A note on additive commutativity-preserving mappings

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Abstract. We characterize additive surjective commutativity-preserving mappings on $M_n$, $n \geq 2$.

The problem of characterizing linear transformations on $M_n$, the algebra of $n \times n$ complex matrices, that preserve some properties, has been considered in a number of papers. It turns out that this kind of mapping is often of the form

\begin{equation}
X \mapsto \sigma AXA^{-1} + f(X)I \quad \text{or} \quad X \mapsto \sigma AX^{tr}A^{-1} + f(X)I,
\end{equation}

where $\sigma$ is a non-zero complex number, $X^{tr}$ denotes the transpose of $X$, and $f$ is a linear functional on $M_n$. It is natural to try to get similar results studying not linear but merely additive preservers. OMLADIČ and ŠEMRL [10], [9] characterized additive spectrum-preserving mappings and additive mappings preserving operators of rank one. We say that $\phi$ preserves commutativity if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$ (briefly $A \leftrightarrow B$), and it preserves commutativity in both directions if also $\phi(A) \leftrightarrow \phi(B)$ implies $A \leftrightarrow B$. Bijective additive mappings preserving commutativity on more general algebras have been described by BREŠAR, MIERS, BANNING and MATHIEU [4], [5], [2]. This note is a continuation of the work of the present author [11], where we obtained the general form of an additive surjective mapping on $M_n$, $n \geq 3$, that preserves commutativity in both directions. The methods we use here are different, and we

Mathematics Subject Classification: 15A27, 15A04.

Key words and phrases: preserving commutativity, additive mappings, complex matrices.
replace the assumption of preserving commutativity in both directions by the weaker one, preserving commutativity in one direction only. Moreover, the characterization of such mappings on \( M_2 \) is included. For \( n \geq 3 \) we obtain as a result the mappings of the form

\[
X \mapsto \sigma T(X) + p(X)I
\]

where \( \sigma \neq 0 \) is a complex constant, \( p \) a complex valued additive mapping on \( M_n \), and \( T : M_n \to M_n \) is defined either by \([x_{ij}] \mapsto A [f(x_{ij})] A^{-1}\), or \([x_{ij}] \mapsto A [f(x_{ij})]^\text{tr} A^{-1}\) for some invertible matrix \( A \) and a ring automorphism \( f \) on \( \mathbb{C} \). The mapping \( \lambda \mapsto \overline{\lambda} \) of a complex number to its conjugate is a nontrivial continuous ring automorphism of \( \mathbb{C} \). Moreover, there exist nowhere continuous ring automorphisms of \( \mathbb{C} \) [1]. It is not surprising that the result for \( n = 2 \) differs essentially from that for \( n \geq 3 \). Even in the linear case the mappings of the form (1) are not the only ones that arise as bijective commutativity preservers on \( M_2 \) [13]. Any mapping of the form (2) can be regarded as a compositum of a linear bijective commutativity-preserving mapping and a ring automorphism \([x_{ij}] \mapsto [f(x_{ij})]\), additively perturbed by a mapping \( X \mapsto p(X)I \). The same holds true in the two dimensional case. The set of all bijective linear mappings \( \phi : M_2 \to M_2 \) satisfying \( \phi(I) = \lambda I \), for some \( \lambda \neq 0 \), is equal to the set of all bijective linear mappings on \( M_2 \) that preserve commutativity. This is a straightforward consequence of the fact that the commutant \( X' \) (the set of all matrices from \( M_n \) commuting with \( X \)) of any non-scalar matrix \( X \in M_2 \) is only two dimensional, i.e.:

\[
X' = \{ \alpha X + \beta I, \ \alpha, \beta \in \mathbb{C} \}.
\]

Before giving the proofs we introduce some notation: \([A, B] = AB - BA\), \( E_{ij} = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. The mapping \( \phi : M_n \to M_n \) is called \( f \)-quasilinear, for some \( f : \mathbb{C} \to \mathbb{C} \), if it is additive, and if the relation \( \phi(\alpha X) = f(\alpha)\phi(X) \) holds for all complex numbers \( \alpha \) and \( X \in M_n \).

**Theorem.** Let \( \phi \) be an additive surjective commutativity-preserving mapping on \( M_n \), \( n \geq 2 \).
If \( n \geq 3 \) then there exists a ring automorphism \( f : \mathbb{C} \to \mathbb{C} \), a non-zero complex constant \( \sigma \), an invertible matrix \( A \) and an additive function \( p : M_n \to \mathbb{C} \) such that \( \phi \) is either of the form

\[
\begin{align*}
(\text{a}) \quad & \phi([x_{ij}]) = \sigma A[f(x_{ij})]A^{-1} + p([x_{ij}])I, \quad [x_{ij}] \in M_n, \\
(\text{b}) \quad & \phi([x_{ij}]) = \sigma A[f(x_{ij})]^\text{tr} A^{-1} + p([x_{ij}])I, \quad [x_{ij}] \in M_n.
\end{align*}
\]

In the case \( n = 2 \) there exists a ring automorphism \( f : \mathbb{C} \to \mathbb{C} \), an additive function \( p : M_2 \to \mathbb{C} \) and a linear mapping \( L : M_2 \to M_2 \) which leaves the subspace \( \{ \lambda I, \lambda \in \mathbb{C} \} \) invariant, such that \( \phi \) is of the form

\[
\phi([x_{ij}]) = L([f(x_{ij})]) + p([x_{ij}])I, \quad [x_{ij}] \in M_2.
\]

Remarks. 1. This note contains also the proof for \( n = 2 \), the case that is exceptional, and was not considered in the previously mentioned papers.

2. If we add the assumption of injectivity, the result for \( n \geq 3 \) follows from \([4]\).

3. Not only do we not need injectivity, in this particular case, studying the mappings on \( M_n \), the proof is much shorter, and involves only simple linear algebra tools.

**Proof.** We will show that \( \phi \) is not “very far” from being linear. As \( \phi \) preserves commutativity we have that

\[
\phi(\alpha X) \leftrightarrow \phi(X)
\]

for all complex numbers \( \alpha \), and \( X \in M_n \). Let \( \mu \in \mathbb{C} \), \( \mu \neq 0 \), be fixed. Since \( \phi \) is surjective, we can get for every pair of indices \( i, j \) a matrix \( Z_{ij} \) with \( \phi(Z_{ij}) = \mu E_{ij} \). All block matrices in the proof will be partitioned according to \( \mathbb{C}^n = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \text{Span}(e_3, \ldots, e_n) \) where \( \{e_i, 1 \leq i \leq n\} \) is the standard basis of \( \mathbb{C}^n \). The blocks that are not of dimension \( 1 \times 1 \) will be denoted using capital letters. We have divided the proof into three steps.

**Step 1.** If \( \phi(X) = \mu E_{ij} + \delta I, \delta \in \mathbb{C}, 1 \leq i, j \leq n \), then \( \phi(\alpha X), \alpha \in \mathbb{C} \), is a sum of a scalar multiple of \( E_{ij} \) and a diagonal matrix \( D = \text{diag}(d_1, d_2, \ldots, d_n) \) satisfying \( d_i = d_j \).
If \( n = 2 \) this is a straightforward consequence of (3) and (4). Let \( n \geq 3 \) and \( i = j \). As \( E_{ii} \) is similar to \( E_{11} \) (by a permutation matrix) we may assume \( i = 1 \) with no loss of generality. Because of (4) \( \phi(\alpha X) \leftrightarrow E_{11} \), and is therefore of the form

\[
\phi(\alpha X) = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.
\]

By the same argument, we have that

\[
\phi(\alpha Z_{22}) = \begin{bmatrix} b_{11} & 0 & B_{13} \\ 0 & b_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix},
\]

and by the additivity of \( \phi \)

\[
\phi(\alpha (X + Z_{22})) = \begin{bmatrix} a_{11} + b_{11} & 0 & B_{13} \\ 0 & a_{22} + b_{22} & A_{23} \\ B_{31} & A_{32} & A_{33} + B_{33} \end{bmatrix}.
\]

For the same reason, \( \phi(\alpha (X + Z_{22})) \) commutes with \( \phi(X) + \phi(Z_{22}) \), and therefore also with \( E_{11} + E_{22} \). This forces \( A_{23} \) and \( A_{32} \) to be zero. Replacing \( Z_{22} \) by \( Z_{kk}, k \geq 3 \), in the previous consideration, we get that \( A_{33} \) is a diagonal matrix. In particular, \( \phi(\alpha Z_{ii}), 1 \leq i \leq n \), is a diagonal matrix for every \( \alpha \in \mathbb{C} \).

Assume now \( i \neq j \). Without loss of generality, we can fix \( (i, j) = (1, 2) \). Since \( \phi(\alpha X) \) commutes with \( E_{12} \), we may write its block matrix as

\[
\phi(\alpha X) = \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix}.
\]

The matrix \( \phi(\alpha (X + Z_{11} + Z_{22})) \), which is of the form

\[
\phi(\alpha (X + Z_{11} + Z_{22})) = \phi(\alpha X) + \phi(\alpha Z_{11}) + \phi(\alpha Z_{22})
\]

\[
= \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix}
\]

\[
= \begin{bmatrix} c_{11} + a_{11} + b_{11} & c_{12} & C_{13} \\ 0 & c_{11} + a_{22} + b_{22} & 0 \\ 0 & C_{32} & C_{33} + A_{33} + B_{33} \end{bmatrix}.
\]
commutes with $\phi (X + Z_{11} + Z_{22}) = \phi (X) + \phi (Z_{11}) + \phi (Z_{22})$, and consequently with $E_{12} + E_{11} + E_{22}$. This implies

$$C_{13} = 0 \quad \text{and} \quad C_{32} = 0.$$ 

In order to get that $C_{33}$ is diagonal, we choose $k$, $3 \leq k \leq n$, and compute

$$\phi (\alpha (X + Z_{kk})) = \phi (\alpha X) + \phi (\alpha Z_{kk})$$

$$= \begin{bmatrix} c_{11} & c_{12} & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} + \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} + d_{11} & c_{12} & 0 \\ 0 & c_{11} + d_{22} & 0 \\ 0 & 0 & C_{33} + D_{33} \end{bmatrix}.$$ 

We know that the last matrix commutes with $\phi (X + Z_{kk}) = \mu (E_{12} + E_{kk}) + \delta I$, and therefore also with $E_{kk}$. Moreover, $D_{33}$ is diagonal, which yields the desired conclusion.

**Step 2.** If $\phi (X) = \mu E_{ij} + \delta I$, $\delta \in \mathbb{C}$, $1 \leq i, j \leq n$, then there exists a ring automorphism $f : \mathbb{C} \to \mathbb{C}$ (independent of $i, j$ and $\delta$) and a complex valued function $v_{ij}$, such that for every $\alpha \in \mathbb{C}$ holds

$$\phi (\alpha X) = f (\alpha) \mu E_{ij} + v_{ij} (\alpha, \delta) I,$$ 

and $v_{ij}$ is additive in the first argument.

It suffices to get (5) for $i = 1$ if $i = j$, and for $(i, j) = (1, 2)$ in the case $i \neq j$. Suppose first $i = j = 1$. If $n \geq 3$ choose $r \geq 3$. Using the additivity of $\phi$ and applying Step 1 leads to

$$\phi (\alpha (X + Z_{2r})) = \phi (\alpha X) + \phi (\alpha Z_{2r})$$

$$= \text{diag} (a_1, \ldots, a_n) + \text{diag} (d_1, \ldots, d_n) + u_{2r} E_{2r}, \quad d_2 = d_r,$$

and because of (4), the matrix $\phi (\alpha (X + Z_{2r}))$ commutes with $E_{11} + E_{2r}$. Therefore, $a_r = a_2$ for all $r \geq 3$. Now, there exist functions $u_{11}$ and $v_{11}$, both additive in the first argument, such that

$$\phi (\alpha X) = u_{11} (\alpha, \delta) E_{11} + v_{11} (\alpha, \delta) I.$$
Next, we shall derive a similar equation to the above one for \((i, j) = (1, 2)\). Let \(\phi(X) = \mu E_{12} + \delta I\). For any \(k\), \(3 \leq k \leq n\), \(\phi(\alpha (X + Z_{1k}))\) commutes with \(E_{12} + E_{1k}\). This gives

\[
0 = [\phi(\alpha (X + Z_{1k})), E_{12} + E_{1k}]_{1k} = \phi(\alpha X)_{11} - \phi(\alpha X)_{kk},
\]

and, applying the assertion of Step 1, we get the existence of the functions \(u_{12}\) and \(v_{12}\), such that

\[
\phi(\alpha X) = u_{12}(\alpha, \delta) E_{12} + v_{12}(\alpha, \delta) I.
\]

Hence, if \(\phi(X) = \mu E_{ij} + \delta I\), \(1 \leq i, j \leq n\), there exist functions \(u_{ij}\) and \(v_{ij}\) such that

\[(7) \quad \phi(\alpha X) = u_{ij}(\alpha, \delta) E_{ij} + v_{ij}(\alpha, \delta) I.\]

Functions \(u_{ij}\) and \(v_{ij}\) are additive in the first argument and unique. In the case \(n = 2\), and \(i \neq j\), the relation (7) is a straightforward consequence of Step 1. If \(n = 2\) and \(i = j\), \(\phi(\alpha X)\) is a diagonal matrix by Step 1, and as \(\mu \neq 0\) was fixed, we get (7) with \(u_{ii}\) and \(v_{ii}\), \(i = 1, 2\), unique.

Since \(\phi\) is surjective, there exist matrices \(X_{ij}\) with \(\phi(X_{ij}) = E_{ij}\). Fix the set \(\{X_{ij}; \phi(X_{ij}) = E_{ij}\}\). In the previous consideration \(\mu \neq 0\) was fixed but arbitrary. In particular, the application of (7) at \(\mu = 1\) and \(\delta = 0\) guaranties the existence of uniquely defined additive functions \(f_{ij}\) and \(g_{ij}\) with

\[(8) \quad \phi(\alpha X_{ij}) = f_{ij}(\alpha) E_{ij} + g_{ij}(\alpha) I.\]

We will now show that the functions \(f_{ij}\) are independent of \(i\) and \(j\). Let \(i \neq j\). By (4), and the additivity of \(\phi\), we have

\[
\phi(\alpha (X_{ii} + X_{ij})) \leftrightarrow E_{ii} + E_{ij},
\]

and

\[
\phi(\alpha (X_{ii} + X_{ij})) = f_{ii}(\alpha) E_{ii} + f_{ij}(\alpha) E_{ij} + (g_{ii}(\alpha) + g_{ij}(\alpha)) I.
\]

Therefore,

\[
0 = [f_{ii}(\alpha) E_{ii} + f_{ij}(\alpha) E_{ij}, E_{ii} + E_{ij}] = (f_{ii}(\alpha) - f_{ij}(\alpha)) [E_{ii}, E_{ij}],
\]
which implies \( f_{ii} = f_{ij} \). Replacing \( E_{ij} \) by \( E_{ji} \) in the above computation, we also obtain \( f_{ii} = f_{ji} \) for all \( 1 \leq i, j \leq n, n \geq 2 \). From now on \( f \) will be written instead of \( f_{ij} \).

Our next goal is to show that

\[
(9) \quad u_{ij}(\alpha, \delta) = f(\alpha)\mu
\]

for all \( 1 \leq i, j \leq n \). Let \( \phi(X) = \mu E_{ij} + \delta I \), and take \( k, k \neq j \). As \( \phi(\alpha(X + X_{ik})) \leftrightarrow \mu E_{ij} + E_{ik} \), we have that

\[
0 = [u_{ij}(\alpha, \delta) E_{ij} + f(\alpha)E_{ik}, \mu E_{ij} + E_{ik}]
= (u_{ij}(\alpha, \delta) - \mu f(\alpha))[E_{ij}, E_{ik}].
\]

Certainly, we can always choose \( k, k \neq j \), such that \([E_{ij}, E_{ik}] \neq 0\), and the desired conclusion now follows.

What is left is to show that \( f \) is multiplicative and surjective. For all complex \( \alpha \) and \( \beta \) we have

\[
(10) \quad \phi(\alpha\beta X_{12}) = f(\alpha\beta)E_{12} + g_{12}(\alpha\beta)I
\]

and

\[
\phi(\beta X_{12}) = f(\beta)E_{12} + g_{12}(\beta)I.
\]

If \( f(\beta) = 0 \) the relation \( \phi(\alpha(\beta X_{12} + X_{lk})) \leftrightarrow \phi(\beta X_{12}) + E_{lk} \), \( 1 \leq l, k \leq n \), implies that \( \phi(\alpha\beta X_{12}) \) is a scalar matrix, and thus \( f(\alpha\beta) = 0 \). Take now \( \mu = f(\beta) \neq 0, \delta = g_{12}(\beta) \) and \( (i, j) = (1, 2) \). Combining equations (7) with \( X \) being replaced by \( \beta X_{12} \), and (9) we obtain

\[
(11) \quad \phi(\alpha(\beta X_{12})) = f(\alpha)f(\beta)E_{12} + v_{12}(\alpha, g_{12}(\beta)) I.
\]

Comparing the last equation to (10) gives the multiplicativity of \( f \).

It is routine to show that the set \( \{X_{ij}, \phi(X_{ij}) = E_{ij}\} \), that has already been fixed before, forms a basis of \( M_n \), \( n \geq 2 \). For details we refer the reader to [11, p. 208]. From the linear independence of the set \( \{X_{ij}\} \), the relation (8) and the surjectivity of \( \phi \), the surjectivity of \( f \) is now easily obtained.

Note that for every \( X \in M_n \) there exist unique numbers \( \alpha_{ij}, 1 \leq i, j \leq n \), such that \( X = \sum_{i,j} \alpha_{ij} X_{ij} \).
**Step 3.** There exists a surjective linear mapping $L : M_n \to M_n$, $n \geq 2$, that preserves commutativity, and an additive function $q$ on $M_n$ such that

$$
\phi ([x_{ij}]) = L ([f (x_{ij})]) + q ([x_{ij}]) I.
$$

Let us first define an additive mapping $\phi_1 : M_n \to M_n$,

$$
\phi_1 (X) = \phi_1 \left( \sum_{i,j} \alpha_{ij} X_{ij} \right) = [f (\alpha_{ij})],
$$

which is surjective (since $f$ is surjective), and preserves commutativity because of

$$
\phi (X) = \phi \left( \sum_{i,j} \alpha_{ij} X_{ij} \right)
$$

(12)

$$
= \phi_1 (X) + \left( \sum_{i,j} g_{ij} (\alpha_{ij}) \right) I
$$

$$
= \phi_1 (X) + p(X) I.
$$

Clearly, $p$ just involved is additive. Furthermore, we observe that $\phi_1$ is $f$-quasilinear as

$$
\phi_1 (\alpha X) = \phi_1 \left( \sum_{i,j} \alpha \alpha_{ij} X_{ij} \right)
$$

$$
= f(\alpha) \left[ f (\alpha_{ij}) \right]
$$

$$
= f(\alpha) \phi_1 (X)
$$

for every $\alpha \in \mathbb{C}$. Let $\psi$ denote the mapping on $M_n$ defined by

$$
\psi ([x_{ij}]) = [f (x_{ij})]
$$

which is additive, bijective and preserves commutativity in both directions. Finally, we define $L = \phi_1 \circ \psi^{-1}$, and observe that it is homogeneous. Indeed,

$$
L(\alpha X) = \phi_1 (\psi^{-1}(\alpha X)) = \phi_1 (f^{-1}(\alpha)\psi^{-1}(X)) = \alpha L(X).
$$
Moreover, $L$ is additive, surjective and preserves commutativity which establishes the assertion of Step 3.

Since $L$ is linear, surjective and preserves commutativity, we then clearly have $L(I) = cI$, for some $c \neq 0$. If $n = 2$, the relation $\phi_1 = L \circ \psi$ substituted in (12) gives the desired conclusion. If $n \geq 3$ we end the proof of the theorem by substituting the well known form of a surjective linear commutativity-preserving mapping [3], [13] in (12).

Acknowledgement. The author is grateful to prof. P. Šemrl for the careful reading of this note, and for many valuable suggestions.

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(Received April 16, 1998; revised April 2, 1999)