdots h_{sr}(d_s)$$

$$= \frac{1}{\pi(x)} \sum_{d_1 \leq a_1x + b_1} \cdots \sum_{d_s \leq a_sp + b_s} h_{1r}(d_1) \cdots h_{sr}(d_s) \sum_{p \leq x} 1_{(p, b_l) = 1, \forall l \mid d_l \mid a_l, p + b_l, \forall l}$$

$$= \frac{1}{\pi(x)} \sum''_{d_l \leq a_lx + b_l, \forall l} h_{1r}(d_1) \cdots h_{sr}(d_s)\pi(x, [d_1, \ldots, d_s], v),$$

where the double prime $''$ means that the summation in taken over all vectors $(d_1, \ldots, d_s) \in \mathbb{N}^s$ for which $(d_l, a_l) = 1$, $(d_l, b_l) = 1$, $\forall l$, and $(d_k, d_l)|\Delta_{kl}$, $1 \leq k < l \leq s$, and where $v$ is the only integer for which

$$a_lv + b_l \equiv 0 \mod d_l, \ \forall l,$$

and $0 \leq v \leq [d_1, \ldots, d_s] - 1$. It is clear also, that $(v, [d_1, \ldots, d_s]) = 1$.

Since

$$P(r) = \sum'' h_{1r}(d_1) \cdots h_{sr}(d_s)\phi([d_1, \ldots, d_s]),$$

we can write

$$R_1 \leq R_{11} + R_{12} + R_{13} + R_{14},$$

where

$$R_{11} = \frac{1}{\pi(x)} \sum''_{d_l \leq x, \forall l} |h_{1r}(d_1) \cdots h_{sr}(d_s)|$$

$$\times \left| \pi(x, [d_1, \ldots, d_s], v) \frac{\text{li} x}{\phi([d_1, \ldots, d_s])} \right|,$$

$$R_{12} = \frac{1}{\pi(x)} \sum''_{d_l \leq x, \forall l} |h_{1r}(d_1) \cdots h_{sr}(d_s)| \left| \text{li} x - \pi(x) \right|,$$

$$R_{13} = \sum_{l=1}^s \sum_{d_l > z}'' |h_{1r}(d_1) \cdots h_{sr}(d_s)| \frac{\phi([d_1, \ldots, d_s])}{\varphi([d_1, \ldots, d_s])},$$
and

\[ R_{14} = \frac{1}{\pi(x)} \sum_{k=1}^{s} \sum'_{d_l \leq a_l x + b_l, \forall l, d_s > z} |h_{1r}(d_1) \ldots h_{sr}(d_s)| \pi(x, [d_1, \ldots, d_s], v) \]

with some \( z, z^s \leq x^{1/3} \), which we shall choose later.

By Lemma 6

\[ R_{11} \ll \frac{1}{\pi(x)} \max_{d \leq z^s} \left| \pi(x, d, v) - \frac{\text{li} x}{\varphi(d)} \right|^\alpha \left( \frac{x}{\log x} \right)^{1-\alpha} \]

\[ \times \sum'_{d_l \leq z, \forall l} \left| h_{1r}(d_1) \ldots h_{sr}(d_s) \right| \left( \varphi([d_1, \ldots, d_s]) \right)^{1-\alpha}, \]

and by Lemma 9 it is

\[ \ll \frac{1}{(\log x)^{(K-1)\alpha}} \sum'_{d_l \leq z, \forall l} \left( \prod_{1 \leq j < k \leq s} (d_j, d_k) \right)^{1-\alpha} \frac{|h_{1r}(d_1) \ldots h_{sr}(d_s)|}{(\varphi(d_1, \ldots, d_s))^{1-\alpha}} \]

\[ \ll \frac{1}{(\log x)^{(K-1)\alpha}} \left( \prod_{1 \leq j < k \leq s} \Delta_{jk} \right)^{1-\alpha} \sum_{d_1=1}^{\infty} \frac{|h_{1r}(d_1)|}{(\varphi(d_1))^{1-\alpha}} \ldots \sum_{d_s=1}^{\infty} \frac{|h_{sr}(d_s)|}{(\varphi(d_s))^{1-\alpha}}. \]

Keeping in mind that

\[ \prod_{1 \leq j < k \leq s} |\Delta_{jk}| \ll \Delta^{s(s-1)/2} \]

and using also the inequality (11), we obtain that

\[ R_{11} \ll \frac{1}{(\log x)^{(K-1)\alpha}} \Delta^{s(s-1)/2} \exp \left( \frac{c_9 sr^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}} \right) \]

\[ \ll \frac{1}{(\log x)^{(K-1)\alpha}} \exp \left( \frac{c_{10} sr^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}} \right). \]

Using the same ideas as in the estimation of \( R_{11} \), from Lemma 8 and the inequality (10) we deduce that

\[ R_{12} \ll \exp(-c_{11} \sqrt{\log x}) \Delta^{s(s-1)/2} \exp \left( c_{12} s (\log r)^A \log \log r \right) \]
and from the inequalities (10) and (9) that

\[ R_{13} \ll \Delta^{s(s-1)/2} \exp \left( c_{13} s \log r \right)^A \log \log r \sum_{l=1}^s \sum_{d_l > z} \frac{|h_{1l}(d_l)|}{\varphi(d_l)} \]
\[ \ll \Delta^{s(s-1)/2} \exp \left( c_{13} s \log r \right)^A \log \log r \sum_{l=1}^s \sum_{d_l > z} \frac{1}{z^\alpha} \exp \left( \frac{c_{14} r^\alpha (\log r)^{A-1}}{(1 - \alpha)^{A+1}} \right) \]
\[ \ll \frac{1}{z^\alpha} \exp \left( \frac{c_{15} r^\alpha (\log r)^{A-1}}{(1 - \alpha)^{A+1}} \right). \]

Analogously

\[ R_{14} \ll \frac{1}{\pi(x)} \sum_{k=1}^s \sum_{d_k \leq a x + b, \forall t \in d_k > z} |h_{1r}(d_1) \ldots h_{sr}(d_s)| \left( \frac{x}{[d_1, \ldots, d_s] + 1} + 1 \right) \]
\[ \ll \left( \frac{1}{z^\alpha} + \frac{(ax + b)^s(1 - \alpha)}{x} \right) \log x \exp \left( \frac{c_{16} r^\alpha (\log r)^{A-1}}{(1 - \alpha)^{A+1}} \right). \]

Let us choose \( z = (\log x)^{c_{17}} \) with sufficiently large constant \( c_{17} \). Then the estimates of \( R_{11}, R_{12}, R_{13}, \) and \( R_{14} \) imply that

\[ R_1 \ll \left( \frac{1}{(\log x)^B} + \frac{(ax + b)^s(1 - \alpha)}{x} \right) \log x \exp \left( \frac{c s r^\alpha (\log r)^{A-1}}{(1 - \alpha)^{A+1}} \right). \]

If \( p > r \), then

\[ w_p - 1 = \sum_{m=1}^\infty \frac{h_1(p^m) + \cdots + h_s(p^m)}{\varphi(p^m)} \]

and it follows from (B), (C) and Lemma 3, that

\[ |w_p - 1| \leq \frac{p}{p - 1} \left( \frac{(|h_1(p)| + \cdots + |h_s(p)|)^2}{p} \right)^{1/2} \frac{1}{p^{1/2}} + \frac{c_{18} s (\log p)^A}{p^2} \]
\[ \leq \frac{p}{p - 1} \left( \frac{s S(r, x)}{p} \right)^{1/2} + \frac{c_{18} s (\log p)^A}{p^2} \leq \frac{p}{p - 1} \frac{1}{2} + \frac{c_{18} (\log p)^A}{p}. \]

Thereby it is clear that

\[ |w_p - 1| \leq \frac{3}{4} \]

(14)
if \( p > p_0 \), where \( p_0 \) is large enough and depends only on the constants from (B).

Without loss of generality we may assume that \( r \geq p_0 \). We estimate the value of \( P(r, x) \) using (14), Lemma 3, Lemma 5 and the conditions (A), (B) and (C). Thus

\[
P(r, x) = \exp \left\{ \sum_{r < p \leq x} \log \left( 1 + \sum_{m=1}^{\infty} \frac{h_1(p^m) + \cdots + h_s(p^m)}{(\varphi(p))^m} \right) \right\}
\]

\[
= \exp \left\{ \sum_{r < p \leq x} \left( \frac{h_1(p) + \cdots + h_s(p)}{p} \right) + O \left( \frac{s(\log p)^A}{p^2} \right) \right\}
\]

\[
+ O \left( \frac{s |h_1(p)|^2 + \cdots + |h_s(p)|^2}{p} \right) \right\}
\]

\[
= \exp \left\{ \sum_{r < p \leq x} \frac{h_1(p) + \cdots + h_s(p)}{p} \right\}
\]

\[
+ O \left( \frac{s(\log r)^{A-1}}{r} + S(r, x) \right) \right\} \ll 1.
\]

Put

\[
P_{r1} = \{ p \mid p \leq x, \exists q > r \text{ and } \exists l, \text{ that } q \mid a_lp + b_l \text{ and } |h_l(p)| > \frac{1}{2} \},
\]

\[
P_{r2} = \{ p \mid p \leq x, \exists q > r \text{ and } \exists l, \text{ that } q^2 \mid a_lp + b_l \} \setminus P_{r1},
\]

\[
P_{r3} = \{ p \mid p \leq x \} \setminus (P_{r1} \cup P_{r2}).
\]

Split further \( R_2 \) into three sums \( R_{21}, R_{22}, R_{23} \) over \( p \) from \( P_{r1}, P_{r2}, P_{r3} \), respectively.

It follows from (B) and (15) that

\[
R_{21} \ll \frac{\Psi(x)}{\pi(x)} \sum_{p \in P_{r1}} 1 \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{1 < q \leq a_l x + b_l} \sum_{|h_l(q)| > 1/2} \sum_{(a_l p + b_l) \equiv 0 \mod q} 1.
\]

Let

\[
y = \exp \left( 1 - (S(r, x)^{1/2}) \right) \log x.
\]
Split the right-hand side of (16) into two sums $R_{211}$ and $R_{212}$ including terms for which $r < q \leq y$ and $y < q \leq a_l x + b_l$, respectively. Then

\[ R_{211} \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{r < q \leq y} \max_{1 \leq v < q} \pi(x, q, v). \]

By Lemma 7

\[ R_{211} \ll \frac{\Psi(x)}{\pi(x)} \frac{x}{q} \sum_{l=1}^{s} \sum_{r < q \leq y} \frac{1}{q} \sum_{|h_l(q)| > 1/2} |h_l(q)|^2 \ll \Psi(x)(S(r, x))^{1/2}. \]

Changing the order of summation in the expression $R_{212}$, we have

\[ R_{212} \ll \frac{\Psi(x)}{\pi(x)} \frac{x}{y} \sum_{l=1}^{s} \sum_{1 \leq k \leq a_l x + b_l} \sum_{y < q \leq a_l x + b_l} \sum_{p \leq x} 1. \]

(17) \[ = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{1 \leq k \leq a_l x + b_l} \sum_{y < q \leq a_l x + b_l} \sum_{p \leq x} 1. \]

Observing now that the inner sum is empty when $(a_l, k) \neq 1$ and using Lemma 11 and a few well-known estimates, we get that

\[ R_{212} \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{1 \leq k \leq a_l x + b_l} \frac{x}{\varphi(k)} \log^2 \frac{x}{k} \varphi(a_l |b_l|) \]

(18) \[ \ll \frac{\Psi(x)}{\pi(x)} s \log \log(a |b| + 2) \frac{x}{\log^2 \frac{ax + b}{y}} \sum_{1 \leq k \leq \frac{ax + b}{y}} \frac{1}{\varphi(k)} \]

\[ \ll \Psi(x) s \log \log(a |b| + 2) \log x \frac{x}{\log^2 \frac{ax + b}{y}} \log \frac{ax + b}{y}. \]
Since
\[
\log \frac{xy}{ax + b} = \log y - \log \left( a + \frac{b}{x} \right) \geq c_{19} \log y \geq \frac{c_{19}}{2} \log x
\]
with sufficiently small positive constant \( c_{19} \), we obtain
\[
R_{212} \ll \Psi(x) s \log \log(a|b| + 2) \left( \frac{\log \left( a + \frac{b}{x} \right)}{\log x} + (S(r, x))^{1/2} \right).
\]
We have now that
\[
(19) \quad R_{21} \ll \Psi(x) s \log \log(a|b| + 2) \left( \frac{\log \left( a + \frac{b}{x} \right)}{\log x} + (S(r, x))^{1/2} \right).
\]
The value of \( R_{22} \) does not exceed
\[
\frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{r \leq q \leq (a_{l}x + b_{l})^{1/2}} \sum_{p \leq x \atop a_{l}p + b_{l} \equiv 0 \mod q^{2}} 1
\leq \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{r \leq q \leq (a_{l}x + b_{l})^{1/2}} \max_{1 \leq v \leq q^{2}} \pi(x, q^{2}, v).
\]
For the estimation of \( \pi(x, q^{2}, v) \) in the range \( r < q \leq x^{1/4} \), we use Lemma 7. In the range \( x^{1/4} < q \leq (a_{l}x + b_{l})^{1/2} \), we estimate \( \pi(x, q^{2}, v) \) trivially. Thus
\[
R_{22} \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \left( \sum_{r \leq q \leq x^{1/4}} \frac{x}{q^{2}} \log \frac{x}{q^{2}} + \sum_{x^{1/4} \leq q \leq x^{1/2}} \frac{x}{q^{2}} + \sum_{x^{1/2} \leq q \leq (a_{l}x + b_{l})^{1/2}} 1 \right).
\]
Now Lemma 5 implies that
\[
(20) \quad R_{23} \ll \Psi(x) \left( (r \log r)^{-1} + x^{-1/4} + \frac{(ax + b)^{1/2}}{x} \right).
\]
For the estimation of \( R_{23} \) we use the inequality
\[
e^{u} - e^{v} \ll |u - v| (|e^{u}| + |e^{v}|)
\]
true for all \( u, v \in \mathbb{C} \). Therefore
\[
(21) \quad R_{23} \ll \frac{\Psi(x)}{\pi(x)} \sum_{p \in F_{r}\cdot} |\log G_{r}^{*}(p) - P(r, x)| \ll R_{231} + R_{232} + R_{233},
\]
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where

$$R_{231} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \in P_{r^3}} \left| \sum_{q \mid a_{1}p + b_{l} \quad q > r} h_{l}(q) - \sum_{r < q \leq \sqrt{x}} \frac{h_{l}(q)}{q} \right|,$$

$$R_{232} = \frac{\Psi(x)}{\pi(x)} \sum_{p \in P_{r^3}} \left| \sum_{r < q \leq x} \frac{h_{1}(q) + \cdots + h_{s}(q)}{q} - \log P(r, x) \right|,$$

$$R_{233} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \in P_{r^3}} \sum_{q \mid a_{1}p + b_{l} \quad q > r} |h_{l}(q)|^2.$$

The sum $R_{231}$ does not exceed

$$R_{2311} + R_{2312} + R_{2313} + R_{2314},$$

where

$$R_{2311} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \leq x} \left| \sum_{q \mid a_{1}p + b_{l} \quad r < q \leq \sqrt{x}} h_{l}(q) - \sum_{r < q \leq \sqrt{x}} \frac{h_{l}(q)}{q} \right|,$$

$$R_{2312} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \leq x} \sum_{q \mid a_{1}p + b_{l} \quad \sqrt{x} < q \leq x^{1-\delta}} |h_{l}(q)|,$$

$$R_{2313} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \leq x} \sum_{q \mid a_{1}p + b_{l} \quad q > x^{1-\delta}} 1,$$

$$R_{2314} = \Psi(x) \sum_{l=1}^{s} \sum_{\sqrt{x} < q \leq x} \frac{|h_{l}(q)|}{q}$$

with $\delta = \left( S(\sqrt{x}, x) \right)^{1/4}$.

By the Cauchy inequality, the inner sum from $R_{231}$ does not exceed

$$\left( \sum_{p \leq x} \left| \sum_{q \mid a_{1}p + b_{l} \quad r < q \leq \sqrt{x}} h_{l}(p) - \sum_{r < q \leq \sqrt{x}} \frac{h_{l}(q)}{q} \right|^2 \right)^{1/2} \left( \sum_{p \leq x} 1 \right)^{1/2}.$$
and then by Lemma 1, it is
\[ \ll \pi(x) \left( \sum_{r < q \leq \sqrt{x}} \frac{|h_l(q)|^2}{q} \right)^{1/2}. \]

Therefore
\[ R_{2311} \ll \Psi(x) \left( s S(r, \sqrt{x}) \right)^{1/2}. \]

Let us estimate \( R_{2312} \). Changing the order of summation, we obtain that
\[ R_{2312} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{\sqrt{x} < q \leq x^{1-\delta}} |h_l(q)| \sum_{p \leq x, ap + b \equiv 0 \mod q} 1. \]

By Lemma 7, the inner sum is
\[ \ll \frac{x}{q \log \frac{x}{q}} \ll \frac{x}{\delta q \log x}. \]

It follows now from the Cauchy inequality that
\[ R_{2312} \ll \frac{\Psi(x)}{\delta} \left( \sum_{\sqrt{x} < q \leq x^{1-\delta}} \left( \frac{|h_1(q)| + \cdots + |h_s(q)|}{q} \right)^2 \right)^{1/2} \left( \sum_{\sqrt{x} < q \leq x^{1-\delta}} \frac{1}{q} \right)^{1/2} \]
\[ \ll \Psi(x) \sqrt{x} \left( S(\sqrt{x}, x) \right)^{1/4}. \]

Similarly as in (17) and (18) (the only difference is that in place of \( y \) we take \( x^{1-\delta} \)), we have
\[ R_{2313} \ll \Psi(x) s \log \log(a|b|+2) \log x \frac{1}{x^{2-\delta}} \log \frac{ax + b}{x^{1-\delta}} \]
\[ \ll \Psi(x) s \log \log(a|b|+2) \left( (S(\sqrt{x}, x))^{1/4} + \log \left( \frac{a + b}{\sqrt{x}} \right) \right). \]

In the same way as in the estimation of \( R_{2312} \), it follows from the Cauchy inequality that
\[ R_{2314} \ll \Psi(x) \left( s S(\sqrt{x}, x) \right)^{1/2}, \]
and therefore

\[
R_{231} \ll \Psi(x)s \log \log (a|b| + 2)
\]

\[
\times \left( (S(r, \sqrt{x}))^{1/2} + (S(\sqrt{x}, x))^{1/4} + \frac{\log (a + \frac{b}{x})}{\log x} \right).
\]

Using (15), we easily get that

\[
R_{232} \ll \Psi(x) \left( \frac{s(\log r)^{A-1}}{r} + S(r, x) \right).
\]

Analogously as in the estimation of \(R_{21}\), we obtain that

\[
R_{233} \ll \Psi(x)s \log \log (a|b| + 2) \left( (S(r, x))^{1/2} + \frac{\log (a + \frac{b}{x})}{\log x} \right).
\]

Collecting the latter estimates into (21), we get the estimate of \(R_{23}\). Then it follows from (19), (20), and (21) that

\[
R_2 \ll \Psi(x)s \left( \frac{(\log r)^{A-1}}{r} + \log \log (a|b| + 2)
\]

\[
\times \left( (S(r, x))^{1/2} + (S(\sqrt{x}, x))^{1/4} + \frac{\log (a + \frac{b}{x})}{\log x} \right) \right).
\]

Finally, putting (13) and (22) into (12) and remembering (15), we obtain that

\[
M_x(G) - P(x) \ll \left( \frac{1}{(\log x)^{\alpha}} + \frac{(ax + b)^{s(1-\alpha)}}{x} \log x \right)
\]

\[
\times \exp \left( csr^\alpha (\log r)^{A-1} \frac{1}{(1 - \alpha)^{A+1}} \right)
\]

\[
+ \Psi(x)s \left( \frac{(\log r)^{A-1}}{r} + \log \log (a|b| + 2) \left( (S(r, x))^{1/2}
\]

\[
+ (S(\sqrt{x}, x))^{1/4} + \frac{\log (a + \frac{b}{x})}{\log x} \right) \right),
\]
and our Theorem is proved.

**Proof** of Corollary 1. In case when \( g_l(n) = \varphi(n)/n \), we have \( h_1(p) = h(p) = -1/p \) and \( h_1(p^m) = 0 \) for \( m \geq 2 \). Hence

\[
wp = \frac{1}{\varphi\left(\sum_{m_1=0}^{1} \ldots \sum_{m_s=0}^{1} h(p^{m_1} \ldots h(p^{m_s})\right)}.
\]

If \( p > s \), it is clear that

\[
w_p = 1 + \frac{s}{p(p - 1)}.
\]

Let \( p \leq s \). Split the numbers 1, \ldots, \( s \) into residue classes mod \( p \). There are \( p - 1 \) residue classes the members of which are coprime to \( p \). Among these classes there are \( \eta \) residue classes with \( \xi + 1 \) members and \( p - \eta - 1 \) residue classes with \( \xi \) members. Let us observe also that \( p|(k - j) \) only if \( j \) and \( k \) belong to the same residue class mod \( p \). Therefore

\[
w_p = 1 + \frac{1}{\varphi(p)} \left( \eta C_{\xi+1}^1 + (p - \eta - 1)C_\xi^1 \right) h(p) + \left( \eta C_{\xi+1}^2 + (p - \eta - 1)C_\xi^2 \right) h^2(p)
\]

\[
+ \cdots + \left( \eta C_{\xi+1}^\xi + (p - \eta - 1)C_\xi^\xi \right) h^\xi(p) + \eta C_{\xi+1}^{\xi+1} h^{\xi+1}(p)
\]

\[
= 1 + \frac{\eta}{p - 1} \left( \left(1 - \frac{1}{p}\right)^{\xi+1} - 1 \right) + \frac{p - \eta - 1}{p - 1} \left( \left(1 - \frac{1}{p}\right)^{\xi} - 1 \right).
\]

The values of \( v_p \) can be evaluated in a similar way.

The estimates of the remainder terms in Corollary 1 can be got from (23) by choosing for example

\[
\alpha = \frac{s - 1 + c_{20}}{s},
\]

\[
r = \begin{cases} 
  c_{21} \left( \log \log x \right)^{1/3} \log \log \log x & \text{in the common case,} \\
  c_{21} \left( \log \log x \log \log \log x \right)^{1/\alpha} & \text{if } s \text{ is fixed,}
\end{cases}
\]

with sufficiently small constants \( c_{20}, c_{21} \) and by making some simple calculations.
The proof of Corollary 2 is based on the method of characteristic functions. This method is well-known. It is explained for example in [6].

The proof of Corollary 4 is based on Weyl's well-known theorem [10] about the uniform distribution mod 1 of a number sequence. This proof can be realized in the same way as in [9].

For the proof of Corollary 5 it is enough to repeat the proof of the analogous result from [9].

References


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