On optimal linear congruences for \( L_2(k, \chi \omega^{1-k}) \)

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Dedicated to Professor Kálmán Győry
on the occasion of his 60th birthday

Abstract. Our purpose in the paper is to investigate divisibility properties of 2-adic \( L \)-functions attached to quadratic characters at integers. Following Uehara’s ideas we extend the linear congruence relations proved in [6], [8] and [10] (see also [3], [4], [5], [6] and [7]). For any two-element subset \( L \) of the set \( \{-1, 0, 1, 2\} \) we determine the so-called optimal linear congruence relations for \( L_2(k, \chi \omega^{1-k}) \), with \( k \in L \).

1. Notation

For prime \( p \) as usual we denote by \( \mathbb{C}_p \) the completion of the algebraic closure of \( \mathbb{Q}_p \). \( \mathbb{Q}_p \) denotes the field of \( p \)-adic numbers. For \( a, b \in \mathbb{C}_p \) and \( \alpha \in \mathbb{Q} \) the notation \( a \equiv b \pmod{p^\alpha} \) means that \( |a - b|_p \leq p^{-\alpha} \). \( | \cdot |_p \) denotes the normalized (such that \( |p|_p = 1/p \)) absolute value on \( \mathbb{C}_p \). For \( a, b \in \mathbb{Z} \) and \( \alpha \in \mathbb{N} \) these congruences are the usual congruences for integral rational numbers. We say that \( a \in \mathbb{C}_p \) is \( p \)-integral if \( a \equiv 0 \pmod{p^0} \). For \( a \in \mathbb{Q} \), if \( a \) is \( p \)-integral in the above sense then its denominator is not divisible by \( p \). We say that \( p \)-integral number \( a \) is divisible by \( p^\alpha \) (\( \alpha \geq 1 \)) if \( a \equiv 0 \pmod{p^\alpha} \). We write \( p^\alpha \mid a \). If for \( p \)-integral number \( a \) we have

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\( a \not\equiv 0 \pmod{p^\alpha} \), we write \( p^\alpha \nmid a \) and say that \( a \) is not divisible by \( p^\alpha \). For \( \alpha \in \mathbb{N} \) if \( p^\alpha \mid a \) and \( p^{\alpha+1} \nmid a \), we set \( p^\alpha \| a \). For \( \alpha, \beta \in \mathbb{N} \) and \( p^\alpha \| a \), we write \( \gcd(p^\beta, a) = p^\alpha \) (resp. \( p^\beta \)) if \( \alpha \leq \beta \) (resp. \( \beta < \alpha \)). If \( a \equiv b \pmod{p^\beta} \), we have

\[
\gcd(p^\beta, a) = \gcd(p^\beta, b).
\]

Moreover if \( m, n \in \mathbb{C}_p \) are \( p \)-integers not divisible by \( p \), we observe that

\[
\gcd(p^\beta, a) = \gcd(p^\beta, \frac{a}{m}) = \gcd(p^\beta, an).
\]

We say that \( a \in \mathbb{C}_2 \) is even if \( a \) is 2-integral and divisible by 2. We say that \( a \) is odd if \( a \) is 2-integral and is not even.

As usual let \( \log = \log_p \), \( \omega = \omega_p \) denote the \( p \)-adic logarithm and the Teichmüller character at \( p \) respectively. For a Dirichlet character \( \chi \) let \( L_p(s, \chi) \) be the Kubota–Leopoldt \( L \)-function. For details see [9].

For \( k \in \mathbb{Z} \) let \( l_k = l_{k,p} \) denote the so-called multilogarithms, which are locally analytic functions on the set \( \mathbb{C}_p - \{1\} \) defined inductively by \( l_0(s) = -s/(1-s) \), \( dl_k(s) = l_{k-1}(s)ds/s \) and \( \lim_{s \to 0} l_k(s) = 0 \). For details, see [1]. Moreover if \( k \leq 0 \), we have \( l_k(s) = s(-1)^k R_{-k}(s)/(1-s)^{1-k} \), where \( R_n \in \mathbb{Z}[x] \) \((n \geq 0)\) are the so-called Frobenius polynomials defined in [2]. If \( k = -1 \) we have \( l_{-1}(s) = s/(1-s)^2 \) in particular. If \( k = 1 \), we have \( l_1(s) = -\log_p(1-s) \).

The main interest of the multilogarithms is that they give the Coleman formulas

\[
L_p(k, \chi \omega^{1-k}) = (1 - \chi(p)p^{-k}) \frac{\chi(\xi_M)M}{\tau(k)} \sum_{a=1}^{M-1} \chi(a) l_{k,p}(\xi_M^a).
\]

Here \( \chi \) is a primitive non-trivial Dirichlet character modulo \( M \) and throughout the paper we denote by \( \xi_M \) a primitive \( M \)-th root of unity in \( \mathbb{C}_p \).

For a fundamental discriminant \( d \neq 1 \) as usual we denote by \( \chi_d \) the associated quadratic character (Kronecker symbol). We set \( \chi_1 = 1 \). Denote by \( T_d \) the set of all fundamental discriminants dividing \( d \). Throughout the paper, for \( t, c \in \mathbb{Z} \) \((t \neq 0, c \geq 1)\) we denote by \( \nu(t) \) the number of distinct prime factors of \( t \) and adopt the notation \( \sum_{a=1}^{c'} \) to a sum taken over integers \( a \) prime to \( c \). As usual \( \phi \) denotes Euler’s phi function.
The proofs of the main theorems of the paper (Theorems 1 and 2) are based on the following lemma.

**Lemma 1** (see [8, Lemma 1], cf. [6, Lemma 3]). Let \( \chi \) be a Dirichlet character modulo \( M > 1 \) and let \( N \) be a multiple of \( M \) such that \( N/M > 0 \) is a rational square-free integer relatively prime to \( M \). For arbitrary natural number \( T \) satisfying \( M | T | N \) we assume that \( \zeta_T = \zeta_M \zeta_T/M \) and set

\[
S_{k,\chi}(T) = \sum_{a=1}^{T} \chi(a)l_k(\zeta_T^a).
\]

Then for any integer \( k \) we have

\[
S_{k,\chi}(N) = (-1)^{\nu(N/M)} \prod_{p | (N/M) \text{ prime}} (1 - \overline{\chi(p)}p^{1-k}) S_{k,\chi}(M).
\]

2. Quadratic fields

If \( d \) is the discriminant of a quadratic field, we denote by \( h(d) \), \( k_2(d) \), \( \varepsilon_d \), resp. \( R_2(d) \) the class number, the order of the \( K_2 \)-group of the integers, the fundamental unit, resp. the second Borel regulator of the field \( \mathbb{Q}(\sqrt{d}) \). For \( k \in \{-1, 0, 1, 2\} \) we have

\[
L(k, \chi_d) = \begin{cases} 
-12w_{-1}^{-1}(d)k_2(d), & \text{if } k = -1 \text{ and } d > 1, \\
2w^{-1}(d)h(d), & \text{if } k = 0 \text{ and } d < 0, \\
2d^{-1/2}h(d)\log \varepsilon_d, & \text{if } k = 1 \text{ and } d > 1, \\
2R_2(d)|d|^{-3/2}k_2(d), & \text{if } k = 2 \text{ and } d < 0,
\end{cases}
\]

where \( w(-3) = 6, w(-4) = 4, w(d) = 2 \) if \( d < -4 \) and \( w_2(8) = 48, w_2(5) = 120, w_2(d) = 24 \) if \( d > 8 \). Here \( L(s, \chi) \) is the classical, complex Dirichlet \( L \)-function attached to \( \chi \). In the case when \( k = 2 \) we assume that the so-called Lichtenbaum conjecture for imaginary quadratic fields holds.
Usually, the complex and \( p \)-adic formulas differ by an Euler factor. Namely we have

\[
L_p(k, \chi_d \omega^{1-k}) = \begin{cases} 
-12w_2^{-1}(d)(1 - \chi_d(p)p)k_2(d), & \text{if } k = -1 \text{ and } d > 1, \\
2w^{-1}(d)(1 - \chi_d(p))h(d), & \text{if } k = 0 \text{ and } d < 0, \\
2d^{-1/2}(1 - \chi_d(p)p^{-1})h(d)p \log \varepsilon_d, & \text{if } k = 1 \text{ and } d > 1, \\
2R_{2,p}(d)|d|^{-3/2}(1 - \chi_d(p)p^{-2})k_2(d), & \text{if } k = 2 \text{ and } d < 0,
\end{cases}
\]

where by analogy \( R_{2,p}(d) \) denotes the second \( p \)-adic regulator of the corresponding field \( \mathbb{Q}(\sqrt{d}) \). In the case when \( k = 2 \) the above equation is the statement of a \( p \)-adic analogue of the Lichtenbaum conjecture for imaginary quadratic fields.

3. The numbers \( W_{k,e}(n) \)

Let \( k, n \in \mathbb{Z} \) and \( e \in T_8 \). For \( n \geq 0 \) write

\[
\gamma_{n,e} = \begin{cases} 
-1, & \text{if } n \equiv 1, 2 \pmod{4} \text{ and } e \in T_8 - T_4, \\
1, & \text{otherwise}
\end{cases}
\]

and

\[
W_{k,e}(n) = \sum_{l=0}^{n} (-1)^{(k+1)(2l+1)}(2n+1)^{1-k} \gamma_{n,e} \left( \frac{2n+1}{n-l} \right).
\]

The numbers \( W_{k,e}(n) \) are 2-integral rational numbers. We have \( \text{ord}_2(W_{k,e}(n)) \geq n \). For details see [10].

4. Uehara’s functions

From now on we assume that \( p = 2, \omega = \omega_2 \) and \( l_k = l_{k,2} \). For any Dirichlet character \( \psi \) modulo \( f \) and \( k \in \mathbb{Z} \) let \( L_{k,\psi} \) denote the so-called Uehara functions. These functions are defined by

\[
L_{k,\psi}(s) = \frac{1}{2}(-1)^{k+1}(l_k(s) - l_k(-s)) \quad (s \neq \pm 1),
\]
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if $\psi$ is the trivial character, and

$$L_{k,\psi}(s) = (-1)^{k+1} \frac{\tau(\overline{\psi}, \zeta_f)}{f} \sum_{a=1}^f \psi(a) l_k(\zeta_f^a s) \quad (s \neq \zeta_f^a)$$

otherwise. For details see [8]. For $\psi = \chi_e$ set $L_{k,\psi} = L_{k,e}$.

The proof of the main result of the paper (Theorem 1) is based on the following properties of Uehara’s functions implied by the identity of Lemma 1 and proved in [8] and [10].

**Lemma 2** (see [6], [8, Lemma 2] and [10, Lemma 1]). Given any odd integer $M$, let $\chi$ by a primitive Dirichlet character modulo $M$. Suppose that $N$ is an odd multiple of $M$ such that $N/M > 0$ is a rational square-free integer relatively prime to $M$. Let $\psi$ be a primitive Dirichlet character being either trivial or of even conductor coprime to $N$. Assume that for arbitrary natural number $T$ satisfying $M | T \mid N$ we have $\zeta_T = \zeta_M \zeta_T/M$. Then for any integer $k$ we have

$$\frac{\tau(\chi, \zeta_M)}{M} \sum_{a=1}^N \chi(a) L_{k,\psi}(\zeta_N^a)$$

$$= (-1)^{\nu(N/M)} \prod_{p | (N/M)} (1 - \chi(p)p^{1-k}) L_2(k, \chi \psi \omega^{1-k}),$$

unless $k = 1$ and the characters $\chi$ and $\psi$ are trivial, in which case we have

$$\sum_{a=1}^N L_{k,\psi}(\zeta_N^a) = \begin{cases} -(\log_2 N)/2, & \text{if } N \text{ is a prime number,} \\ 0, & \text{otherwise.} \end{cases}$$

**Remark.** In the formulation of Lemma 2 of [8] there is a small error, which implies the same error in Lemma 1 of [10]. The right hand sides of the identities of the lemmas should be multiplied by $(-1)^{k+1}$.

**Lemma 3.** Let $c$ (>$1$) be an odd natural number. If $k \neq 0, 1$ we have

$$\sum_{a=1}^c l_k(\zeta_c^a) = (-1)^{k+1+\nu(c)} (1 - 2^{-k})^{-1} \prod_{p | c} (1 - p^{1-k}) L_2(k, \omega^{1-k}).$$
If $k = 0$ or $1$ we have

$$\sum_{a=1}^{c} l_k(\zeta_c^a) = \begin{cases}
-\frac{1}{2} \phi(c), & \text{if } k = 0, \\
- \log_2 c, & \text{if } k = 1 \text{ and } c \text{ is a prime number}, \\
0, & \text{otherwise}
\end{cases}$$

**Proof.** Given $r \in \mathbb{N}$ we have

$$\frac{1}{r} \sum_{\zeta^r = 1} l_k(\zeta) = \frac{l_k(z^r)}{r^k}$$

(see [1, Proposition 6.1]). Applying this formula with $r = 2$ we obtain

$$\mathcal{L}_{k,1}(s) = (-1)^{k+1}(l_k(s) - 2^{-k}l_k(s^2)) \quad (s \neq \pm 1).$$

Hence we have

$$\sum_{a=1}^{c} l_k(\zeta_c^a) = (-1)^{k+1}(1 - 2^{-k})^{-1} \sum_{a=1}^{c} \mathcal{L}_{k,1}(\zeta_c^a)$$

because

$$\left(1 - 2^{-k}\right) \sum_{a=1}^{c} l_k(\zeta_c^a) = (-1)^{k+1} \sum_{a=1}^{c} \left((-1)^{k+1}(l_k(\zeta_c^a) - l_k(\zeta_c^{2a}))\right)$$

$$= (-1)^{k+1} \sum_{a=1}^{c} \mathcal{L}_{k,1}(\zeta_c^a).$$

Thus Lemma 3 in the case when $k \neq 0$ follows easily from Lemma 2.

If $k = 0$ we have

$$\sum_{a=1}^{c} l_0(\zeta_c^a) = \sum_{a=1}^{c} \frac{\zeta_c^a}{1 - \zeta_c^a} = \sum_{a=1}^{c} \frac{1}{1 - \zeta_c^a} - \phi(c) = \frac{1}{2} \phi(c) - \phi(c) = -\frac{1}{2} \phi(c),$$

which completes the proof. \hfill \square

**Lemma 4** (cf. [6, Lemma 2]). Given $d \neq 1$ an odd fundamental discriminant we have

$$\sum_{a=1}^{\lfloor |d| \rfloor} \chi_d(a) l_0(\zeta_{|d|}^a) = \begin{cases}
\frac{|d|h(d)}{\tau(\chi_d, \zeta_{|d|})}, & \text{if } d < 0, \\
0, & \text{otherwise}.
\end{cases}$$
Proof. By the definition of $l_0$ we have
\[
\sum_{a=1}^{|d|} \chi_d(a) l_0(\zeta_d^a) = \sum_{a=1}^{|d|} \frac{\chi_d(a) \zeta_d^a}{1 - \zeta_d^a} = \sum_{a=1}^{|d|} \frac{\chi_d(a)}{1 - \zeta_d^a} - \sum_{a=1}^{|d|} \chi_d(a).
\]
Hence and from Lemma 2 [6] the identity of the hypothesis of Lemma 4 follows immediately. \(\square\)

In Lemmas 5 and 6 $\xi$ (≠ 1) denotes a primitive $N$th root of unity, where $N$ is an odd natural number.

Lemma 5 (see [6] and [8, Lemma 4]). For any $e \in \mathcal{T}_8$ write $\alpha = \text{sgn } e$ and set
\[
w_\alpha = \frac{\alpha \xi}{1 + \alpha \xi^2}.
\]
Then we have
\[
\mathcal{L}_{-1,e}(\xi) = \sum_{k=0}^\infty (4\alpha)^k w_\alpha^{2k+1}, \quad \mathcal{L}_{0,e}(\xi) = \omega_{-\alpha},
\]
\[
\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^\infty \frac{(4\alpha)^k \omega_{-\alpha}^{2k+1}}{2k+1}, \quad \mathcal{L}_{2,e}(\xi) = \sum_{k=0}^\infty \left( \frac{-16\alpha}{2k+1} \right)^{2k} \left( \frac{2k}{k} \right)^{-1},
\]
if $e \in \mathcal{T}_4$, and
\[
\mathcal{L}_{-1,e}(\xi) = -\sum_{k=0}^\infty (2\alpha)^k (2k-1) \omega_{\alpha}^{2k+1}, \quad \mathcal{L}_{0,e}(\xi) = \sum_{k=0}^\infty (-2\alpha)^k \omega_{\alpha}^{2k+1},
\]
\[
\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^\infty \frac{(2\alpha)^k \omega_{\alpha}^{2k+1}}{2k+1},
\]
\[
\mathcal{L}_{2,e}(\xi) = \sum_{k=0}^\infty \left( \frac{-16\alpha}{2k+1} \right)^{2k} \left( \frac{2k}{k} \right)^{-1} \sum_{l=0}^k \frac{2l}{l} \sum_{l=0}^k 2^{-3l},
\]
if $e \in \mathcal{T}_8 - \mathcal{T}_4$.

Remark. Uehara in a letter to the author has observed that the formulas for $\mathcal{L}_{-1,e}(\xi)$ and $\mathcal{L}_{2,e}(\xi)$ given in the above lemma can be deduced easily from his formulas for $\mathcal{L}_{0,e}(\xi), \mathcal{L}_{1,e}(\xi)$, and differential properties of Coleman’s multilogarithms. The details of the proof are left to the reader as an exercise.
Lemma 6 (see [10, Lemma 3]). For any \( e \in T_8 \) and \( m \in \mathbb{Z} \) write
\[
\alpha = (-1)^{m+1} \text{sgn } e \quad \text{and let}
\]
\[
w_\alpha = \frac{\alpha \xi}{1 + \alpha \xi^2}.
\]
Then we have
\[
L_{m,e}(\xi) = \sum_{k=0}^{\infty} \frac{\alpha^k W_{m,e}(k)}{2k+1} w_\alpha^{2k+1}.
\]

5. Some special sequences

Let \( K \) be a finite non-empty subset of the rational integers. We will consider linear combinations of Uehara’s functions at \( \xi \) with 2-adic integral coefficients
\[
x = \{ x_{k,e} \}_{(k,e) \in K \times T_8} \subseteq \mathbb{C}_2.
\]
For any \( L \subseteq K \) the \( x \) is said to be defined on \( L \) if \( x_{k,e} = 0 \) for \( k \not\in L \). Let
\[
\alpha_k = \binom{2k}{k}^{-1} \quad \text{and} \quad \beta_k = \binom{2k}{k}^{-1} \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l}.
\]

Given 2-adic integers \( a_{k,e}(n) \in \mathbb{C}_2 \) with \( k \in K \), \( e \in T_8 \), \( n \geq 0 \) we consider some sequences of linear combinations of \( x_{k,e} \) of the form
\[
y_n(x) = \sum_{(k,e) \in K \times T_8} a_{k,e}(n) x_{k,e}, \quad n \geq 0.
\]

For any \( L \subseteq K \) the sequence \( (y_n)_{n \geq 0} \) of this form is said to be defined on \( L \), if the sum is taken over \( (k,e) \in L \times T_8 \).

For \( x = \{ x_{k,e} \}_{(k,e) \in K \times T_8} \) we consider two sequences \( z = (z_n)_{n \geq 0} \) and \( u = (u_n)_{n \geq 0} \) of the form (5.3). The sequences are defined on \( K = \{-1, 0, 1, 2\} \) in the former case and on any finite subset \( K \) of \( \mathbb{Z} \) in the latter case by
\[
z_0 = \sum_{(k,e) \in K \times T_8} x_{k,e}, \quad z_1 = 2 \sum_{(k,e) \in K \times T_8 \text{ sgn } e = (-1)^k} x_{k,e},
\]
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\[ z_{2l+\varrho} = 2^{l+\varrho} \left( 2^l(2l+1)^2((1-\varrho)x_{-1,1} + x_{-1,-4}) \right. \]
\[ - (2l-1)(2l+1)^2((1-\varrho)x_{-1,8} + x_{-1,-8}) \]
\[ + (2l+1)^2((1-\varrho)x_{0,-8} + x_{0,8}) \]
\[ + 2^l(2l+1)((1-\varrho)x_{1,1} + x_{1,-4}) \]
\[ + (2l+1)((1-\varrho)x_{1,8} + x_{1,-8}) \]
\[ + 2^{3l}\alpha_1((1-\varrho)x_{2,-4} + x_{2,1}) \]
\[ + 2^{3l}\beta_1((1-\varrho)x_{2,-8} + x_{2,8}) \left. \right), \]

if $l \geq 1$, $\varrho \in \{0, 1\}$, and

\[ u_{2l+\varrho} = 2^\varrho \sum_{k,e} (-1)^{(k+1)}(2l+1)^{1-k}\gamma_{l,e}x_{k,e}, \quad l \geq 0, \; \varrho \in \{0, 1\}, \]

where the sum in the latter case is taken over all $(k, e) \in K \times T_8$ if $\varrho = 0$, and over $(k, e) \in K \times T_8$ with $\text{sgn} \; e = (-1)^k$ if $\varrho = 1$.

Let $y = (y_n)_{n \geq 0}$ be a sequence of the form (5.3). Let $c = c(y)$ be a non-negative number such that there exist 2-adic integers $x_{k,e}$ not all even satisfying

\[ y_n(x) \equiv 0 \pmod{2^c}, \quad n \geq 0, \]

and if for some 2-adic integers $x_{k,e}$ we have

\[ y_n(x) \equiv 0 \pmod{2^{c+1}}, \quad n \geq 0, \]

then all the numbers $x_{k,e}$ are even.

**Lemma 7** (see [8, Lemma 5]). Let $K = \{-1, 0, 1, 2\}$ and let $L$ be a non-empty subset of $K$. Write $c(L) = c(z)$, where $z = (z_n)_{n \geq 0}$ is the sequence given above, defined on $L$. Then we have

\[ c(L) = 12, \; 9, \; 5, \; \text{resp.} \; 2, \]

if $\text{card}(L) = 4, 3, 2, \; \text{resp.} \; 1$, unless $L = \{-1, 1\}$ or $\{0, 2\}$, in which cases

\[ c(L) = 6. \]
**Lemma 8** (see [10, Lemma 5]). Let \( m \geq 1 \) be an integer and let
\[
K = \{-m + 2, -m + 3, \ldots, 1\}.
\]
Then we have
\[
c(u_n) = 3m - 1 + \text{ord}_2 ((m - 1)!).
\]

**Remark.** Lemma 8 is also valid for any set consisting of \( m \) consecutive integers. In order to prove it we apply the same reasoning as in the proof of Lemma 5 [10].

### 6. Linear combinations of \( L_{k,e}(\xi) \)

Recall that \( N \) is an odd natural number and \( \xi (\neq 1) \) is a primitive \( N \)th root of unity in \( \mathbb{C}_2 \). Given 2-adic integers \( \{x_{k,e}\}_{(k,e) \in K \times T_8} \subseteq \mathbb{C}_2 \) not all even, defined on a non-empty subset \( L \) of \( K \), our purpose is to evaluate the linear combinations
\[
\sum_{(k,e) \in K \times T_8} x_{k,e} L_{k,e}(\xi),
\]
modulo powers of 2. In order to obtain the congruences stated in Lemma 9 we appeal to Lemmas 5 and 7. Combining the obtained congruences with Lemmas 1 and 2 we shall derive some new congruences for linear combinations of the values of 2-adic \( L \)-functions \( L_2(k; \chi \omega_{1-k}) \) with arbitrary 2-adic integral coefficients, where \( \chi \) are primitive quadratic Dirichlet characters.

**Lemma 9** (see [8, Lemma 5]). Set \( K = \{-1, 0, 1, 2\} \). Let \( x_{k,e} \ (k \in K, \ e \in T_8) \) be 2-adic integers not all even defined on a non-empty subset \( L \) of \( K \). Then we have
\[
\sum_{(k,e) \in L \times T_8} x_{k,e} L_{k,e}(\xi) \equiv 0 \pmod{2^\lambda},
\]
where \( 2^\lambda \) is the greatest common divisor of
\[
2^{c(L)} \text{ and } z_n, \quad 0 \leq n \leq \max(2c(L) - 4, 2),
\]
and
\[
c(L) = 12, 9, 5, \text{ resp. } 2,
\]
On optimal linear congruences for \( L_2(k, \chi^1\omega^{-k}) \)

if \( \text{card}(L) = 4, 3, 2, \) resp. 1, unless \( L = \{-1, 1\} \) or \( \{0, 2\} \), in which cases
\[ c(L) = 6. \]

**Proof.** We first observe that for \( n \) even
\[ 2z_n = z_{n+1} + \tilde{z}_{n+1}, \]
where the \( \tilde{z}_{n+1} \) comes from \( z_{n+1} \) by replacing \( x_{k,-4} \) (resp. \( x_{k,1} \), \( x_{k,-8} \) or \( x_{k,8} \)) by \( x_{k,1} \) (resp. \( x_{k,-4} \), \( x_{k,8} \) or \( x_{k,-8} \)).

In [8, Lemma 5] the congruence of Lemma 9 was proved modulo the greatest common divisor of \( 2^{c(L)} \) and \( z_n, 0 \leq n \leq 2c(L)-2 \). Now it suffices to use the congruences
\[ z_{2l+1} \equiv 2^{l+1}\eta \pmod{2^{l+2}}, \quad \tilde{z}_{2l+1} \equiv 2^{l+1}\tilde{\eta} \pmod{2^{l+2}}, \]
\[ z_{2l} \equiv 2^l(\eta + \tilde{\eta}) \pmod{2^{l+1}}, \]
where \( l \geq 1 \) and
\[ \eta = x_{1,-8} + x_{0,8} + x_{1,-8} + x_{2,8}. \]
These congruences follow immediately by the definition of the \( z_{2l+\varphi} \). Indeed we have
\[ z_{2l+\varphi} \equiv 2^{l+\varphi}\left(\left((1-\varphi)x_{-1,8} + x_{-1,-8}\right) + \left((1-\varphi)x_{0,-8} + x_{0,8}\right) + \left((1-\varphi)x_{1,8} + x_{1,-8}\right) + \left((1-\varphi)x_{2,-8} + x_{2,8}\right)\right) \pmod{2^{l+\varphi+1}} \]
because \( \text{ord}_2(2^{3l}a_l) \geq 2l \) and \( \text{ord}_2(2^{3l}b_l) = 0 \).

By the above, we have
\[ z_{2c(L)-2} \equiv 2^{c(L)-1}(\eta + \tilde{\eta}) \pmod{2^{c(L)}} \]
\[ z_{2c(L)-3} \equiv 2^{c(L)-1}\eta \pmod{2^{c(L)}} \]
\[ z_{2c(L)-4} \equiv 2^{c(L)-2}(\eta + \tilde{\eta}) \pmod{2^{c(L)-1}}, \]
\[ z_{2c(L)-5} \equiv 2^{c(L)-2}\eta \pmod{2^{c(L)-1}}, \]
provided \( c(L) > 2 \). Therefore we may ignore \( z_{2c(L)-2} \) and \( z_{2c(L)-3} \) if \( c(L) > 2 \).

Appealing to Lemmas 6 and 8 we obtain:
Lemma 10 (see [10, Lemma 6]). Let $m \geq 1$ be an integer and let
\[ K = \{-m + 2, -m + 3, \ldots, 1\}. \]
Let $x_{k,e}$ $(k, e \in K, e \in T_8)$ be integers in $\mathbb{C}_2$ not all even. Then we have

(i) \[ \sum_{(k,e) \in K \times T_8} x_{k,e} L_{k,e}(\xi) \equiv 0 \pmod{2^\lambda}, \]

where $2^\lambda$ is the greatest common divisor of $2^{c(u_n)}$ and $u_n$, $0 \leq n \leq 4m - 1$,

(ii) for an arbitrary integer $s$
\[ \sum_{(k,e) \in K \times T_8} x_{k,e} L_{k+s,e}(\xi) \equiv 0 \pmod{2^\lambda}. \]

7. Main theorems

In this section we extend linear congruence relations proved in [8] and [10]. We follow Uehara’s ideas from [6] and give a further generalization of the Gras–Uehara type congruence for linear combinations of the values of 2-adic $L$-functions $L_2(k, \chi \omega^{1-k})$, where $\chi$ is a quadratic Dirichlet character. We restrict our attention to the cases when $k$ is taken over an arbitrary non-empty subset $L$ of the set $K = \{-1, 0, 1, 2\}$ or when $k$ is taken over an arbitrary finite set of consecutive integers. These cases were considered in [8] and [10] respectively. It appears to be still an open problem to find the Gras–Uehara type congruence when $k$ is taken over any finite subset of the rational integers.

Let $d$ be an odd fundamental discriminant and let $m > 1$ be a natural number. Throughout the paper let $\Psi, \Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ if $s \mid m$. Let $\delta_{X,Y}$ denote the Kronecker delta function, that is, $\delta_{X,Y} = 1$ if $X = Y$ and is zero otherwise. For $k \in \mathbb{Z}$ and $e \in T_8$ we write
\[ L_2^{[m,\Theta]}(k, \chi_{c,d}\omega^{1-k}) = 0 \]
On optimal linear congruences for $L_2(k, \chi_{\omega^{1-k}})$

if $d = e = k = 1$, and

$$L_2^{[m, \Theta]}(k, \chi_{ed\omega^{1-k}})$$

$$= \left( \prod_{p|m} \left( 1 - \chi_{ed}(p)\Theta(p)p^{1-k} \right) - \delta_{d,1} \prod_{p|m} \left( 1 - \Theta(p) \right) \right) L_2(k, \chi_{ed\omega^{1-k}})$$

otherwise. Set

$$L_2^{[m, \Theta]}(k, \chi_{d\omega^{1-k}})$$

$$= \begin{cases} h(d), & \text{if } k = 0 \text{ and } d < 0, \\ 0, & \text{if } k = 0 \text{ and } d > 0, \\ (1 - \chi_{d}(2)2^{-k})^{-1}L_2^{[m, \Theta]}(k, \chi_{d\omega^{1-k}}), & \text{otherwise}. \end{cases}$$

If $\Theta(s) = 1$ for $s \mid m$, we have $L_2^{[m, \Theta]}(k, \chi_{ed\omega^{1-k}}) = L_2^{[m]}(k, \chi_{ed\omega^{1-k}})$ and

$$L_2^{[m]}(k, \chi_{ed\omega^{1-k}})$$

$$= \begin{cases} 0, & \text{if } d = e = k = 1, \\ \prod_{p|m} \left( 1 - \chi_{ed}(p)p^{1-k} \right) L_2(k, \chi_{e\omega^{1-k}}), & \text{otherwise}. \end{cases}$$

Now we are ready to extend the main theorems of the papers [8] and [10]. Let $m, s > 1$ be square-free natural numbers with $s \mid m$. We shall apply the following identity

$$(7.4) \sum_{t \mid s} \Theta(t) \prod_{p \mid (s/t)} \left( 1 - \Theta(p) \right) \prod_{p \mid t} \left( 1 - \Phi(p) \right) = \prod_{p \mid s} \left( 1 - \Phi(p)\Theta(p) \right),$$

see [6, (3.1)].

**Theorem 1** (cf. [8, Main Theorem], [10, Theorem]). Let $m > 1$ be a square-free odd natural number having $\nu$ prime factors and let $\Psi, \Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions satisfying $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ if $s \mid m$. Let $K$ have the same meaning as in Lemma 9 (resp. Lemma 10)
and let $x = \{x_{k,e}\}_{(k,e) \in K \times T}$ be a set of 2-adic integers not all even. Set
\[
\Lambda_1(m, \Theta) = \frac{1}{2} \sum_{p|m \text{ prime}} \Theta(p) \log_2 p \prod_{q|(m/p)} (1 - \Theta(q)).
\]

Then the number
\[
\Lambda(x, m, \Psi, \Theta) := \sum_{(k,e) \in K \times T} x_{k,e} \sum_{d \in T_m} \Psi([d]) L_2^{[m, \Theta]}(k, \chi_{de} \omega^{1-k}) + x_{1,1} \Lambda_1(m, \Theta)
\]
is a 2-adic integer divisible by $2^\nu + \lambda$, where $\lambda$ has the same meaning as in Lemma 9 if $K = \{-1, 0, 1, 2\}$ and $x$ is defined on a non-empty finite subset $L$ of $K$ (resp. Lemma 10 if $K$ is a finite set of consecutive integers).

**Proof.** Write
\[
\Lambda_2(x, m, \Theta) = \prod_{p|m \text{ prime}} (1 - \Theta(p)) \sum_{(k,e) \in K \times T, (k,e) \neq (1,1)} x_{k,e} L_2(k, \chi_{de} \omega^{1-k}).
\]

and
\[
L_2'(k, \chi_{de} \omega^{1-k}) = \begin{cases} 
0, & \text{if } e = d = k = 1, \\
L_2(k, \chi_{de} \omega^{1-k}), & \text{otherwise}.
\end{cases}
\]

We proceed in the same manner as in the proof of the Main Theorem in [8] (resp. the Theorem in [10]). Making use of (7.4), for any multiplicative function $\Phi : \mathbb{N} \to \mathbb{C}_2$ and fixed $u, s$ with $u \mid s$ we obtain
\[
\Theta^{-1}(u) \sum_{t|s} \Theta(t) \prod_{p|(s/t)} (1 - \Theta(p)) \prod_{p|(t/u)} (1 - \Phi(p))
\]
\[
= \prod_{p|(s/u)} (1 - \Phi(p)) \Theta(p)).
\]

This follows from (7.4) by a simple induction on the number of prime factors of $s/u$. We observe that for any functions $f$ and $g$
\[
\sum_{d|m} f(d) \sum_{c|d} g(c) h(d, c) = \sum_{d|m} g(d) \sum_{d|c|m} f(c) h(c, d).
\]
Therefore we have

\[
\Lambda(x, m, \Psi, \Theta) - x_1 \Lambda_1(m, \Theta) + \Lambda_2(x, m, \Theta) = \sum_{(k,e) \in K \times T_8} x_{k,e} \sum_{d \in T_m} \Psi(|d|) \prod_{p \mid (m/d)} (1 - \Theta(p) \chi_{ed}(p)p^{1-k}) L'_{2}(k, \chi_{ed}\omega^{1-k})
\]

\[
= \sum_{(k,e) \in K \times T_8} x_{k,e} \sum_{d \in T_m} \Psi(|d|) \prod_{p \mid (m/d)} (1 - \Theta(p) \chi_{ed}(p)p^{1-k}) L'_{2}(k, \chi_{ed}\omega^{1-k})
\]

\[
\times \sum_{c \in T_d} \mu(|c|) \Theta(|c|) \prod_{p \mid (m/c)} (1 - \Theta(p)) \prod_{p \mid (c/d)} (1 - \chi_{ed}(p)p^{1-k}) L'_{2}(k, \chi_{ed}\omega^{1-k})
\]

\[
= \sum_{c \in T_d} \mu(|c|) \Theta(|c|) \prod_{p \mid (m/c)} (1 - \Theta(p)) \prod_{p \mid (c/d)} (1 - \chi_{ed}(p)p^{1-k}) L'_{2}(k, \chi_{ed}\omega^{1-k})
\]

Consequently appealing to Lemma 2 we obtain

\[
\Lambda(x, m, \Psi, \Theta) = \sum_{1 \neq d \in T_m} \Theta(|d|) \mu(|d|) \prod_{p \mid (m/d)} (1 - \Theta(p)) \sum_{a=1}^{\frac{|d|}{2}} \left( \sum_{k \in K} x_{k,e} L'_{k,d}(\zeta_{|d|}) \right)
\]

\[
\times \left( \sum_{e \in T_d} \mu(|c|) \Psi(|c|) \Theta^{-1}(|c|) \tau(\chi_c, \zeta_{|c|}) |c|^{-1} \chi_c(a) \right)
\]

\[
= \sum_{1 \neq d \in T_m} \Theta(|d|) \mu(|d|) \prod_{p \mid (m/d)} (1 - \Theta(p)) \sum_{a=1}^{\frac{|d|}{2}} \left( \sum_{k \in K} x_{k,e} L'_{k,d}(\zeta_{|d|}) \right)
\]

\[
\times \left( \prod_{p \mid (d)} (1 - \tau(\chi_{p^*}, \zeta_{p^*})p^{-1} \Psi(p) \Theta^{-1}(p) \chi_{p^*}(a)) \right),
\]

where \(p^* = (-1)^{(p-1)/2}p\) and \(\zeta_{|d|} = \prod_{p \mid (d)} \zeta_p\).
Now Theorem 1 follows from Lemma 9 when $K = \{-1, 0, 1, 2\}$ or from Lemma 10 when $K$ is a set of consecutive integers. □

The Main Theorem in [8] and Theorem in [10] are special cases of Theorem 1 when $\Theta(s) = 1$ for $s \mid m$.

We now extend Theorem 2 [6] (a supplement of Theorem 1 [6]). Let $m (> 1)$ be a square-free odd natural number. Denote by $I(m)$ the set of $k \in \mathbb{Z}$ such that $l_k(\zeta_a^c)$ are 2-adic integers for any $c$ and $a$ with $c \mid m$, $c \neq 1$, $1 \leq a \leq c$ and $\gcd(a, c) = 1$. By definition, we have $1 \in I(m)$ and $r \in I(m)$ for any integer $r \leq 0$. The question whether $I(m) = \mathbb{Z}$ remains to be open.

**Theorem 2** (cf. [6, Theorem 2]). Let $m > 1$ be a square-free odd natural number having $\nu$ prime factors and let $\Psi, \Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions satisfying $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ if $s \mid m$. Set

$$
\Lambda_{0,*}(m, \Theta) = \frac{1}{2} \left( \prod_{p \mid m, \text{ prime}} (1 - \Theta(p)p) - \prod_{p \mid m, \text{ prime}} (1 - \Theta(p)) \right)
$$

and

$$
\Lambda_{1,*}(m, \Theta) = \sum_{p \mid m, \text{ prime}} \Theta(p) \log_2 p \prod_{q \mid (m/p), \text{ prime}} (1 - \Theta(q)).
$$

For $k \in I(m)$ the number

$$
\Lambda_*(k, m, \Psi, \Theta) := \sum_{d \in T_m} \Psi(|d|) L_2^{[m, \epsilon]}(k, c_d \omega^{1-k})
$$

$$
+ \delta_{k,0} \Lambda_{0,*}(m, \Theta) + \delta_{k,1} \Lambda_{1,*}(m, \Theta)
$$

is a 2-adic integer divisible by $2^\nu$.

**Proof.** Write

$$
\Lambda'(k, m, \Theta) = \begin{cases} 
(1 - 2^{-k})^{-1} L_2^{[m, \epsilon]}(k, \omega^{1-k}), & \text{if } k \neq 0, 1, \\
\Lambda_k,*(m, \Theta), & \text{otherwise}
\end{cases}
$$

and

$$
\Lambda''(k, m, \Psi, \Theta) = \sum_{d \in T_m} \Psi(|d|) \prod_{p \mid (m/d)} \left(1 - \Theta(p)c_d p^{1-k} \right) L_2''(k, c_d \omega^{1-k}),
$$

$$
L_2'(k, \omega^{1-k})
\right)
\right)
\right).
$$

For $k \in I(m)$ the number

$$
\Lambda_*(k, m, \Psi, \Theta) := \sum_{d \in T_m} \Psi(|d|) L_2^{[m, \epsilon]}(k, c_d \omega^{1-k})
$$

$$
+ \delta_{k,0} \Lambda_{0,*}(m, \Theta) + \delta_{k,1} \Lambda_{1,*}(m, \Theta)
$$

is a 2-adic integer divisible by $2^\nu$.
where

\[
L''_2(k, \chi_d \omega^{1-k}) = \begin{cases} 
    h(d), & \text{if } k = 0 \text{ and } d < 0, \\
    0, & \text{if } k = 0 \text{ and } d > 0, \\
    (1 - \chi_d(2)2^{-k})^{-1}L_2(k, \chi_d \omega^{1-k}), & \text{otherwise}.
\end{cases}
\]

We first observe that

\[
\Lambda_*(k, m, \Psi, \Theta) = \Lambda'(k, m, \Theta) + \Lambda''(k, m, \Psi, \Theta).
\]

On the other hand, by virtue of (7.4) we have

\[
\Lambda'(k, m, \Theta) = (1 - 2^{-k})^{-1} \sum_{d \in T_m, d \neq 1} \Theta(|d|) \prod_{p | m/d, p \text{ prime}} (1 - \Theta(p)) \\
\times \prod_{p | d, p \text{ prime}} (1 - p^{1-k})L_2(k, \omega^{1-k}),
\]

if \( k \neq 0, 1 \) and

\[
\Lambda'(0, m, \Theta) = \frac{1}{2} \sum_{d \in T_m, d \neq 1} (-1)^{v(d)} \Theta(|d|) \phi(|d|) \prod_{p | m/d, p \text{ prime}} (1 - \Theta(p)).
\]

Moreover by virtue of (7.5) we have

\[
\Lambda''(k, m, \Psi, \Theta) = \sum_{d \in T_m} \Psi(|d|) L''_2(k, \chi_d \omega^{1-k}) \Theta^{-1}(|d|) \sum_{d | m} \Theta(|c|) \\
\times \prod_{p | m/c, p \text{ prime}} (1 - \Theta(p)) \prod_{p | c/d, p \text{ prime}} (1 - \chi_d(p)p^{1-k}),
\]

and so in view of (7.6) we obtain

\[
\Lambda''(k, m, \Psi, \Theta) = \sum_{d \in T_m} \Theta(|d|) \prod_{p | m/d, p \text{ prime}} (1 - \Theta(p)) \\
\times \sum_{c \in T_d} \Psi(|c|) \Theta^{-1}(|c|) \prod_{p | d/c, p \text{ prime}} (1 - \chi_c(p)p^{1-k})L''_2(k, \chi_c \omega^{1-k}).
\]
Therefore appealing to Lemmas 3 and 4 we deduce that

\[ \Lambda'(k, m, \Theta) = (-1)^{k+1} \sum_{d \in T_m, d \neq 1} (-1)^{\nu(d)} \Theta(|d|) \prod_{p \mid (m/d), p \text{ prime}} (1 - \Theta(p)) \sum_{b=1}^{|d|} l_k(\zeta_{d|d}^b) \]

and

\[ \Lambda''(k, m, \Psi, \Theta) = (-1)^{k+1} \sum_{d \in T_m, d \neq 1} \Theta(|d|) \prod_{p \mid (m/d), p \text{ prime}} (1 - \Theta(p)) \times \sum_{c \in T_d} \Psi(|c|) \Theta^{-1}(|c|) \tau(\chi_c, \zeta_{|c|}) |c| \prod_{p \mid (d/c), p \text{ prime}} (1 - \chi_c(p)p^{1-k}) \sum_{b=1}^{|c|} \chi_c(b) l_k(\zeta_{c|c}^b). \]

Thus in view of Lemma 1 we have

\[ \Lambda_*(k, m, \Psi, \Theta) = (-1)^{k+1} \sum_{d \in T_m, d \neq 1} (-1)^{\nu(d)} \Theta(|d|) \prod_{p \mid (m/d), p \text{ prime}} (1 - \Theta(p)) \times \sum_{c \in T_d} \Psi(|c|) \Theta^{-1}(|c|) \mu(|c|) \tau(\chi_c, \zeta_{|c|}) |c| \sum_{b=1}^{|d|} \chi_c(b) l_k(\zeta_{d|d}^b) \]

\[ = (-1)^{k+1} \sum_{d \in T_m, d \neq 1} (-1)^{\nu(d)} \Theta(|d|) \prod_{p \mid (m/d), p \text{ prime}} (1 - \Theta(p)) \sum_{b=1}^{|d|} l_k(\zeta_{d|d}^b) \times \sum_{c \in T_d} \mu(|c|) \Psi(|c|) \Theta^{-1}(|c|) \tau(\chi_c, \zeta_{|c|}) \chi_c(b) \]

\[ = (-1)^{k+1} \sum_{d \in T_m, d \neq 1} \Theta(|d|) \prod_{p \mid (m/d), p \text{ prime}} (1 - \Theta(p)) \sum_{b=1}^{|d|} l_k(\zeta_{d|d}^b) \times \prod_{p \mid \omega d, p \text{ prime}} \left( \tau(\chi_\omega, \zeta_\omega)p^{\omega-1}\Psi^{-1}(p)\chi_\omega(b) - 1 \right), \]

which proves Theorem 2. \(\square\)
8. Optimal linear congruences

The congruences in the hypothesis of Theorem 1

\[
\sum_{(k,e) \in K \times T_8} x_{k,e} \sum_{d \in T_m} \Phi(|d|) L_2^{[m,\Theta]}(k, \chi_{ed}\omega^{1-k}) + x_{1,1} \Lambda_1(m, \Theta) \equiv 0 \pmod{2^{\nu+\lambda}}
\]

are said to be optimal if \(\lambda = c(L)\) (resp. \(\lambda = c(u_n)\)). The 2-adic integers \(x_{k,e} (k \in K, e \in T_8)\) determining an optimal linear congruence are called optimal for \(K\). For example, the congruences proved in [4], [7] or resp. [5] are optimal for \(K = \{0\}, K = \{-1,0\}\) or resp. \(K = \{-m, \ldots, -1,0\}\) \((m \geq 0)\).

Optimal linear congruences exist for any non-empty subset \(L\) of \(K = \{-1,0,1,2\}\) and when \(K\) is a finite subset of consecutive integers. Such a congruence was given explicitly in the proof of Lemma 5 in [8] in the former case and inductively in the proof of Lemma 6 in [10] in the latter case.

9. Applications of Theorem 1

When \(L = \{0,1\}\) Theorem 1 gives the congruences of Gras [3] and Uehara [6] for class numbers of quadratic fields which are modulo \(2^{\nu+\lambda}\), where \(\lambda \leq 5\). When \(L = \{-1,0\}\) (resp. \(L = \{0\}\)) we obtain congruences for the same objects as those in [7] (resp. [4]). The obtained congruences are modulo \(2^{\nu+\lambda}\), where \(\lambda \leq 6\) (resp. \(\lambda \leq 2\)). When \(2 \in L\) the congruences implied by Theorem 1 are quite new and especially interesting. They produce, via a 2-adic version of the Lichtenbaum conjecture, some new congruences for the conjectured orders of \(K_2\)-groups of the integers of imaginary quadratic fields. We present these congruences in a general form in Theorem 3.

For the discriminant \(\mathcal{D}\) of a quadratic field, we write

\[
H(\mathcal{D}) = L_2(k, \chi_\mathcal{D}\omega^{1-k}) \quad \text{(resp. } K_2(\mathcal{D}) = 2L_2(k, \chi_\mathcal{D}\omega^{1-k})\text{)}.
\]

if \(k = 0, \mathcal{D} < 0\) or \(k = 1, \mathcal{D} > 1\) (resp. \(k = -1, \mathcal{D} > 1\) or \(k = 2, \mathcal{D} < 0\)). We have

\[
H(\mathcal{D}) = \begin{cases} 
2^{-1}(\mathcal{D})(1 - \chi_\mathcal{D}(2)) h(\mathcal{D}), & \text{if } \mathcal{D} < 0, \\
(2 - \chi_\mathcal{D}(2)) \mathcal{D}^{-1/2} h(\mathcal{D}) \log_2 \epsilon_\mathcal{D}, & \text{if } \mathcal{D} > 1,
\end{cases}
\]
and

\[
K_2(D) = \begin{cases} 
-24\omega_2^{-1}(D)(1 - \chi_D(2))k_2(D), & \text{if } D > 1, \\
(4 - \chi_D(2))|D|^{-3/2}R_{2,2}(D)k_2(D), & \text{if } D < 0.
\end{cases}
\]

In the formula for \(K_2(D)\) when \(D < 0\) we assume that the 2-adic Lichtenbaum conjecture for imaginary quadratic fields holds. Now we are ready to extend results of [8, Applications]. We rewrite Theorem 1 with \(K = \{-1, 0, 1, 2\}\) in the form:

**Theorem 3** (cf. [8, Applications]). Let \(m > 1\) be a square-free odd natural number having \(\nu\) prime factors and let \(\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2\) be multiplicative functions such that \(\Theta(s) \equiv \Psi(s) \equiv 1 \pmod{2}\) if \(s | m\). Set \(K = \{-1, 0, 1, 2\}\) and let \(L\) be a non-empty subset of \(K\). Given a set \(x = \{x_{k,e}\}_{(k,e)\in K \times T_8}\) of 2-adic integers not all even defined on \(L\), set

\[
\Lambda = \Lambda_{-1} + \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_{-1}' + \Lambda_1',
\]

where

\[
\Lambda_{-1} = \frac{1}{2} \sum_{e \in T_8} x_{-1,e} \sum_{d \in T_m} \Psi(|d|) \\
\times \left( \prod_{p|m\atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p^2) - \delta_{d,1} \prod_{p|m\atop p \text{ prime}} (1 - \Theta(p)) \right) K_2(ed),
\]

\[
\Lambda_0 = \sum_{e \in T_8} x_{0,e} \sum_{d \in T_m\atop ed < 0} \Psi(|d|) \\
\times \left( \prod_{p|m\atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p) - \delta_{d,1} \prod_{p|m\atop p \text{ prime}} (1 - \Theta(p)) \right) H(ed),
\]

\[
\Lambda_1 = \sum_{e \in T_8} x_{1,e} \sum_{d \in T_m\atop ed > 1} \Psi(|d|) \\
\times \left( \prod_{p|m\atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)) - \delta_{d,1} \prod_{p|m\atop p \text{ prime}} (1 - \Theta(p)) \right) H(ed),
\]

\[
\Lambda_2 = \frac{1}{2} \sum_{e \in T_8} x_{2,e} \sum_{d \in T_m\atop ed < 0} \Psi(|d|)
\]
On optimal linear congruences for $L_2(k, \chi \omega^{1-k})$ 

\[
\times \left( \prod_{p|m, p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p^{-1}) - \delta_{d,1} \prod_{p|m, p \text{ prime}} (1 - \Theta(p)) \right) K_2(ed),
\]

\[
\Lambda_{-1} = \frac{1}{12} x_{-1,1} \left( \prod_{p|m, p \text{ prime}} (1 - \Theta(p)p^2) - \prod_{p|m, p \text{ prime}} (1 - \Theta(p)) \right),
\]

\[
\Lambda_1 = -\frac{1}{2} x_{1,1} \sum_{p|m, p \text{ prime}} \Theta(p) \log_2 p \prod_{q|(m/p), q \text{ prime}} (1 - \Theta(q)).
\]

Then the number $\Lambda$ is a 2-adic integer divisible by $2^{\nu+\lambda}$, where $\lambda$ has the same meaning as in Theorem 1.

10. The case $L = \{0, 1\}$

Hardy and Williams [4] discovered a new type of linear congruence relating class numbers of imaginary quadratic fields. A general linear congruence relating class numbers and units both of real and imaginary quadratic fields was discovered by Gras [3]. Gras derived his congruence using 2-adic measure theory. Uehara [6] reproved Gras’ congruence using elementary 2-adic arguments. Both Gras and Uehara used the 2-adic analogue of Dirichlet’s class number formulas. Urbanowicz and Wójcik [8] and Wójcik [10] indicated how Uehara’s techniques may be used to obtain more general congruences among the values of 2-adic $L$-functions. Gras and Uehara’s congruences are special cases of Theorems 1 and 2.

**Theorem 4** (see [6, Theorem 1]). Let $m > 1$ be an odd square-free integer having $\nu$ prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ for any divisor $s | m$. In the notation of Theorem 3, for any 2-adic integers $x_{0,e}, x_{1,e} (e \in \mathbb{T}_8)$ not all even we have

\[
\sum_{e \in \mathbb{T}_8} x_{0,e} \sum_{d \in \mathbb{T}_m, ed < 0} \Psi(|d|) \left( \prod_{p|m, p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p) - \delta_{d,1} \prod_{p|m, p \text{ prime}} (1 - \Theta(p)) \right) H(ed)
\]

\[
+ \sum_{e \in \mathbb{T}_8} x_{1,e} \sum_{d \in \mathbb{T}_m, ed > 1} \Psi(|d|) \left( \prod_{p|m, p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)) - \delta_{d,1} \prod_{p|m, p \text{ prime}} (1 - \Theta(p)) \right) H(ed)
\]

\[
- \frac{1}{2} x_{1,1} \sum_{p|m, p \text{ prime}} \Theta(p) \log_2 p \prod_{q|(m/p), q \text{ prime}} (1 - \Theta(q)) \equiv 0 \pmod{2^{\nu+\lambda}},
\]

where $2^\lambda$ is the greatest common divisor of the eight integers $s_i$ $(0 \leq i \leq 7)$ defined by

\[
\begin{align*}
    s_0 &= x_{0,-8} + x_{0,-4} + x_{0,1} + x_{0,8} + x_{1,-8} + x_{1,-4} + x_{1,1} + x_{1,8}, \\
    s_1 &= 2(x_{0,1} + x_{0,8} + x_{1,-8} + x_{1,-4}), \\
    s_2 &= 2(3x_{0,-8} + 3x_{0,8} + x_{1,-8} + 2x_{1,-4} + 2x_{1,1} + x_{1,8}), \\
    s_3 &= 4(3x_{0,8} + x_{1,-8} + 2x_{1,-4}), \\
    s_4 &= 4(5x_{0,-8} + 5x_{0,8} + x_{1,-8} + 4x_{1,-4} + 4x_{1,1} + x_{1,8}), \\
    s_5 &= 8(x_{0,8} + x_{1,-8}), \\
    s_6 &= 8(x_{0,-8} + x_{0,8} - x_{1,-8} - x_{1,8}), \\
    s_7 &= 32.
\end{align*}
\]

Remark. The proof of Theorem 4 is straightforward. We see at once that $\gcd(z_i, 32) = \gcd(s_i, 32), 0 \leq i \leq 6$, which is clear from (1.1) and (1.2) (with $p = 2$).

Theorem 4 is the main result of [6]. This theorem and its supplement stated in [6, Theorem 2] include the congruences proved in [3, Théorèmes (1.3), (1.4)] and [4]. For details and other applications see [6].

In fact Uehara has provided a general method of producing such congruences. It is a simple matter to determine linear congruence relations with given $\lambda$. We will look more closely at the case when $\lambda = 5$.

Corollary 1. The congruence in the hypothesis of Theorem 4 is optimal if and only if

\[
\begin{align*}
    x_{0,-8} &= a, \\
    x_{0,-4} &= a + 32b - 16c - 24d + 4e + 4f + 2g,
\end{align*}
\]
On optimal linear congruences for $L_2(k,\chi^{1-k})$

$x_{0,1} = -a + 16c + 16d - 4e - 4f - 2g + 2h,

x_{0,8} = -a + 16d - 4f + 2h,

x_{1,-8} = a - 16d + 4f + 4g - 2h,

x_{1,-4} = a - 16d + 4e + 4f - 2g - 2h,

x_{1,1} = -a - 8d - 4e + 4f + 2g,

x_{1,8} = -a + 32d - 8f - 4g,

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with $a$ odd.

**Proof.** The congruence in the hypothesis of Theorem 4 is valid modulo $2^{\nu+5}$ if and only if

\begin{equation}
\begin{aligned}
s_0 &= 32b, 
s_1 &= 32c, 
s_2 &= 32d, 
s_3 &= 32e, 
s_4 &= 32f, 
s_5 &= 32g, 
s_6 &= 32h
\end{aligned}
\end{equation}

(10.7)

for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{0,-4}, x_{0,1}, x_{0,8}, x_{1,-8}, x_{1,-4}, x_{1,1}, x_{1,8}$ and determinant $-8$. An easy computation gives the formulas of Corollary 1 at once. \hfill \Box

**Corollary 2.** If the congruence in the hypothesis of Theorem 4 is optimal then all the $x_{0,e}, x_{1,e}$ ($e \in T_8$) are odd. None of these coefficients can vanish in particular.

**11. The case $L = \{-1,0\}**

In this case the obtained congruences extend those of [7] for the orders of $K_2$-groups of the integers of real quadratic fields and class numbers of imaginary quadratic fields. We leave it to the reader to show that Theorem 5 implies the Theorem in [7]. In the case when $L = \{-1,0\}$ we have $c(L) = 5$ and the congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \leq 5$.

**Theorem 5.** Let $m > 1$ be an odd square-free integer having $\nu$ prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv
\(\Theta(s) \equiv 1 \pmod{2}\) for any divisor \(s \mid m\). In the notation of Theorem 3, for any 2-adic integers \(x_{-1,e}, x_{0,e} \ (e \in T_8)\) not all even we have

\[
\sum_{e \in T_8} x_{-1,e} \sum_{d \in T_m, \text{ed} > 1} \Psi(|d|) \left( \prod_{p|m, \text{p prime}} (1 - \chi_{ed}(p)\Theta(p)p^2) - \delta_{d,1} \prod_{p|m, \text{p prime}} (1 - \Theta(p)) \right) K_2(ed)
\]

\[+ 2 \sum_{e \in T_8} x_{0,e} \sum_{d \in T_m, \text{ed} < 0} \Psi(|d|) \left( \prod_{p|m, \text{p prime}} (1 - \chi_{ed}(p)\Theta(p)p) - \delta_{d,1} \prod_{p|m, \text{p prime}} (1 - \Theta(p)) \right) H(ed)
\]

\[+ \frac{1}{6} x_{-1,1} \left( \prod_{p|m, \text{p prime}} (1 - \Theta(p)p^2) - \prod_{p|m, \text{p prime}} (1 - \Theta(p)) \right) \equiv 0 \pmod{2^{\nu + \lambda + 1}},
\]

where \(2^\lambda\) is the greatest common divisor of the eight integers \(s_i \ (0 \leq i \leq 7)\) defined by

\[
s_0 = x_{-1,-8} + x_{-1,-4} + x_{-1,1} + x_{-1,8} + x_{0,-8} + x_{0,-4} + x_{0,1} + x_{0,8},
\]
\[
s_1 = 2(x_{-1,-8} + x_{-1,-4} + x_{0,1} + x_{0,8}),
\]
\[
s_2 = 2(-x_{-1,-8} + 2x_{-1,-4} + 2x_{-1,1} - x_{-1,8} + x_{0,-8} + x_{0,8}),
\]
\[
s_3 = 4(-x_{-1,-8} + 2x_{-1,-4} + x_{0,8}),
\]
\[
s_4 = 4(-3x_{-1,-8} + 4x_{-1,-4} + 4x_{-1,1} - 3x_{-1,8} + x_{0,-8} + x_{0,8}),
\]
\[
s_5 = 8(x_{-1,-8} + x_{0,8}),
\]
\[
s_6 = 8(-x_{-1,-8} - x_{-1,8} + x_{0,-8} + x_{0,8}),
\]
\[
s_7 = 32.
\]

**Proof.** The proof is immediate. We apply (1.1) and (1.2) again. \(\square\)

**Corollary 1.** The congruence in the hypothesis of Theorem 5 is optimal if and only if

\[
x_{-1,-8} = a,
\]
\[
x_{-1,-4} = a + 4e - 2g,
\]
On optimal linear congruences for $L_2(k, \chi_{1-k})$

\[ x_{-1,1} = -a + 8d - 4e + 2g - 2h, \]
\[ x_{-1,8} = -a + 6d - 4f - 2h, \]
\[ x_{1,-8} = a + 6d - 4f - 4g + 2h, \]
\[ x_{1,-4} = a + 32b - 16c - 40d + 4e + 8f + 2g + 2h, \]
\[ x_{1,1} = -a + 16c - 4e - 2g, \]
\[ x_{1,8} = -a + 4g, \]

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with odd $a$.

**Proof.** The congruence in the hypothesis of Theorem 5 is valid modulo $2^{\nu+5}$ if and only if $s_0$, $s_1$, $s_2$, $s_3$, $s_4$, $s_5$, $s_6$ satisfy (10.7) for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{-1,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}$, $x_{-1,1}$, $x_{-1,8}$, $x_{0,-8}$, $x_{0,-4}$, $x_{0,1}$, $x_{0,8}$ and determinant $-8$. A standard computation gives the formulas of Corollary 1 at once. \qed

**Corollary 2.** If the congruence in the hypothesis of Theorem 5 is optimal then all the $x_{-1,e}$, $x_{0,e}$ ($e \in T_8$) are odd. None of these coefficients can vanish in particular.

12. The case $L = \{-1, 2\}$

In the case when $L = \{-1, 2\}$ we derive linear congruences among the conjectured orders of $K_2$-groups of the integers of quadratic fields. In this case the obtained congruence provides an analogue of the Gras and Uehara congruence in $K_2$-theory. Here $c(L) = 5$ and the congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \leq 5$.

**Theorem 6.** Let $m > 1$ be an odd square-free integer having $\nu$ prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ for any divisor $s \mid m$. In the notation of Theorem 3, for
any 2-adic integers \( x_{-1,e}, x_{2,e} \) \( (e \in T_8) \) not all even we have

\[
\sum_{e \in T_8} x_{-1,e} \sum_{d \in T_m \atop ed > 1} \Psi(|d|) \left( \prod_{p|m \atop p \text{ prime}} \left( 1 - \chi_{ed}(p) \Theta(p) p^2 \right) \right) - \delta_{d,1} \prod_{p|m \atop p \text{ prime}} \left( 1 - \Theta(p) \right) K_2(ed) \\
+ \sum_{e \in T_8} x_{2,e} \sum_{d \in T_m \atop ed < 0} \Psi(|d|) \left( \prod_{p|m \atop p \text{ prime}} \left( 1 - \chi_{ed}(p) \Theta(p) p^{-1} \right) \right) - \delta_{d,1} \prod_{p|m \atop p \text{ prime}} \left( 1 - \Theta(p) \right) K_2(ed),
\]

\[
+ \frac{1}{6} x_{-1,1} \left( \prod_{p|m \atop p \text{ prime}} \left( 1 - \Theta(p) p^2 \right) - \prod_{p|m \atop p \text{ prime}} \left( 1 - \Theta(p) \right) \right) \equiv 0 \pmod{2^{\nu+\lambda+1}},
\]

where \( 2^\lambda \) is the greatest common divisor of the eight integers \( s_i \) \( (0 \leq i \leq 7) \) defined by

\begin{align*}
 s_0 &= x_{-1,-8} + x_{-1,-4} + x_{-1,1} + x_{-1,8} + x_{2,-8} + x_{2,1} + x_{2,8}, \\
 s_1 &= 2(x_{-1,-8} + x_{-1,-4} + x_{2,1} + x_{2,8}), \\
 s_2 &= 2(7x_{-1,-8} + 2x_{-1,-4} + 2x_{-1,1} + 7x_{-1,8} + 5x_{2,-8} \\
 & \quad + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}), \\
 s_3 &= 4(-x_{-1,-8} + 2x_{-1,-4} + 4x_{2,1} + 5x_{2,8}), \\
 s_4 &= 4(5x_{-1,-8} + 4x_{-1,-4} + 4x_{-1,1} + 5x_{-1,8} + x_{2,-8} + x_{2,8}), \\
 s_5 &= 8(x_{-1,-8} + x_{2,8}), \\
 s_6 &= 8(3x_{-1,-8} + 3x_{-1,8} + x_{2,-8} + x_{2,8}), \\
 s_7 &= 32.
\end{align*}

**Proof.** In order to obtain the above formulas for \( s_i \), \( 0 \leq i \leq 6 \) we make use of (1.1) and (1.2). \( \square \)
Corollary 1. The congruence in the hypothesis of Theorem 6 is optimal if and only if

\[
\begin{align*}
x_{-1,-8} &= a, \\
x_{-1,-4} &= -3a + 32c - 4e + 2g, \\
x_{-1,1} &= 3a + 64b - 32c - 8d + 4e - 2g + 2h, \\
x_{-1,8} &= -a - 128b + 16d + 4f - 6h, \\
x_{2,-8} &= a + 384b - 48d - 12f - 4g + 22h, \\
x_{2,-4} &= -3a - 288b + 16c + 40d - 4e + 8f + 6g - 18h, \\
x_{2,1} &= 3a - 16c + 4e - 6g, \\
x_{2,8} &= -a + 4g,
\end{align*}
\]

where \(a, b, c, d, e, f, g, h \in \mathbb{C}_2\) are integers with \(a\) odd.

Proof. We proceed in the same way as in the proof of Corollary 1 to Theorem 5. Taking \(x_{-1,-8} = a\) we obtain a system of seven linear equations with seven unknowns \(x_{-1,-4}, x_{-1,1}, x_{-1,8}, x_{2,-8}, x_{2,-4}, x_{2,1}, x_{2,8}\) and determinant 8. An easy verification gives the above formulas immediately. \(\square\)

Corollary 2. If the congruence in the hypothesis of Theorem 6 is optimal then all the \(x_{-1,e}, x_{2,e}\) \((e \in T_8)\) are odd. None of these coefficients can vanish in particular.

13. The case \(L = \{1, 2\}\)

In the case when \(L = \{1, 2\}\) we obtain linear congruences for class numbers of real quadratic fields and the orders of \(K_2\)-groups of the integers of imaginary quadratic fields. In this case \(c(L) = 5\) and the obtained congruences are valid modulo \(2^{\nu + \lambda + 1}\), where \(\lambda \leq 5\).

Theorem 7. Let \(m > 1\) be an odd square-free integer having \(\nu\) prime factors, and let \(\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2\) be multiplicative functions such that \(\Psi(s) \equiv 1\mod{2}\). Then...
\[ \Theta(s) \equiv 1 \pmod{2} \text{ for any divisor } s \mid m. \text{ In the notation of Theorem 3, for any } 2\text{-adic integers } x_{1,e}, x_{2,e} \ (e \in T_8) \text{ not all even we have} \]

\[ 2 \sum_{e \in T_8} x_{1,e} \sum_{d \mid m \atop d > 1} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)) \right) \]

\[ - \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) \big) H(ed) \]

\[ + \sum_{e \in T_8} x_{2,e} \sum_{d \mid m \atop d < 0} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p^{-1}) \right) \]

\[ - \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) \big) K_2(ed), \]

\[ - x_{1,1} \sum_{p \mid m \atop p \text{ prime}} \Theta(p) \log_2 p \prod_{q \mid (m/p) \atop q \text{ prime}} (1 - \Theta(q)) \equiv 0 \pmod{2^{\nu + \lambda + 1}}, \]

where \(2^\lambda\) is the greatest common divisor of the eight integers \(s_i\) \((0 \leq i \leq 7)\) defined by

\[ s_0 = x_{1,-8} + x_{1,-4} + x_{1,1} + x_{2,-8} + x_{2,-4} + x_{2,1} + x_{2,8}, \]

\[ s_1 = 2(x_{1,-8} + x_{1,-4} + x_{2,1} + x_{2,8}), \]

\[ s_2 = 2(3x_{1,-8} + 6x_{1,-4} + 6x_{1,1} + 3x_{1,8} + 5x_{2,-8} + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}), \]

\[ s_3 = 4(x_{1,-8} + 2x_{1,-4} + 4x_{2,1} - x_{2,8}), \]

\[ s_4 = 4(-3x_{1,-8} + 4x_{1,-4} + 4x_{1,1} - 3x_{1,8} + x_{2,-8} + x_{2,8}), \]

\[ s_5 = 8(x_{1,-8} + x_{2,8}), \]

\[ s_6 = 8(3x_{1,-8} + 3x_{1,8} + x_{2,-8} + x_{2,8}), \]

\[ s_7 = 32. \]

**Proof.** It follows from (1.1) and (1.2) that

\[ \gcd(z_3, 32) = \gcd(4(3x_{1,-8} + 6x_{1,-4} + 4x_{2,1} + 5x_{2,8}), 32) = \gcd(s_3, 32) \]
and the corollary follows easily from Theorem 7. □

Corollary 1. The congruence in the hypothesis of Theorem 7 is optimal if and only if

\begin{align*}
x_{1,-8} &= a, \\
x_{1,-4} &= a + 32c - 4e - 10g, \\
x_{1,1} &= -a + 192b - 32c - 24d + 4e + 8f + 10g + 2h, \\
x_{1,8} &= -a + 128b - 16d + 4f + 2h, \\
x_{2,-8} &= a - 384b + 48d - 12f - 4g - 2h, \\
x_{2,-4} &= a + 96b + 16c - 8d - 4e - 6g - 2h, \\
x_{2,1} &= -a - 16c + 4e + 6g, \\
x_{2,8} &= -a + 4g,
\end{align*}

where \(a, b, c, d, e, f, g, h \in \mathbb{C}_2\) are integers with \(a\) odd.

Proof. The proof is standard. We proceed in the same way as in the proof of Corollary 1 to Theorem 5. Taking \(x_{1,-8} = a\) we obtain a system of seven linear equations with seven unknowns \(x_{-1,-4}, x_{1,1}, x_{1,8}, x_{2,-8}, x_{2,-4}, x_{2,1}, x_{2,8}\) and determinant \(-8\). The details are left to the reader. □

Corollary 2. If the congruence in the hypothesis of Theorem 7 is optimal then all the \(x_{1,e}, x_{2,e} (e \in T_8)\) are odd. None of these coefficients can vanish in particular.

14. The cases \(L = \{-1, 1\}\) and \(L = \{0, 2\}\)

In the case when \(L = \{-1, 1\}\) (resp. \(L = \{0, 2\}\)) we obtain linear congruences between class numbers and the orders of \(K_2\)-groups of the integers of real (resp. imaginary) quadratic fields. In both the cases \(c(L) = 6\) and the obtained congruences are valid modulo \(2^{\nu+\lambda+1}\), where \(\lambda \leq 6\).
Theorem 8. Let $m > 1$ be an odd square-free integer having $\nu$ prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \mod 2$ for any divisor $s \mid m$. In the notation of Theorem 3, for any 2-adic integers $x_{-1,e}, x_{1,e}$ ($e \in T_8$) not all even we have

\[
\sum_{e \in T_8} x_{-1,e} \sum_{d \in T_m \atop ed>1} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p^2) - \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) \right) K_2(ed)
\]

\[
+ 2 \sum_{e \in T_8} x_{1,e} \sum_{d \in T_m \atop ed>1} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)) - \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) \right) H(ed)
\]

\[
+ \frac{1}{6} x_{-1,1} \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)p^2) - \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) \right)
\]

\[- x_{1,1} \sum_{p \mid m \atop p \text{ prime}} \Theta(p) \log_2 p \prod_{q \mid (m/p) \atop q \text{ prime}} (1 - \Theta(q)) \equiv 0 \mod 2^{\nu + \lambda + 1},
\]

where $2^\lambda$ is the greatest common divisor of the eight integers $s_i$ ($0 \leq i \leq 7$) defined by

- $s_0 = x_{-1,-8} + x_{-1,-4} + x_{-1,1} + x_{-1,8} + x_{1,-8} + x_{1,1} + x_{1,8},$
- $s_1 = 2(x_{-1,-8} + x_{-1,-4} + x_{1,-8} + x_{1,1}),$
- $s_2 = 2(-3x_{-1,-8} + 6x_{-1,-4} + 6x_{-1,1} - 3x_{-1,8} + x_{1,-8} + 2x_{1,-4} + 2x_{1,1} + x_{1,8}),$
- $s_3 = 4(-3x_{-1,-8} + 6x_{-1,-4} + x_{1,-8} + 2x_{1,-4}),$
- $s_4 = 4(5x_{-1,-8} + 4x_{-1,-4} + 4x_{-1,1} + 5x_{-1,8} + 5x_{1,-8} + 4x_{1,-4} + 4x_{1,1} + 5x_{1,8}),$
- $s_5 = 8(5x_{-1,-8} + 4x_{-1,-4} + 5x_{1,-8} + 4x_{1,-4}).$
On optimal linear congruences for $L_2(k, \chi_{1-k})$

$s_6 = 8(3x_{-1,-8} + 3x_{-1,8} - x_{1,-8} - x_{1,8})$,

$s_7 = 64$.

**Proof.** Note that in the case when $L = \{-1, 1\}$ we have

$$z_8 \equiv -2z_6 \pmod{64}, \quad z_7 \equiv -2z_5 \pmod{64},$$

and in consequence we may ignore $z_8$ and $z_7$ (the $z_n$ with $n = 2c(L) - 4$, $2c(L) - 5$). In order to obtain formulas for $s_i$, $0 \leq i \leq 6$ we use (1.1) and (1.2). For example, we have

$$\gcd(z_3, 64) = \gcd(4(-9x_{-1,-8} + 2x_{-1,-4} + 3x_{1,-8} + 6x_{1,-4}), 64) = \gcd(s_3, 64).$$

The corollary follows easily from Theorem 8. $\square$

**Corollary 1.** The congruence in the hypothesis of Theorem 8 is optimal if and only if

$$x_{-1,-8} = a,$$

$$x_{-1,-4} = a - 48c + 4e + 2g,$$

$$x_{-1,1} = -a - 160b + 48c + 8d - 4e + 8f - 2g + 2h,$$

$$x_{-1,8} = -a - 64b + 4f + 2h,$$

$$x_{1,-8} = -a - 128c + 8g,$$

$$x_{1,-4} = -a + 208c - 4e - 10g,$$

$$x_{1,1} = a + 480b - 208c - 8d + 4e - 24f + 10g - 2h,$$

$$x_{1,8} = a - 192b + 128c + 12f - 8g - 2h,$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with $a$ odd.

**Proof.** The congruence in the hypothesis of Theorem 8 is valid modulo $2^\nu + 6$ if and only if

$$s_0 = 64b, \quad s_1 = 64c, \quad s_2 = 64d, \quad s_3 = 64e,$$

$$s_4 = 64f, \quad s_5 = 64g, \quad s_6 = 64h$$

(14.8)
for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}, x_{-1,1}, x_{-1,8}, x_{1,-8}, x_{1,-4}, x_{1,1}, x_{1,8}$ and determinant $-64$. A standard computation gives the formulas of Corollary 1 at once. \hfill \Box

**Corollary 2.** If the congruence in the hypothesis of Theorem 8 is optimal then all the $x_{k,e}$ are odd. None of these coefficients can vanish in particular.

**Theorem 9.** Let $m > 1$ be an odd square-free integer having $\nu$ prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod{2}$ for any divisor $s \mid m$. In the notation of Theorem 3, for any 2-adic integers $x_{0,e}, x_{2,e} (e \in T_8)$ not all even we have

$$2 \sum_{e \in T_8} x_{0,e} \sum_{d \in T_m \atop ed < 0} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p) \right)$$

$$- \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) H(ed)$$

$$+ \sum_{e \in T_8} x_{2,e} \sum_{d \in T_m \atop ed < 0} \Psi(|d|) \left( \prod_{p \mid m \atop p \text{ prime}} (1 - \chi_{ed}(p)\Theta(p)p^{-1}) \right)$$

$$- \delta_{d,1} \prod_{p \mid m \atop p \text{ prime}} (1 - \Theta(p)) K_2(ed),$$

$$\equiv 0 \pmod{2^{\nu+\lambda+1}},$$

where $2^\lambda$ is the greatest common divisor of the eight integers $s_i \ (0 \leq i \leq 7)$ defined by

$$s_0 = x_{0,-8} + x_{0,-4} + x_{0,1} + x_{0,8} + x_{2,-8} + x_{2,-4} + x_{2,1} + x_{2,8},$$

$$s_1 = 2(x_{0,1} + x_{0,8} + x_{2,1} + x_{2,8}),$$

$$s_2 = 2(9x_{0,-8} + 9x_{0,8} + 5x_{2,-8} + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}),$$

$$s_3 = 4(9x_{0,8} + 4x_{2,1} + 5x_{2,8}),$$

$$s_4 = 4(x_{0,-8} + x_{0,8} + x_{2,-8} + x_{2,8}).$$
On optimal linear congruences for $L_2(k, \chi_{1-k})$

$s_5 = 8(x_{0,8} + x_{2,8}),$

$s_6 = 8(5x_{0,-8} + 5x_{0,8} + x_{2,-8} + x_{2,8}),$

$s_7 = 64.$

**Proof.** Note that in the case when $L = \{0, 2\}$ we have

$$z_8 \equiv 2z_6 \pmod{64}, \quad z_7 \equiv 2z_5 \pmod{64},$$

and in consequence we may ignore the $z_8$ and $z_7$ (the $z_n$ with $n = 2c(L) - 4,$ $2c(L) - 5$). We apply (1.1) and (1.2) again. □

**Corollary 1.** The congruence in the hypothesis of Theorem 9 is optimal if and only if

$$x_{0,-8} = a,$$

$$x_{0,-4} = a + 64b - 32c - 8d + 4e + 4f - 2g,$$

$$x_{0,1} = -a + 32c - 4e - 4f + 2g + 2h,$$

$$x_{0,8} = -a - 4f + 2h,$$

$$x_{2,-8} = -a + 16f - 8g,$$

$$x_{2,-4} = -a + 8d - 4e - 20f + 10g,$$

$$x_{2,1} = a + 4e + 4f - 10g - 2h,$$

$$x_{2,8} = a + 4f + 8g - 2h,$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with $a$ odd.

**Proof.** The congruence in the hypothesis of Theorem 9 is valid modulo $2^\nu + 6$ if and only if $s_0, s_1, s_2, s_3, s_4, s_5, s_6$ satisfy (14.8) for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{0,-4}, x_{0,1}, x_{0,8}, x_{2,-8}, x_{2,-4}, x_{2,1}, x_{2,8}$ and determinant 64. An easy verification gives the formulas of Corollary 1 at once. □

**Corollary 2.** If the congruence in the hypothesis of Theorem 9 is optimal then all the $x_{0,e}, x_{2,e} (e \in T_8)$ are odd. None of these coefficients can vanish in particular.
15. Concluding remarks

Uehara’s approach used in [8] and [10] gives a method of producing linear congruences. It would be interesting to use this method to find for given \( \lambda \) explicit formulas for the \( x_{k,e} \), such that the linear congruences are valid modulo \( 2^{\nu + \lambda} \). This approach should yield many new congruences between class numbers and the orders of \( K_2 \)-groups of the rings of integers of quadratic fields. In the case of the orders of \( K_2 \)-groups for imaginary quadratic fields such congruences would be completely new. The detailed results will appear in forthcoming publications.

Another direction for further investigation would be to extend WÓJCİK’s congruence [10] by giving a congruence for a linear combination of the values \( L_2(k, \chi \omega^{1-k}) \), where the numbers \( k \) are taken from any finite subset of the integers. Wójcik’s congruence involved the case when this subset consisted of consecutive integers. URBANOWICZ and WÓJCİK [8] found such a congruence for any subset of the set \( \{-1, 0, 1, 2\} \).

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References

On optimal linear congruences for $L_2\left(k, \chi \omega^{1-k}\right)$


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