Generalized pseudo-contractions and nonlinear variational inequalities

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Abstract. Based on a modified iterative algorithm, the solvability of a class of nonlinear variational inequality problems involving Lipschitzian generalized pseudo-contractions is presented on convex sets in Hilbert spaces.

1. Introduction

General variational inequalities have been applied to many problems in applied mathematics, physics, engineering sciences, and others. A closely associated notion of the complementarity involves several problems in mathematical programming, game theory, economics, and mechanics. There are situations where both concepts are equivalent, especially on a closed convex cone. For more details on variational inequalities, we advise to consult [2–4, 7–12].

Let $H$ be a real Hilbert space and let $K$ be a nonempty closed convex subset of $H$. Let $\langle u, v \rangle$ and $\|u\|$ denote, respectively, the inner product and norm on $H$ for $u, v$ in $H$. Let $P_K$ be the projection of $H$ onto $K$. For an operator $T : K \to H$, we consider the nonlinear variational inequality (NVI) problem (Pl): Find an element $x$ in $K$ such that

\begin{equation}
\langle (I - T)x, y - x \rangle \geq 0 \quad \text{for all } y \text{ in } K,
\end{equation}

where $I$ is the identity.

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The NVI problem (1) is equivalent to a complementarity problem when $K$ is a closed convex cone ([9]).

Next, we consider an important concept of the generalized pseudo-contractivity – a mild generalization of the pseudo-contractivity introduced by Browder and Petryshyn in [1]. Generalized pseudo-contractions are more general than Lipschitz continuous operators and unify certain class of operators.

**Definition 1.1.** An operator $T : H \to H$ is said to be a generalized pseudo-contraction if, for all $x, y$ in $H$, there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \leq r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2. \quad (2)$$

It is easy to check that (2) is mutually equivalent to

$$\langle Tx - Ty, x - y \rangle \leq r\|x - y\|^2. \quad (3)$$

Clearly, this implies that

$$\langle (I - T)x - (I - T)y, x - y \rangle \geq (1 - r)\|x - y\|^2, \quad (4)$$

that is, $I - T$ is strongly monotone for $r < 1$. Here $I$ is the identity.

For $r = 1$ in (2), we arrive at the usual concept of the pseudo-contractivity of $T$ introduced by Browder and Petryshyn in [1], that is,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2. \quad (5)$$

An operator $T : H \to H$ is called Lipschitz continuous if there is a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\| \quad \text{for all } x, y \in H. \quad (6)$$

Clearly, (6) implies

$$\langle Tx - Ty, x - y \rangle \leq s\|x - y\|^2. \quad (7)$$
Remark 1.1. We note that (2) and (3) are mutually equivalent, whereas (6) and (7) are not (since (7) does not imply (6)). That is why the generalized pseudo-contractions are more general than the Lipschitz continuous operators.

Here our aim is to present, based on a modified iterative algorithm, the solution of the NVI problem (1) involving the generalized pseudoccontrastions which are Lipschitz continuous. The obtained results generalize, especially the results on pseudo-contractive and Lipschitz continuous operators in Hilbert space settings. For selected recent research works on the pseudo-contractivity, we advise [5, 6].

2. Nonlinear variational inequalities

We are just about ready to present the result on the solvability of the NVI problem (1).

Lemma 2.1 [4]. Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Then for an element $z$ in $H$, an element $x$ in $K$ satisfies
\begin{align}
\langle x - z, y - x \rangle & \geq 0 \quad \text{for all } y \in K \quad \text{iff} \quad x = P_K z. 
\end{align}

Theorem 2.1. Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. Then NVI problem (1) has a solution $x$ in $K$ iff $x$ in $K$ satisfies the relation
\begin{align}
x &= P_K [x - t(x - Tx)],
\end{align}
where $t > 0$ is arbitrary.

Proof. Assume an element $u$ in $K$ is a solution of the NVI problem (1). Then $u$ in $K$ is such that
\begin{align}
\langle u - Tu, y - u \rangle & \geq 0 \quad \text{for all } y \in K.
\end{align}
Now for any $t > 0$, it follows that
\begin{align}
\langle u - (u - t(u - Tu)), y - u \rangle & \geq 0 \quad \text{for all } y \in K.
\end{align}
By Lemma 2.1, we find that
\begin{align}
u &= P_K [u - t(u - Tu)].
\end{align}
Conversely, if \( u \) satisfies the relation
\[
u = P_K[u - t(u - Tu)],
\]
then \( u \) belongs to \( K \) and, by Lemma 2.1, we obtain
\[
\langle u - (u - t(u - Tu)), y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.
\]
Since \( t > 0 \), this implies that
\[
\langle u - Tu, y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.
\]
Hence \( u \) is a solution of the NVI problem (1).

**Theorem 2.2.** Let \( H \) be a real Hilbert space and \( K \) be a nonempty closed convex subset of \( H \). Let \( T : K \to H \) be generalized pseudo-contractive (with constant \( r > 0 \)) and Lipschitz continuous (with constant \( s \geq 1 \)). Let \( \{a_n\} \) be an increasing sequence in \([0,1)\) such that
\[
\sum_{n=0}^{\infty} a_n = \infty \quad \text{for all } n \geq 0.
\]
If, for an element \( x_0 \) in \( K \), the sequence \( \{x_n\} \) is generated by an iterative algorithm
\[
x_{n+1} = (1 - a_n)x_n + a_nP_K[(1 - t)x_n + tTx_n] \quad \text{for all } n \geq 0,
\]
then the sequence \( \{x_n\} \) converges to a unique solution of the NVI problem (1) for \( 0 < t < 2(1 - r)/(1 - 2r + s^2) \), and \( r < 1 \).

For \( \{a_n\} = 1 \), Theorem 2.2 reduces to

**Corollary 2.1.** Let \( T : K \to H \) be generalized pseudo-contractive and Lipschitz continuous, and let \( r > 0 \) and \( s \geq 1 \) be constants of the generalized pseudo-contractivity and Lipschitz continuity of \( T \), respectively. Then the sequence \( \{x_n\} \), generated by an iterative algorithm
\[
x_{n+1} = P_K[(1 - t)x_n + tTx_n] \quad \text{for an element } x_0 \text{ in } K
\]
and for all \( t \) such that \( 0 < t < 2(1 - r)/(1 - 2r + s^2) \), converges to a unique solution of the NVI problem (1).

**Proof** of Theorem 2.2. Suppose that \( z \) is a solution of the NVI problem (1). Then by Theorem 2.1, we have
\[
z = P_K[(1 - t)z + tTz].
\]
Since $P_K$ is nonexpansive, we find that
\begin{align}
\|x_{n+1} - z\| &= \|(1 - a_n)x_n + a_n P_K [(1 - t)x_n + t Tx_n] - z\| \\
&\leq (1 - a_n)\|x_n - z\| + a_n \|t(Tx_n - Tz) + (1 - t)(x_n - z)\|. 
\end{align}

(16)

Now, since $T$ is generalized pseudo-contractive (and hence equivalent to (3)) and Lipschitz continuous, it follows that
\begin{align}
\|t(Tx_n - Tz) + (1 - t)(x_n - z)\|^2 &= (1 - t)^2\|x_n - z\|^2 + 2t(1 - t)(Tx_n - Tz, x_n - z) + t^2\|Tx_n - Tz\|^2 \\
&\leq (1 - t)^2\|x_n - z\|^2 + 2t(1 - t)r\|x_n - z\|^2 + t^2s^2\|x_n - z\|^2 \\
&= (1 - t)^2 + 2t(1 - t)r + t^2s^2 \|x_n - z\|^2. 
\end{align}

(17)

Applying (17) to (16), we get
\begin{align}
\|x_{n+1} - z\| &\leq [1 - a_n + a_n (1 - t)^2 + 2t(1 - t)r + t^2s^2)^{1/2}] \|x_n - z\| \\
&= [1 - (1 - k)a_n] \|x_n - z\| \leq \prod_{j=0}^{n} [1 - (1 - k)a_j] \|x_0 - z\|, 
\end{align}

(18)

where $0 < k = [(1 - t)^2 + 2t(1 - t)r + t^2s^2]^{1/2} < 1$ for all $t$ such that $0 < t < 2(1 - r)/(1 - 2r + s^2)$, $r < 1$ and $s \geq 1$. Since $\sum_{j=0}^{\infty} a_j = \infty$ and $k < 1$, this implies that $\lim_{n \to \infty} \prod_{j=0}^{n} [1 - (1 - k)a_j] = 0$. Hence $\{x_n\}$ converges to $z$.

To show the uniqueness of the solution, let $x_1$ and $x_2$ be two solutions of the NVI problem (1). Then we have
\begin{align}
\langle (I - T)x_1, y - x_1 \rangle \geq 0 \quad \text{for all } y \text{ in } K, 
\end{align}

(19)

and
\begin{align}
\langle (I - T)x_2, y - x_2 \rangle \geq 0 \quad \text{for all } y \text{ in } K.
\end{align}

(20)

If we replace $y$ in (19) by $x_2$ and $y$ in (20) by $x_1$, and add, we obtain
\begin{align}
\langle (I - T)x_1 - (I - T)x_2, x_1 - x_2 \rangle \leq 0.
\end{align}

(21)
Since $I - T$ is strongly monotone with constant $1 - r$, we find on applying (21) that

$$ (22) \quad (1 - r)\|x_1 - x_2\|^2 \leq \langle (I - T)x_1 - (I - T)x_2, x_1 - x_2 \rangle \leq 0. $$

This implies that $x_1 = x_2$, and this completes the proof.

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References


