Bäcklund transformations of $n$-dimensional constant torsion curves

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Abstract. The Bäcklund transformation of two surfaces of $\mathbb{R}^3$ with the same constant negative Gaussian curvature transforms an asymptotic line of one surface into an asymptotic line of the other. Since by Enneper the asymptotic lines of such a surface have constant torsion, it is natural to restrict the Bäcklund transformations to such curves. This idea was developed by Annalisa Calini and Thomas Ivey in [2]. We shall prove the converse of their theorem and generalize the transformation for the $n$-dimensional case.

0. Introduction

By the work of Bianchi and Lie it is possible to compute the Gaussian curvature of the focal surfaces of a line congruence in terms of the coefficients of the first fundamental form for the spherical representation and the distance between the corresponding limit points of these surfaces (see [3]).

Bäcklund proved that for pseudospherical congruences satisfying the two additional conditions that the distance $r$ between corresponding limit points is constant and that the normals of the focal surfaces at these points form a constant angle $\theta$, the curvatures must be equal to the same negative constant $-\sin^2 \theta / r^2$ (see [3]). By Ennepers relation between the first, second and third fundamental forms of a surface of negative Gaussian curvature the relation $\tau^2 = -K$ between the torsion $\tau$ of an asymptotic line

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and the curvature $K$ holds. When the curvature is constant the torsion cannot change. Bearing in mind that under pseudospherical congruences asymptotic lines correspond, these provide a method for restricting the Bäcklund transformation to constant torsion curves. This was done by Annalisa Calini and Thomas Ivey in [2]. They constructed a constant torsion curve from a given one. We prove the converse of their theorem, namely: If there is a correspondence $\nu$ between the points of two unit speed curves $c, \tilde{c}$ having the property that the line joining the corresponding points $c(s)$ and $\tilde{c}(s) = \nu(c(s))$ is the intersection of the osculating planes of these curves, and this intersection has the same angle with the curves, the line segment $c(s)\tilde{c}(s)$ has constant length $r$ and the binormals in corresponding points form a constant angle $\theta$, then the curves have the same constant torsion $\sin \theta/r$.

We could not find a connection between the sectional curvature of an $n$-dimensional manifold of constant negative curvature in the $2n - 1$-dimensional euclidean space and the curvatures of its asymptotic lines in order to restrict the generalized Bäcklund transformation (see [4], and [5]) to curves in $2n - 1$-dimensional euclidean spaces.

However if we consider just the transformation of Annalisa Calini and Thomas Ivey [2] for 3-dimensional constant torsion curves we can generalize it for higher dimensions.

1. Bäcklund transformations of 3-dimensional constant torsion curves

In [3] A. Calini and T. Ivey constructed a curve of constant torsion from a given one. We shall prove that under some assumptions made for a transformation between two curves they must have the same constant torsion.

**Theorem 1.1.** Suppose that $\nu$ is a transformation between two curves $c$ and $\tilde{c}$ of $\mathbb{R}^3$ with $\tilde{c}(s) = \nu(c(s))$, where $s$ is the arc length of $c$, such that in corresponding points we have:

1. The line joining these points is the intersection of the osculating planes of the curves, such that the line segment $c(s)\tilde{c}(s)$ has constant length $r$.

2. The vector $\tilde{c}(s) - c(s)$ forms the same angle $\beta \neq \pi/2$ with the tangent vectors of the curves.
(3) The binormals of the curves form the same constant angle \( \theta \neq 0 \).

Then the torsions of the curves are equal to the same constant \( \sin \theta/r \).

**Proof.** Denote by \((e_1, e_2, e_3)\) the Frenet frame of \(c\) and by \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) that of \(\tilde{c}\) in the corresponding points \(c(s)\) and \(\tilde{c}(s)\). If we denote by \(f_1\) the unit vector of \(\tilde{c}(s) - c(s)\), then we can complete \(f_1, e_3\) and \(f_1, \tilde{e}_3\) to the positively oriented orthonormal frames \((f_1, f_2, e_3)\) and \((f_1, \tilde{f}_2, \tilde{e}_3)\) respectively. Let \(f_3 = e_3, \tilde{f}_3 = \tilde{e}_3\) and \(-\beta\) be the angle between \(f_1\) and \(e_1\). Then the angle between \(f_1\) and \(\tilde{e}_1\) is also \(-\beta\). Thus we can obtain the frames \((f_1, f_2, f_3)\) and \((f_1, \tilde{f}_2, \tilde{f}_3)\) by rotating the frames \((e_1, e_2, e_3)\) and \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) around \(e_3\) and \(\tilde{e}_3\) respectively with angle \(-\beta\). Analytically this can be written as:

\[
\begin{align*}
 f_1 &= \cos \beta e_1 + \sin \beta e_2, \\
 f_2 &= -\sin \beta e_1 + \cos \beta e_2, \\
 f_3 &= e_3,
\end{align*}
\]

and

\[
\begin{align*}
 \tilde{f}_1 &= \cos \beta \tilde{e}_1 + \sin \beta \tilde{e}_2, \\
 \tilde{f}_2 &= -\sin \beta \tilde{e}_1 + \cos \beta \tilde{e}_2, \\
 \tilde{f}_3 &= \tilde{e}_3.
\end{align*}
\]

Since \(f_3 = e_3, \tilde{f}_3 = \tilde{e}_3\) and the angle between \(e_3, \tilde{e}_3\) is the constant \(\theta\), we can obtain the frame \((f_1, \tilde{f}_2, \tilde{f}_3)\) by rotating the frame \((f_1, f_2, f_3)\) around \(f_1\) with angle \(\theta\). Thus we have:

\[
\begin{align*}
 \tilde{f}_2 &= \cos \theta f_2 - \sin \theta f_3, \\
 \tilde{f}_3 &= \sin \theta f_2 + \cos \theta f_3.
\end{align*}
\]

Using (1.1), (1.2) and (1.3) we can express \(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\) in terms of \(e_1, e_2, e_3\) as follows:

\[
\begin{align*}
 \tilde{e}_1 &= e_1 + (1 - \cos \theta) \sin \beta (\cos \beta e_2 - \sin \beta e_1) + \sin \theta \sin \beta e_3, \\
 \tilde{e}_2 &= e_2 - (1 - \cos \theta) \cos \beta (\cos \beta e_2 - \sin \beta e_1) - \sin \theta \cos \beta e_3, \\
 \tilde{e}_3 &= \cos \theta e_3 + \sin \theta (\cos \beta e_2 - \sin \beta e_1).
\end{align*}
\]

Bearing in mind that the distance between \(c(s)\) and \(\tilde{c}(s)\) is the constant \(r\) and \(f_1 = \cos \beta e_1 + \sin \beta e_2\), we have:

\[
\tilde{c}(s) = c(s) + r(\cos \beta e_1 + \sin \beta e_2).
\]
Differentiating \((\mathbf{c} - \mathbf{c})^2 = r^2\) we obtain \(2(\mathbf{c} - \mathbf{c})(\dot{\mathbf{c}}|\dot{\mathbf{c}}|\mathbf{e}_1 - \mathbf{e}_1) = 0\). Since \(\mathbf{c} - \mathbf{c} = r\mathbf{f}_1\), this yields \(\dot{\mathbf{c}}|\cos \beta - \cos \beta = 0\), so that \(\mathbf{c}\) is also of unit speed.

Using the Frenet formulae for \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\) it is easy to see that \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\) given by (1.4) satisfy the Frenet formulae if and only if: \(\tilde{K}_1 = K_1 - 2C \sin \beta\), \(\tilde{K}_2 = K_2\) and \(d\beta/ds = C \sin \beta - K_1\), where \(K_1, \tilde{K}_1\) and \(K_2, \tilde{K}_2\) are the curvatures and torsions of \(\mathbf{c}\) and \(\mathbf{c}\) respectively, and \(C = K_2 \tan \theta/2\).

By (1.5) we have:

\[
\tilde{\mathbf{e}}_1 = \left(1 - rK_2 \tan \frac{\theta}{2} \sin^2 \beta\right) \mathbf{e}_1 + rK_2 \tan \frac{\theta}{2} \sin \beta \cos \beta \mathbf{e}_2 + rK_2 \sin \beta \mathbf{e}_3.
\]

Comparing this with (1.4)_1 we obtain \(K_2 = \sin \theta/r\). Thus the the curves \(\mathbf{c}\) and \(\mathbf{c}\) have the same constant torsion

\[
\tilde{K}_2 = K_2 = \frac{\sin \theta}{r},
\]

and the transformation can be given by

\[
\tilde{\mathbf{c}} = \mathbf{c} + 2C \frac{2C}{C^2 + K_2^2} (\cos \beta \mathbf{e}_1 + \sin \beta \mathbf{e}_2),
\]

where

\[
\frac{d\beta}{ds} = C \sin \beta - K_1.
\]

The last two equations are the defining relations of the transformation given by A. Calini and T. Ivey in Theorem 1.1 of [3]. Examples for such transformation are given in [3].

2. Bäcklund transformations of \(n\)-dimensional constant torsion curves

From now on we mean by the torsion of a curve its last curvature. Let \(\mathbf{c}\) and \(\mathbf{c}\) be two curves in \(\mathbb{R}^n\), with curvatures \(K_1, \ldots, K_{n-1}\) and \(\tilde{K}_1, \ldots, \tilde{K}_{n-1}\) respectively. Then the main theorem of Section 1 can be generalized as follows:

**Theorem 2.1.** Suppose that \(\nu\) is a transformation between \(\mathbf{c}\) and \(\mathbf{c}\) with \(\tilde{\mathbf{c}}(s) = \nu(\mathbf{c}(s))\), where \(s\) is the arclength of \(\mathbf{c}\) such that for correspond-
ing points we have:

(1) The line joining these points is contained in the intersection of the osculating hyperplanes and the line segment \(\bar{c}(s)\bar{c}(s)\) has constant length \(r\).

(2) (a) The angle between the vectors \(f_1\) and \(e_{n-1}\) is complementary to the angle between the vectors \(e_1\) and \(f_{n-1}\), where \(f_1\) is the unit vector of \(\bar{c}(s) - c(s)\), \((f_1, \ldots, f_{n-2})\) and \((f_1, \ldots, f_{n-2}, f_{n-1}, e_n)\) are positively oriented frames of the intersection of the osculating planes and the whole space respectively and \((e_1, \ldots, e_n)\) is the Frenét frame of \(c\).

(b) \(\langle e_1, f_1 \rangle \neq 0\), where \(\langle ., . \rangle\) is the standard scalar product of \(\mathbb{R}^n\).

(3) The Frenét frame of \(\bar{c}\) can be obtained from that of \(c\) by a rotation with constant angle \(\theta \neq 0\) around a plane which contains \(e_n\).

Then the curves have the same constant torsion \(\sin \theta / r\). Moreover for \(n \geq 4\) we have that

\[ K_1 = \bar{K}_1, \ldots, K_{n-3} = \bar{K}_{n-3}. \]

**Proof.** From (3) we have

\[ \bar{E} = A^T \Theta AE, \]

and \(E^T = (e_1, \ldots, e_n)\) and \(\bar{E}^T = (\bar{e}_1, \ldots, \bar{e}_n)\) are the Frenét frames of \(c\) and \(\bar{c}\).

\[ \Theta = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \]

and \(A \in SO(n)\) such that \(a_{in} = a_{ni} = \delta_{in}\), where \(\delta_{ij}\) is the Kronecker symbol. In terms of the entries this can be written as:

\[
\begin{cases}
\bar{e}_i = \sum_{j=1}^{n-1} [\delta_{ij} - a_{n-1,i}a_{n-1,j}(1 - \cos \theta)]e_j - a_{n-1,i}\sin \theta e_n; \\
i = 1, n-1,
\end{cases}
\]

\[ \bar{e}_n = \sin \theta \sum_{j=1}^{n-1} a_{n-1,j}e_j + \cos \theta e_n. \]

(2.1)

Since the first \(n - 2\) columns and rows of \(\Theta\) form an identity matrix and \(a_{in} = a_{ni} = \delta_{in}\), \(A\bar{E} = \Theta AE\) implies that the first \(n - 2\) rows of \(AE\) are equal to the first \(n - 2\) columns of \(A\bar{E}\) and form the basis \((f_1, \ldots, f_{n-2})\) for the intersection of the osculating hyperplanes. Thus \(F^T = (f_1, \ldots, f_{n-1}, e_n)\), where \(F = AE\).
By (2)(a) we also have

\[ a_{1,n-1} = -a_{n-1,1}, \]

where \( A = (a_{ij})_{1 \leq i,j \leq n} \). Differentiating (2.1) and using the Frenét formulæ for \( E, \tilde{E} = (\tilde{e}_1, \ldots, \tilde{e}_n) \) satisfies the Frenét formulæ of \( \tilde{c} \) if and only if the following groups of relations hold:

(I) \[ a_{n-1,2} = a_{n-1,n-1} = 0, \quad \text{if } n \geq 4, \]
\[ \tilde{K}_{n-2} = K_{n-2} + 2K_{n-1} \tan \frac{\theta}{2} a_{n-1,n-2}, \]
\[ \begin{cases} 
  a_{n-1,1} = -K_1 a_{n-1,2} + K_{n-1} a_{n-1,n-1} a_{n-1,1} \tan \frac{\theta}{2}, \\
  a_{n-1,j} = K_j a_{n-1,j+1} - K_{j-1} a_{n-1,j-1} \\
  + K_{n-1} a_{n-1,n-1} a_{n-1,j} \tan \frac{\theta}{2}; \quad j = 2, n-2, \\
  a_{n-1,n-1} = -K_{n-2} a_{n-1,n-2} - K_{n-1} \tan \frac{\theta}{2} (1 - a_{n-1,n-1}^2), 
\end{cases} \]

(III) \[ K_i = \tilde{K}_i; \quad i = 1, n-3, \quad \text{if } n \geq 4. \]

Since the distance between corresponding points is the constant \( r \), we have

\[ \tilde{c} = c + rf_1. \]

Using \( F = AE \) we have

\[ f_1 = \sum_{j=1}^{n-1} a_{1j} e_j, \]

which implies

\[ \tilde{c} = c + r \sum_{j=1}^{n-1} a_{1j} e_j. \]

We have already seen that the first \( n-2 \) rows of \( A\tilde{E} \) coincide with the first \( n-2 \) columns of \( AE \). Thus

\[ \langle f_1, \tilde{e}_1 \rangle = \langle f_1, e_1 \rangle = a_{11}. \]
Differentiating \((\ddot{c} - c)^2 = r^2\) we obtain \(2(\dot{c} - c, |\dot{c}|\ddot{e}_1 - e_1) = 0\), which by (2.2) and (2.3) becomes \(|\dot{c}| = 1\), hence \(\dot{c}\) is also of unit speed. Differentiating (2.3) and using the Frenet formulae for \(e_j\) we obtain

\[
\ddot{e}_1 = (1 + ra_{11} - K_1a_{12})e_1 + r \sum_{j=2}^{n-2} (a_{1j} - K_ja_{1,j+1} + K_{j-1}a_{1,j-1})e_j
\]

\[
+ r (a_{1,n-1} + a_{1,n-2}K_{n-2})e_{n-1} + ra_{1,n-1}K_{n-1}e_n.
\]

Comparing this with (2.1) and using \(a_{1,n-1} = -a_{n-1,1}\), we obtain

\[
\begin{align*}
K_1 &= \frac{ra_{11} + a_{n-1,1}^2(1 - \cos \theta)}{ra_{12}}, \\
K_j &= \frac{a_{1j} + a_{n-1,1}a_{n-1,j}(1 - \cos \theta) + K_{j-1}a_{1,j-1}}{a_{1,j+1}}, \\
K_{n-1} &= \frac{\sin \theta}{r}.
\end{align*}
\]

In conclusion, (III) and (IV) are exactly the assertions of our theorem.

For \(n > 4\) if we fix a unit speed curve \(c\) in \(\mathbb{R}^n\) with a given constant torsion, the system \((0) + (I) + (II) + (IV)\) is underdetermined. But this system is equivalent to the conditions (1), (2), (3) of Theorem 3.1. In conclusion, for every curve \(c \subset \mathbb{R}^n\); \(n > 4\) with \(K_{n-1} = a\), where \(a\) is a given constant, and every vector \(v \in T_{c(0)}\mathbb{R}^n\), we can find an infinite number of curves \(\tilde{c} \subset \mathbb{R}^n\) satisfying the conditions (1), (2), (3) of Theorem 3.1 and such that \(\sin \theta = ar\) and \(\tilde{c}(0) - c(0) = v\), where \(T_{c(0)}\mathbb{R}^n\) is the tangent space of \(\mathbb{R}^n\) in \(c(0)\).

**Remark 2.2.** The case \(n = 4\) is a special one, since for this dimension the condition (I) implies that the matrix \(A\) must be of the form

\[
A = \begin{bmatrix}
0 & \cos \beta & -\sin \beta & 0 \\
0 & \sin \beta & \cos \beta & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

hence the discussion of this case reduces to the 3-dimensional one.
Example 2.3. The following example will be given for $n = 5$. Let 
$r > 0, a, b, c, \theta$, be five constants, such that 

$$4(a^2 + b^2)\sin^4 \frac{\theta}{2} = c^2 r^2.$$ 

Consider the curve $c : s \mapsto \exp(s \Omega)$, where

$$\Omega = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\
-a & 0 & b & 0 & 0 \\
0 & -b & 0 & c & 0 \\
0 & 0 & -c & 0 & \sin \frac{\theta}{r} \\
0 & 0 & 0 & -\frac{\sin \theta}{r} & 0 \end{bmatrix}.$$ 

By the Frenet formulae, the curvatures of $c$ are $K_1 = a$, $K_2 = b$, $K_3 = c$, $K_4 = r^{-1} \sin \theta$. Integrating the system $(0) + (I) + (II) + (IV)$, we obtain

$$\tilde{c}(s) = c(s) - \left[ \frac{4}{r^3} \frac{b^2}{a^2} \sin^6 \frac{\theta}{2} s - a\alpha(s) \right] e_1$$

$$+ \dot{\alpha}(s)e_2 + \left[ \left( \frac{2}{r} \frac{b}{a} \sin^2 \frac{\theta}{2} - \frac{4}{r^2} \sin^6 \frac{\theta}{2} \right) s - b\alpha(s) \right] e_3 + \frac{2}{r} \frac{b}{ac} \sin \frac{\theta}{2} e_4,$$

and

$$a_{43} = -\frac{2}{r} \frac{1}{c} \sin^2 \frac{\theta}{2},$$

where $\alpha$ satisfies the following differential equation:

$$\dot{\alpha}^2(s) + \left[ \frac{4}{r^3} \frac{b^2}{a^2} \sin^6 \frac{\theta}{2} s - a\alpha(s) \right]^2$$

$$+ \left[ \left( \frac{1}{r} \frac{2b}{a} \sin^2 \frac{\theta}{2} - \frac{4}{r^2} \sin^6 \frac{\theta}{2} \right) s - b\alpha(s) \right]^2 + \frac{1}{r^2} \frac{4b^2}{a^2c^2} \sin^4 \frac{\theta}{2} = 1$$

In particular the constant solutions of this equation can be found explicitly by solving a quadratic polynomial equation.

Using the formulae $\dot{K}_1 = K_1, \ldots, \dot{K}_{n-3} = K_{n-3}, \dot{K}_{n-1} = K_{n-2}$ and $\dot{K}_{n-2} = K_{n-2} + 2K_{n-1} \tan \frac{\theta}{2} a_{n-1,n-2}$, we obtain

$$\dot{K}_1 = a, \quad \dot{K}_2 = b, \quad \dot{K}_4 = \frac{\sin \theta}{r},$$
and
\[ \tilde{K}_3 = c - \frac{8}{r^2} \frac{1}{c} \sin^4 \frac{\theta}{2}, \]
respectively.

Hence, if we impose the initial conditions \( \tilde{E}(0) = I \), where \( I \) is the identical matrix, the Frenet formulae of \( \tilde{c} \) yields
\[ \tilde{c}(s) = \exp s \tilde{\Omega}, \]
where
\[
\tilde{\Omega} = \begin{bmatrix}
0 & a & 0 & 0 & 0 \\
-a & 0 & b & 0 & 0 \\
0 & -b & 0 & c - \frac{4}{c} \frac{8}{r^2} \sin^4 \frac{\theta}{2} & 0 \\
0 & 0 & -c + \frac{4}{c} \frac{8}{r^2} \sin^4 \frac{\theta}{2} & 0 & \frac{\sin \theta}{r} \\
0 & 0 & 0 & -\frac{\sin \theta}{r} & 0
\end{bmatrix}.
\]

References


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