Abstract. The properties of indecomposable nonperfect groups are investigated. It is shown that an indecomposable solvable group is a $p$-group. The characterization of minimal non-“hypercentral-by-finite” groups are obtained.

0. Introduction

A $\mathcal{ZAF}$-group $G$ is a group which is not hypercentral-by-finite, while all proper subgroups of $G$ are hypercentral-by-finite. The group constructed by Čarin [1] and the groups of Heineken–Mohamed type [2–8] (i.e. the non-nilpotent groups with all proper subgroups nilpotent and subnormal) are examples of $\mathcal{ZAF}$-groups. The class of $\mathcal{ZAF}$-groups contains the $\mathcal{NF}$-groups (respectively the $\mathcal{AF}$-groups), i.e. the groups which are minimal non-“nilpotent-by-finite” (respectively minimal non-“abelian-by-finite”). The $\mathcal{AF}$-groups are independently described by V.V. Belyaev [9] and B. Bruno [10]. As it is proved in [9] each locally finite $\mathcal{AF}$-group $G$ is either an indecomposable metabelian group or the Čarin group. After a while in [11] it was proved that the periodic indecomposable metabelian groups are related in the some sense to the groups of Heineken–Mohamed type and (as it is well-known [3–5]) there exist an uncountable family of pair-wise nonisomorphic $p$-groups of Heineken–Mohamed type. The $\mathcal{NF}$-groups are studied in [12–14].

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Recall that a group $G$ is called indecomposable if any two proper subgroups of $G$ generate a proper subgroup of $G$, and is called decomposable otherwise. The decomposable groups are related to the groups which have a proper factorization. According to [17] we say that $G$ has a proper factorization if there are proper subgroups $A$ and $B$ such that $G = AB$.

Recall also one construction from [9], which is a generalization of the construction from [1]. Let $p$ and $q$ be distinct primes, $\mathbb{Z}_q$ the field with $q$ elements, $\mathbb{Z}_q(\alpha)$ will indicate the subfield of the algebraic closure of $\mathbb{Z}_q$ generated by $\alpha$. If $\epsilon_i$ is a primitive $p^i$-th root of 1 ($i = 0, 1, 2, \ldots$), put $F_i = \mathbb{Z}_q(\epsilon_i)$ and $F = \bigcup_{i=0}^{\infty} F_i$. Let $A$ be the additive group of $F$, $B$ be the multiplicative group which contains the $p^i$-th roots of 1 where $i = 0, 1, 2, \ldots$. The rule

$$bab^{-1} = b^{p^m} \cdot a$$

where $a \in A$, $b \in B$ and $b^{p^m} \cdot a$ is the product of the elements $b^{p^m}$ and $a$ in the field $F$, $m$ is some nonnegative integer, defines the action of $B$ on $A$. Constructed in this manner the group $G = A \rtimes B$ is called a Čarın group.

Throughout this paper $p$ will always denote a prime number, $G'$, $G''$, \ldots will indicate the terms of derived series of $G$ and by $\mathbb{C}_p\approx$ stands for the quasicyclic $p$-group. For any group $G$, $F(G)$ means the Fitting subgroup of $G$, $\Phi(G)$ the Frattini subgroup of $G$, and $Z(G)$ the center of $G$.

Most of the standard notation used comes from [18] and [19].

1. In this part we establish some properties of indecomposable groups which we shall need in the sequel.

The following theorem gives the answer to Question 1 [17] for nonperfect groups.

**Theorem 1.1.** Let $G$ be an infinite nonperfect nonabelian group. The following are equivalent:

1. $G$ is an indecomposable group;
2. $G$ has no proper factorization;
3. $G$ is countable, the commutator subgroup $G'$ of $G$ is not properly supplemented in $G$ and the quotient group $G/G'$ is a $p$-quasicyclic group for some prime $p$. 
Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1). Suppose the group $G$ has no proper factorization, but $G = \langle A, B \rangle$ for some proper subgroups $A, B$ of $G$. Then since $G'A \neq G$ and $G'B \neq G$, we conclude that $G = (G'A)(G'B)$, a contradiction.

(1) ⇒ (3) is immediate (see [11]).

(3) ⇒ (1). Suppose the contrary and let $G$ be a countable group with not properly supplemented subgroup $G'$ and $G/G' \cong \mathbb{C}_{p^\infty}$, but $G = \langle H, T \rangle$ for some proper subgroups $H, T$ of $G$. Then

$$\overline{G} = G/G' = (HG'/G') (TG'/G') \cong \mathbb{C}_{p^\infty},$$

whence we conclude that

$$\overline{G} = TG'/G' \text{ and } HG' \leq G'$$

in consequence of which $G = TG' = T$, a contradiction.

**Lemma 1.2.** If $G$ is an indecomposable group then $[G, G'] = G'$.

Indeed, if $G$ is a noncyclic group then this follows from quasicyclity of quotient group $G/G'$.

**Corollary 1.3.** Any nonperfect noncyclic $p$-group of finite exponent is decomposable.

**Lemma 1.4.** An indecomposable periodic solvable group $G$ is a countable (i.e. is finite or countable infinite) $p$-group for some prime $p$.

Proof. If $G$ is cyclic then the result follows from [11]. Thus, suppose that $G$ is noncyclic group satisfying the conditions of lemma. Then $G/G' \cong \mathbb{C}_{p^\infty}$ for some prime $p$. If, further, $G$ is not $p$-group there is a positive integer $m$ such that the quotient groups $G^{(m)}/G^{(m+1)} = P_1 \times Q_1$ with the Sylow $p$-subgroup $P_1$ and a $p'$-subgroup $Q_1$ and therefore $G/G^{(m+1)} = Q_1 \times P$ for some $p$-subgroup $P$. We have a contradiction with indecomposability of $G$. It follows that all factors $G^{(i)}/G^{(i+1)}$ of the derived series of $G$ are $p$-groups; this completes the proof.

**Lemma 1.5.** Let $G$ be an indecomposable locally finite group. If every proper subgroup of $G$ is almost locally solvable, then $G$ is nonsimple.

Proof. Let $G$ be a group satisfies the assumptions. If, further, $G$ is simple then by Corollary A1 [20] $G$ must be linear. Since the locally finite simple groups which are linear must be of Lie type (see [21]), the group $G$ must be decomposable, a contradiction. Hence $G$ is nonsimple and lemma is proved.
Lemma 1.6. Let $G$ be an indecomposable group. Then the following statement are equivalent.

(1) $G$ is nonperfect $p$-group with every proper subgroup nilpotent;
(2) $G$ is a non-nilpotent group with all subgroups subnormal and the commutator subgroup $G'$ of $G$ is nilpotent.

Proof. (1) $\Rightarrow$ (2). If $G$ is an indecomposable nonperfect group whose proper subgroups are nilpotent and $K$ is any proper subgroup of $G$ then by Theorem 1.1 $G'K$ is also a proper subgroup of $G$. Hence $K$ is a subnormal subgroup of $G$.

(2) $\Rightarrow$ (1). Suppose that $G$ is an indecomposable non-nilpotent group with all subgroups subnormal and the commutator subgroup $G'$ of $G$ is nilpotent. Then $G/G' \cong C_{p^\infty}$ and, further, $G$ is a $p$-group by Lemma 1.4. Note that

$$K/K \cap G' \cong G'K/G'$$

is an abelian group of finite exponent for every proper subgroup $K$ of $G$ and so by Proposition 1.2 [22] the subgroups $G'K$ and $K$ are nilpotent.

Relative to Corollary 1 [2] we argue

Lemma 1.7. Any torsion-free (and consequently non-periodic) group $G$ with every proper subgroup nilpotent (respectively hypercentral) is also nilpotent (respectively hypercentral).

Proof. Since nilpotence and hypercentrality are properties of countable character [19, p. 355], we have that $G$ with a noncountable group $G$ with all subgroups nilpotent (respectively hypercentral) is nilpotent (respectively hypercentral). Therefore suppose that $G$ is countable. If $G$ is torsion-free then by Lemma 2 [23] $G$ coincides with the isolator

$$I_G(K) = \{x \in G \mid \exists n \in \mathbb{N} : x^n \in K\}$$

of some proper subgroup $K$ of $G$ and so (see [24]) $G$ is nilpotent (respectively hypercentral).

Suppose now that $G$ is not torsion-free. Then as stated above the quotient group $\overline{G} = G/\tau G$ of $G$ (here $\tau G$ is the periodic part of $G$) is nilpotent (respectively hypercentral). Further, if $G$ is indecomposable then $\overline{G}/\overline{G}' \cong C_{p^\infty}$ and consequently the isolator $I_{\overline{G}}(\overline{G}')$ of $\overline{G}'$ coincides with $\overline{G}$, a contradiction. Thus, $G = \langle A, B \rangle$ for some proper subgroups $A, B$ of $G$. 
Moreover, the image $\overline{A}$ of $A$ (respectively $\overline{B}$ of $B$) in $\overline{G}$ is contained in a proper normal subgroup $\overline{A}_1$ (respectively $\overline{B}_1$) of $\overline{G}$. Then if $A_1$ and $B_1$ are the inverse images of $\overline{A}_1$ and $\overline{B}_1$ in $G$, respectively, $G = A_1B_1$ is a product of two nilpotent (respectively hypercentral) normal subgroups and consequently $G$ is nilpotent (respectively hypercentral).

We consider the question on linearity of indecomposable groups. In view of well-known theorem of Zassenhaus [19, Th. 15.1.3] any group of matrices (over field) with subnormal (respectively hypercentral) proper subgroups is solvable. From the results of Mal’cev [25], Kargapolov [26–27] and Theorem 8 [23], Lemmas 1.4, 1.6, 1.7 we conclude the following

**Corollary 1.8.** Let $G$ be an indecomposable locally solvable periodic group of matrices (over field). Then $G$ is either the cyclic $p$-group $C_{p^n}$ or $C_{p^\infty}$.

Thus, neither the groups of Heineken–Mohamed type nor the minimal non-hypercentral groups are not linear (over field).

**Proposition 1.9.** Let $G$ be a countable group with the hypercentral commutator subgroup $G'$ and the quasicyclic quotient group $G/G'$. Then the group $G$ that satisfies the normalizer condition is an indecomposable $p$-group.

**Proof.** Without restricting generality, suppose $G$ is a metabelian group. Suppose the assertion is false and $G$ is decomposable. Then clearly $G = G'V$ for some proper subgroup $V$ of $G$, whence

$$\overline{G} = G/G' \cap V = \overline{G'} \rtimes \overline{V}.$$  

It is easy to see that $\overline{V} \cong C_{p^\infty}$, $\overline{1} \neq N_{\overline{G'}}(\overline{V}) \leq Z(\overline{G})$ and every proper homomorphic image of $\overline{G}$ has a nontrivial centre. This means [19, Example 12.2.2] that $\overline{G}$ is hypercentral. But then by Lemma 1.18 [18, p. 63] the group $\overline{G}$ is abelian, a contradiction. Thus $G$ is indecomposable. Further, it is easy to see that $G$ is a $p$-group.

**Corollary 1.10.** The quotient group $G/G'$ of a nonabelian countable hypercentral group $G$ is not quasicyclic.
Proposition 1.11. If the commutator subgroup $G'$ of nonabelian indecomposable $p$-group $G$ is abelian (respectively nilpotent of finite exponent) then $G$ satisfies the normalizer condition.

Proof. Pick an arbitrary proper subgroup $K$ of $G$. Clearly without loss generality we may assume that $G' \not\subseteq K$ and $K \not\subseteq G'$. Obviously $G' K$ is a proper subgroup of $G$ and there is an element $a$ of $G$ such that $G' K = G' \langle a \rangle$. Then the subgroup $G' \langle a \rangle$ is hypercentral (see [18, Proposition 1.7] and [28], respectively) and $N_G(K) \geq N_{G' K}(K) \neq K$, as desired.

The following lemma is obvious.

Lemma 1.12. Let $G$ be a group in which every proper subgroup satisfies the normalizer condition. Then the one of the following statements holds.

1. $G$ satisfies the normalizer condition.
2. $G$ is a finitely generated group with the simple quotient group $G/N$ for some normal subgroup $N$ of $G$.

2. In this section we establish some properties of groups without a proper factorization (see [17, Question 1]).

It is well-known that there are finite nonsolvable groups without proper factorization. The following lemma is due to [29].

Lemma 2.1. Let $G$ be a nonabelian finite group. The following statements are equivalent.

1. $G$ has no proper factorization.
2. $F(G) = \Phi(G) = Z(G)$ and the quotient group $G/Z(G)$ is a simple group without proper factorization.

Proof is immediate.

Lemma 2.2. Let $G$ be a nonabelian finitely generated group. If $G$ is a decomposable group without proper factorization then $G$ has a simple quotient group. Further, if $\Phi(G) = 1$ then $G$ is simple.

Proof. Suppose $G = \langle A, B \rangle$ for some proper subgroups $A$ and $B$ of $G$. Without restricting generality, let $A$ and $B$ be maximal subgroups of $G$. Then it is easy to see that $H \triangleleft G$ implies $H \leq A \cap B$. If $F$ is a subgroup of $G$ generated by all normal subgroups of $G$ then the quotient group $G/F$ is simple, and this completes the proof.

Obviously, any non-finitely generated group with a proper subgroup of finite index has a proper factorization. Then we state the following
Corollary 2.3. Let $G$ be a non-finitely generated group. If $G$ contains a nontrivial normal finite subgroup then either $G$ has a proper factorization or the centre $Z(G)$ of $G$ is nontrivial.

Corollary 2.4. (i) An abelian group $G$ has no proper factorization if and only if either $G$ is a cyclic $p$-group or $G$ is a quasicyclic $p$-group.
(ii) A nonperfect nonabelian finite group has a proper factorization.

We shall prove the following theorem.

Theorem 2.5. An indecomposable solvable group $G$ is a locally finite $p$-group.

For the proof of 2.5 we need the following lemma.

Lemma 2.6. Let $G$ be a locally finite group and $M \neq \{0\}$ be a $\mathbb{Z}G$-module which is torsion-free as a group. Then for any finite set $\pi$ of primes, there is a $\mathbb{Z}G$-submodule $N$ of $M$ such that the quotient module $M/N$ is periodic as a group and, for all $p$ in $\pi$, contains an element of order $p$.

Proof of 2.6 is analogous with proof of Lemma 2.3 [14]. We notice only that the group ring $\mathbb{Q}G$ is a (Von Neumann) regular ring by Theorem 1.5 [30, p. 68].

Proof of Theorem 2.5. Suppose that $G$ is a solvable group with derived length $n + 1$, the quotient group $G/G^{(n)}$ is periodic and $G^{(n)}$ has an element of infinite order. Let $T$ be the torsion subgroup of $G^{(n)}$. Put $H = G/T$. Obviously $H/H' \simeq \mathbb{Z}_{p^\infty}$ for some prime $p$. Now $H^{(n)}$ and $H/H^{(n)}$ satisfy the hypotheses of Lemma 2.6 (with $M = H^{(n)}$ and $G = H/H^{(n)}$); hence there exist $N$ normal in $H$, $N \leq H^{(n)}$ such that the quotient group $H^{(n)}/N$ is periodic and contains the elements of order $r$ and $q$ for two distinct primes $r$ and $q$ different from $p$. Clearly, $H/N$ is an indecomposable periodic solvable $p$-group by Lemma 1.4, a contradiction. The proof of Theorem 2.5 is complete.

Corollary 2.7. Any non-periodic solvable group has a proper factorization.
3. This section contains several characterizations of $\mathcal{ZAF}$-groups.

**Remark 3.1.** An abelian-by-(periodic abelian) $R$-group is abelian.

**Remark 3.2.** If $G$ is a $\mathcal{ZAF}$-group then the one of the following holds:

1. $G$ is a finitely generated group with a normal subgroup $N$ such that the quotient group $G/N$ is simple.
2. $G$ is a locally graded group.

Indeed, if each homomorphic image of a finitely generated $\mathcal{ZAF}$-group $G$ is nonsimple then the group $G$ is hypercentral-by-finite, a contradiction. On the other hand, if $G$ is not finitely generated then it is readily verified that $G$ is locally nilpotent-by-finite.

**Lemma 3.3.** Let $G$ be a $\mathcal{ZAF}$-group. Then each normal subgroup of $G$ is an extension of a hypercentral group, which is normal in $G$, by a finite abelian group. If, further, the group $G$ is nonperfect and indecomposable then every subgroup of $G$ is hypercentral-by-(finite abelian).

**Proof.** Pick an arbitrary normal subgroup $N$ of $G$. It is now easy to verify that there is a hypercentral subgroup $H$ of $N$ that is a normal subgroup of $G$ with $|N:H| < \infty$. Put $\overline{G} = G/H$. Then $N = N/H$ is a finite normal subgroup of $\overline{G}$ and consequently

$$|\overline{G} : C_{\overline{G}}(N)| = |N_{\overline{G}}(N) : C_{\overline{G}}(N)| < \infty.$$

Further, since $\overline{G}$ contains no proper subgroup of finite index, we have $\overline{G} = C_{\overline{G}}(N)$ and $N$ is abelian.

Suppose now that the group $G$ is indecomposable and nonperfect. Then $G'K$ is a proper subgroup of $G$ for each subgroup $K$ of $G$ and consequently $G'K$ contains a hypercentral subgroup $F$ of finite index which is normal in $G$. Moreover,

$$K/K \cap F \simeq KF/F \leq G'K/F$$

and as stated above $G'K/F$ is abelian; this completes the proof.
Lemma 3.4. If $G$ is a nonperfect $ZAF$-group then the commutator subgroup $G'$ of $G$ is hypercentral and $G/G' \simeq \mathbb{C}_p^\infty$.

Proof. Since $G$ is a nonperfect $ZAF$-group, the quotient group $G/G'$ is obviously indecomposable and so $G/G' \simeq \mathbb{C}_p^\infty$ (see [11]).

Suppose now that the commutator subgroup $G'$ of $G$ is non-hypercentral. Then $G'$ contains a subgroup $F$ of finite index which is normal in $G$. Put $G = G/F$. Clearly, $|G'| < \infty$ and $G/G' \simeq \mathbb{C}_p^\infty$, whence by Lemma 1.15 [18] $G$ is abelian, a contradiction.

Corollary 3.5. Any nonperfect $ZAF-p$-group $G$ is indecomposable.

Indeed, it is easy to see that the quotient group $G/G''$ is an $AF$-group and hence (see [9] or [10]) it is indecomposable.

Corollary 3.6. Any nonperfect $ZAF-p$-group $G$ is a minimal non-hypercentral group if and only if $G$ satisfies the normalizer condition.

Proof. Part “if” follows from Remark 3.2 and Lemma 1.12.

“Only if”. Let $K$ be an arbitrary proper subgroup of $G$. Then $K$ is hypercentral by Lemma 3.3 and Lemma 2 [31, p. 396], as desired.

Lemma 3.7. Let $G = K \times Q$ be a $ZAF$-group, $Q \simeq \mathbb{C}_p^\infty$ and $K$ a hypercentral subgroup. Then $Z(K) = K' = \Phi(K)$ and $K$ is a $q$-group for a prime $q$ different from $p$.

Proof. Let $A$ be an arbitrary proper $G$-invariant subgroup of $K$. Then $AQ$ contains a normal hypercentral subgroup $F$ of finite index and as follows from $|Q : Q \cap F| < \infty$ we conclude $Q \leq F$ and $AQ = AF$ is hypercentral. Thus, $Q \leq C_G(A)$. Since $G$ is nonabelian, the subgroup $T$ generated by all proper $G$-invariant subgroups of $K$ is a proper $G$-invariant subgroup of $K$.

Suppose, first, that a subgroup $K$ is abelian. If, further, $K$ is nonperiodic then without loss of generality we can assume that $K$ is torsion-free. Since by Theorem of Zaitsev [32] $K/T$ is an abelian $q$-group of exponent $q$ for some prime $q$. From $[a, t] = b$ with some $1 \neq b, a \in K$ and $t \in Q$ we conclude that $1 = [a^q, t] = b^q$, a contradiction. Hence $K$ is a periodic group and so $K$ is an abelian $q$-group of exponent $q$. Consequently $\Phi(K) = 1$. Moreover, Corollary 3.5 implies that $p$ and $q$ are distinct. Since $K = [K, Q] \times C_K(Q)$, we have $C_K(Q) = 1$ and so $T = 1$. Therefore $K$ is a minimal $G$-invariant subgroup of $G$.

Suppose next that $K$ is nonabelian. Since obviously $K' \leq T$ and as proved before $K/K'$ is minimal $G$-invariant abelian subgroup of exponent $q$, we have $T = K' = \Phi(K) = Z(K)$. The proof is completed.

The following lemma is due to [14].
Lemma 3.8. Any nonperfect $\mathcal{ZAF}$-group $G$ is periodic.

Proof. Let $G$ be a $\mathcal{ZAF}$-group. Clearly $G/G' \simeq \mathbb{C}_{p^{\infty}}$. Suppose that it is not periodic; then $G'$ is not periodic. By Lemma 3.4 $G'$ is hypercentral and application of [19, 12.2.6] shows that $G''/G'''$ is not periodic. Put $H = G/G''$ and let $T/G'''$ be the torsion part of $H'$. Obviously, $T$ is properly contained in $G'$. Thus $K = G/T$ is an $\mathcal{AF}$-group and so by Theorem 2.1 [14] $K$ is periodic, a contradiction. Thus $G$ must be periodic, and the proof is completed.

Theorem 3.9. Let $G$ be a decomposable nonperfect group. Then the following statements are equivalent.

1. $G$ is a $\mathcal{ZAF}$-group.
2. $G = M \rtimes Q$, $Q \simeq \mathbb{C}_{q^{\infty}}$, $M$ is a $p$-group, $p$ and $q$ are the distinct primes, $Z(M) = M' = \Phi(M)$, $Q$ acts trivially on the Frattini subgroup $\Phi(M)$ and irreducibly on $M/\Phi(M)$, and, further, $G/\Phi(M)$ is a Čarin group.
3. $G$ is a $\mathcal{NF}$-group.

Proof. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are almost obvious. Therefore we prove only (1) $\Rightarrow$ (3). Since by assumption a nonperfect $\mathcal{ZAF}$-group $G$ is decomposable, there are two nontrivial proper subgroups $U$ and $V$ of $G$ such that $G = \langle U, V \rangle$. Then by Lemma 3.4, for example, $G = G'V$. It follows by Corollary 3.5 that $G$ is not $p$-group. Since $V$ contains a hypercentral subgroup $K$ of finite index then $|G : G'K| < \infty$ and $G = G'K$. By Lemma 3.8 $G$ (and so $K$) is periodic and by Lemma 3.4 there is a $p$-subgroup $K_1$ of $K$ such that $G = G'K_1$. If is easy to see that $G'$ is a $r$-group for some prime $r$. An application of Lemma 3.7 completes the proof.

Remark 3.10. The Theorem 1 of [2] implies that if $p$ and $q$ are the primes from Theorem 3.9 then $q \neq 2$ and the order of $p$ modulo $q$ is an even number.

Remark 3.11. An example of the decomposable nonperfect $\mathcal{NF}$-group (and consequently $\mathcal{ZAF}$-group) which is not an $\mathcal{AF}$-group is constructed in [12].
Lemma 3.12. Any indecomposable nonperfect ZAF-group $G$ is a $p$-group.

Proof. By Lemma 3.4 $G/G' \simeq \mathbb{C}_p\infty$. Put $\overline{G} = G/G''$. It is easy to see that $\overline{G}$ is an indecomposable $AF$-group and so it is a $p$-group. Therefore a hypercentral subgroup $G'$ (and so $G$) is a $p$-group, too.

Theorem 3.13. Let $G$ be a nonperfect group. Then the following statements are equivalent.

1. $G$ is an indecomposable ZAF-group.
2. $G$ is a countable $p$-group and has an infinite normal subgroup $N$ not supplemented nontrivially in $G$ with $G/N \simeq \mathbb{C}_p\infty$, $N^p \neq N$ and the quotient group $G/G''$ is a minimal non-hypercentral group.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.4 $G'$ is a hypercentral subgroup and $G/G' \simeq \mathbb{C}_p\infty$. In view of indecomposability of $G$ the commutator subgroup $G'$ is not supplemented nontrivially in $G$ and $G$ is a $p$-group. From [9] and [11] it follows that $(G')^p \neq G'$. An application of the Proposition 1.7 [18] completes the proof of this part.

(2) $\Rightarrow$ (1). By Theorem 1.1 the group $G$ is indecomposable. If $K$ is an arbitrary proper subgroup of $G$ then $G'K$ is a proper subgroup of $G$. Obviously $G'K$ (and as consequence $K$) is a hypercentral-by-finite, but $G$ is not almost hypercentral. This completes the proof of Theorem.

Note that, it follows from what is proved before that, in particular, every nonperfect minimal non-nilpotent group is a countable solvable $p$-group of Heineken–Mohamed type.

4. In this section we are concerned with the perfect ZAF-groups.

The next result is due to [14, Proposition 3.1].

Proposition 4.1. A perfect ZAF-group $G$ must be countable hyperabelian $p$-group. Moreover, $G$ is the join of an ascending sequence of hypercentral normal subgroups and all proper subgroups of $G$ are hypercentral and ascendant (hence $G$ satisfies the normalizer condition).

This runs along the same lines as proof of Proposition 3.1 [14], replacing “nilpotent” by “hypercentral” and “subnormal” by “ascendant”. Moreover, by Lemma 1.5 $G$ do not have infinite simple images. Since hypercentrality is a property of countable character [19, p. 355] then $G$ is countable. Finally, from Lemma 1.7 follows that $G$ is $p$-group.

From Proposition 4.1 it follows that a non-“locally nilpotent” ZAF-group is not perfect. Whether or not there are perfect ZAF-groups remains an open question.
References


On indecomposable groups and groups with hypercentral-by-finite . . .


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