The quasiasymptotic expansion at zero and generalized Watson lemma for Colombeau generalized functions

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Abstract. Quasiasymptotic expansion at zero in the Colombeau algebra of generalized functions and its coherence with this notion for Schwartz distributions is given. A version of the Watson lemma related to the expansion of the Laplace transformation of an appropriate generalized Colombeau function is proved. In particular, the asymptotic expansion of $\delta^2$ and the expansion of its Laplace transformation is given.

1. Introduction

Asymptotic analysis is an old subject (cf. [1]) which has a lot of applications in applied mathematics, physics and engineering. It approximates integral expressions or solutions of differential equations.

Since a generalized function $g$ is represented by an $\varepsilon$-net of smooth functions $g_\varepsilon$ with the power order growth with respect to $\varepsilon$, $(\varepsilon \to 0)$, the order growth of $\varepsilon$ reflects in some sense its singularity. We found that the singularity at zero is characterized through the analysis of the behaviour of $g_\varepsilon(\varepsilon x)$ as $\varepsilon \to 0$. This was already done for Schwartz distributions, using the quasiasymptotics and quasiasymptotic expansion at zero [9], but since the Colombeau space $\mathcal{G}$ contains elements which are no distributions ($\delta^2$, for example) we reconsider the concept of quasiasymptotic expansion.
in \( \mathcal{G} \). Note that the quasiasymptotic behaviour in \( \mathcal{G} \) is used in [8] for the analysis of a non-linear Cauchy problem.

Generalized asymptotic expansion in the space of Schwartz distributions is studied in [1–3], [5], [9], (see also references in [9]), and the asymptotic expansion related to geometric optics in \( \mathcal{G} \) is analyzed in [6].

The results of this paper are the following.

First, by using the definition of quasiasymptotic behaviour at zero of Colombeau generalized functions [8], we derive the definition and the properties of quasiasymptotic expansions at zero for Colombeau generalized functions. We compare this notion with the corresponding one for Schwartz distributions. An \( f \in \mathcal{D}' \) has a quasiasymptotic expansion if and only if the embedded Colombeau generalized function has a quasiasymptotic expansion in \( \mathcal{G} \).

Second, we give an Abelian-type result for the Laplace transformation of an \( f \in \mathcal{G} \) which has appropriate quasiasymptotic expansion. This is a generalized version of the classical Watson lemma [1].

Third, we find the quasisasymptotic expansion of the generalized function \( \delta^2 \in \mathcal{G} \setminus \mathcal{D}' \) and the asymptotic expansion of its Laplace transformation.

2. Preliminaries

Notation

Schwartz spaces of test functions and distributions on the real line \( \mathbb{R} \) are denoted by \( \mathcal{D} \) and \( \mathcal{D}' \), respectively; \( \mathcal{S} \) is the space of rapidly decreasing functions and its dual \( \mathcal{S}' \) is the space of tempered distributions. Also, we use the notions \( \mathcal{D}(\Omega) \) and \( \mathcal{D}'(\Omega) \) where \( \Omega \) is an open subset of \( \mathbb{R}^n \).

Let \( \alpha \in \mathbb{R} \). Denote

\[
 f_{\alpha+1}(x) = \begin{cases} 
 x^{\alpha}H(x) / \Gamma(\alpha + 1), & \alpha > -1 \\
 f_{\alpha+n+1}^{(n)}(x), & \alpha \leq -1,
\end{cases}
\]

where \( n \) is the smallest integer for which \( \alpha + n > -1 \), and \( H \) is the Heaviside function.

Recall that \( C^\infty(\Omega) \) is a topological vector space whose topology is given by a countable set of seminorms

\[
 \mu_k(\varphi) = \sup \left\{ \left| \varphi^{(i)}(x) \right| : |i| \leq k, x \in \Omega_k, k \in \mathbb{N} \right\}.
\]
Colombeau algebra of generalized functions

We define $E_M$ as the space of locally bounded functions $R(\varepsilon) = R_\varepsilon : (0, 1) \to C^\infty(\Omega)$ such that for every $k \in \mathbb{N}$ there exists $a \in \mathbb{R}$ such that

$$\sup \left\{ \left| R_\varepsilon^{(i)}(x) \right| ; |i| \leq k, x \in \Omega_k \right\} = O(\varepsilon^a), \quad \Omega_k \subset \subset \Omega_{k+1}, \quad \bigcup_{k=1}^\infty \Omega_k = \Omega.$$ 

By $N$ is denoted the space of all elements $H_\varepsilon \subset E_M$ with the property that for every $k \in \mathbb{N}$ and $a \in \mathbb{R}$

$$\sup \left\{ \left| R_\varepsilon^{(i)}(x) \right| ; |i| \leq k, x \in \Omega_k \right\} = O(\varepsilon^a).$$

The quotient space $G = E_M / N$ is a Colombeau space.

In an appropriate way ($R_\varepsilon$ and $H_\varepsilon$ above do not depend on $x$) are defined spaces of moderate complex numbers $E_0$, null spaces $N_0$, and the space of generalized complex numbers $C = E_0 / N_0$.

It is easy to verify that $G(\Omega)$ is a differential algebra, where derivations are defined by $R(\alpha) = [R(\alpha)] (\cdot)$ (denotes equivalence class).

The support of a generalized function $H$, supp $H$, is defined as the complement of the largest open subset $\Omega'$ such that $H|_{\Omega'} = 0$.

It is said that $f$ belongs to $E_{t,M} (\mathbb{R}^n)$ if for any $k \in \mathbb{N}$ there exist $a \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that

$$\sup_{|\alpha| \leq k} \sup_{\mathbb{R}^n} \langle x \rangle^{-m} |\partial^\alpha f_\varepsilon(x)| = O(\varepsilon^a).$$

(note $\langle x \rangle \overset{\text{def}}{=} (1 + |x|^2)^{1/2}$).

The space of elements $g$ of $E_{t,M} (\mathbb{R}^n)$ with the property that for every $k$ there exists $m \in \mathbb{N}$ such that (1) holds for every $a \in \mathbb{R}$, is denoted by $N_t(\mathbb{R}^n)$. It is an ideal of $E_{t,M}(\mathbb{R}^n)$. The quotient space $G_t(\mathbb{R}^n) = E_{t,M} (\mathbb{R}^n) / N_t(\mathbb{R}^n)$ is called Colombeau space of tempered generalized functions. Note that $G_t$ is not a subspace of $G$ because

$$\mathcal{N}(\mathbb{R}^n) \cap E_{t,M}(\mathbb{R}^n) \neq N_t(\mathbb{R}^n),$$

but there is a canonical mapping: $G_t(\mathbb{R}^n) \to G(\mathbb{R}^n)$, and $[G_\varepsilon] = [G_\varepsilon + \mathcal{N}(\mathbb{R}^n)]$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}(\hat{\phi}) = \hat{\phi} \in \mathcal{D}(\mathbb{R}^n)$ and $\hat{\phi} \equiv 1$ on be neighbourhood of zero. Put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^n$, $\varepsilon \in (0, 1)$. We call $\phi$ the “vision” function.
If \( g \in \mathcal{D}' \), then by
\[
\left\langle g(\xi), \epsilon^{-2n} \phi\left(\frac{x - \xi}{\epsilon^2}\right)\right\rangle, \quad x \in \mathbb{R}^n
\]
is denoted the representative of the corresponding element in \( \mathcal{G} \). Its class is called Colombeau regularization of \( g \) and it is denoted by \( Cd \, g \).

**Quasiasymptotic behaviour**

Let \( \mathcal{K} \) be a set of positive measurable functions \( c \) defined on \((0, 1)\) with the following property:
\[
A^{-1} \epsilon^r < c(\epsilon) < A \epsilon^{-r}, \quad \epsilon \in (0, 1)
\]
for some \( A > 0 \) and \( r > 0 \).

The notion of quasiasymptotic behaviour at zero in \( \mathcal{G}(\Omega) \) is introduced in [8].

Let \( F \in \mathcal{G}(\Omega) \). If for every \( \psi \in \mathcal{D}(\Omega) \) there is \( C_\psi \in \mathbb{C}, \, C_\psi \neq 0 \), such that
\[
\lim_{\epsilon \to 0^+} \left\langle \frac{F_\epsilon(x)}{c(\epsilon)}, \, \psi(x) \right\rangle = C_\psi,
\]
then it is said that \( F \) has quasiasymptotics at zero with respect to \( c(\epsilon) \in \mathcal{K} \).

The consequences of this definition are contained in Proposition 2 and Proposition 3 of [8].

**3. The quasiasymptotic expansion at zero of Colombeau generalized functions**

We denote by \( \Lambda \) the set \( \mathcal{N} \) or a finite set of the form \( \{1, 2, \ldots, N\} \), \( N \in \mathbb{N} \).

Let \( c_k \in \mathcal{K}, \, k \in \Lambda \), such that
\[
\lim_{\epsilon \to 0^+} \frac{c_{k+1}(\epsilon)}{c_k(\epsilon)} = 0, \quad k = 1, \ldots, N - 1 \quad \text{(or } k \in \mathbb{N}, \text{ if } \Lambda = \mathbb{N})
\]
and
\[
P_k = [P_\epsilon] \in \mathcal{G}(\Omega), \quad k \in \Lambda.
\]
Then $G = [G_\varepsilon] \in \mathcal{G}(\Omega)$ has quasiasymptotic expansion (strong asymptotic expansion) at zero and this equals $\sum_{k \in \Lambda} P_k$ with respect to $\{c_k(\varepsilon); \ k \in \Lambda\}$ if

$$\left( \frac{G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}}{c_m(\varepsilon)} \right)(\varepsilon x) \rightarrow 0, \ \varepsilon \rightarrow 0^+, \ \text{in} \ \mathcal{D}'(\Omega) \ \text{for every} \ m \in \Lambda$$

$$\left( \frac{G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}}{c_m(\varepsilon)} \right) \rightarrow 0, \ \varepsilon \rightarrow 0^+ \ \text{for every} \ x \in \Omega, \ \text{and every} \ m \in \Lambda.$$ 

In this case we write

$$G(\varepsilon x) \overset{q.e.c.}{\sim} \sum_{k \in \Lambda} P_k(\varepsilon x) \text{ with respect to } \{c_k(\varepsilon); \ k \in \Lambda\}$$

and say that $G$ has quasiasymptotic (strong asymptotic) expansion in the Colombeau sense.

If $G \in \mathcal{G}_t(\mathbb{R})$ and $P_k \in \mathcal{G}_t(\mathbb{R})$, $k \in \Lambda$, then with the same definitions we obtain the quasiasymptotic expansion in $\mathcal{G}_t(\mathbb{R})$.

One can simply prove that this definition does not depend on representatives.

In the sequel (if no additional condition is given), we will always assume that $\{c_k(\varepsilon); \ k \in \Lambda\}$ is a subset of $\mathcal{K}$ satisfying (1).

**Remark 1.** Let $G \in \mathcal{G}(\mathbb{R})$ ($G \in \mathcal{G}_t(\mathbb{R})$) and $P_k \in \mathcal{G}(\mathbb{R})$ ($P_k \in \mathcal{G}_t(\mathbb{R})$), $k \in \Lambda$. We define the strong asymptotic expansion at $\infty$ as follows:

$G$ has strong asymptotic expansion at $\infty$ with respect to $\{c_k(\varepsilon); \ k \in \Lambda\}$ if

$$\left( \frac{G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}}{c_m(\varepsilon)} \right)(x) \rightarrow 0, \ \varepsilon \rightarrow 0^+,$$

for every $x > 0$ and every $m \in \Lambda$.

In this case we write

$$G \left( \frac{x}{\varepsilon} \right) \overset{s.e.c.}{\sim} \sum_{k \in \Lambda} P_k \left( \frac{x}{\varepsilon} \right) \at \infty \text{ with respect to } \{c_k; \ k \in \Lambda\}.$$ 

We will use this definition in Proposition 3 below.
Remark 2. We give a definition related to a special choice of $P_k$, $k \in \Lambda$. Let $\alpha_k$, $k \in \Lambda$ be an increasing sequence of real numbers, and $G = [G_\epsilon] \in \mathcal{G}_t(\Omega)$.

Then $G$ has quasiasymptotic expansion at zero as $\sum_{k\in \Lambda} A_k F_{\alpha_k + 1}$ with respect to $\{c_k(\epsilon); k \in \Lambda\}$ if there are complex numbers $A_k \neq 0$, $k \in \Lambda$, such that for any $m \in \Lambda$

\[
\frac{(G_\epsilon - \sum_{k=1}^m A_k F_{\alpha_k + 1, \epsilon})(\epsilon x)}{c_m(\epsilon)} \rightarrow 0, \quad \epsilon \rightarrow 0^+ \text{ in } \mathcal{D}'(\Omega),
\]

where $F_{\alpha+1, \epsilon} = f_{\alpha+1} * \phi_{\epsilon^2}$. We write

\[G(\epsilon x) \overset{q.e.c.}{\sim} \sum_{k \in \Lambda} A_k F_{\alpha_k + 1}(\epsilon x) \text{ with respect to } \{c_k(\epsilon); k \in \Lambda\}.
\]

In an adequate way one defines the strong asymptotic expansion at zero as $\sum_{k \in \Lambda} A_k F_{\alpha_k + 1}$.

Proposition 1. If (2) holds, then

a) $G'(\epsilon x) \overset{q.e.c.}{=} \sum_{k \in \Lambda} P'_k(\epsilon x)$ with respect to $\{c_k(\epsilon); k \in \Lambda\}$.

b) $\varphi G(\epsilon x) \overset{q.e.c.}{=} \sum_{k \in \Lambda} \varphi(0) P_k(\epsilon x)$ with respect to $\{c_k(\epsilon); k \in \Lambda\}$

where $\varphi \in C^\infty(\mathbb{R})$.

c) The strong asymptotic expansion at zero with respect to $\{c_k(\epsilon); k \in \Lambda\}$ implies the quasiasymptotic expansion at zero in the Colombeau sense if for every compact set $K \subset \subset \Omega$ and $m \in \Lambda$ there exists $\epsilon_m > 0$ such that

\[
\sup \left\{ \left| \frac{G_\epsilon(\epsilon x) - \sum_{k=1}^m P_k(\epsilon x)}{c_m(\epsilon)} \right|; \ x \in \Omega, \ \epsilon \in (0, \epsilon_m) \right\} < \infty.
\]

Proof. Assertion a) is obvious, c) follows by Lebesgue’s theorem on dominated convergence and b) follows from the equivalence of weak and strong convergence in $\mathcal{D}'(\Omega)$ since for every $\psi \in \mathcal{D}(\mathbb{R})$, $\{\psi(x) \varphi(\epsilon x); \ \epsilon \in (0, 1)\}$ is a bounded set in $\mathcal{D}(\mathbb{R})$. \qed
Recall [5] that if $f \in \mathcal{E}'$ and
\[
\left( f - \sum_{k=1}^{N} A_k f_{\alpha_k + 1} \right) \frac{(\varepsilon x)}{c_N(\varepsilon)} \to 0, \quad \varepsilon \to 0^+, \quad \text{for every } N \in \Lambda,
\]
then we say that
\[
f(\varepsilon x) \overset{q.e.}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k + 1}(\varepsilon x) \quad \text{at zero in } D'
\]
with respect to the scale $\{c_k(\varepsilon); \ k \in \Lambda\}$.

The next assertion shows the coherence of quasiasymptotic expansions in Colombeau and Schwartz spaces.

A measurable and positive function $L$ defined on $(0, M), \ M > 0$, is called slowly varying at 0 if
\[
\lim_{\varepsilon \to 0} \frac{L(\varepsilon t)}{L(\varepsilon)} = 1
\]
uniformly for $t \in [a, b] \subset (0, M)$. A function of the form $\rho(\varepsilon) = \varepsilon^\alpha L(\varepsilon), \ x \in (0, M)$ is called a regularly varying function.

**Proposition 2.** Let $f(x) \in \mathcal{E}'(\Omega)$. Then $f(\varepsilon x) \overset{q.e.}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k + 1}(\varepsilon x)$ with respect to $\{\varepsilon^{\alpha_k} L_k(\varepsilon); \ k \in \Lambda\}$ if and only if
\[
Cd f(\varepsilon x) \overset{q.e.}{\sim} \sum_{k \in \Lambda} A_k F_{\alpha_k + 1}(\varepsilon x) \quad \text{with respect to } \{\varepsilon^{\alpha_k} L_k(\varepsilon); \ k \in \Lambda\}.
\]

**Proof.** It is proved in [7] that an $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behaviour at zero in the sense of convergence in $\mathcal{S}'$ if and only if it has quasiasymptotic behaviour at zero in the sense of convergence in $D'$.

The same result is true for the quasiasymptotic expansion at zero. If $f \in \mathcal{S}'(\mathbb{R})$ then $f(\varepsilon x) \overset{q.e.}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k + 1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon); \ k \in \Lambda\}$ in the sense of convergence in $\mathcal{S}'$ iff $f(\varepsilon x) \overset{q.e.}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k + 1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon); \ k \in \Lambda\}$ in the sense of convergence in $D'$. 

Let $\alpha \in \mathcal{S}$, $\eta > 0$, $\varepsilon > 0$. We have

$$\left\langle \left( f * \phi_\eta \right)(\varepsilon x) - \sum_{k=1}^{m} A_k (f_{\alpha k+1} * \phi_\eta)(\varepsilon x) \right, \alpha(x) \right\rangle$$

$$= \left\langle \left( f(x) - \sum_{k=1}^{m} A_k f_{\alpha k+1}(x) \right) \right, \frac{\check{\phi}_\eta(t) * \alpha(t/\varepsilon))(x)}{\varepsilon c_m(\varepsilon)} \right\rangle$$

and for $\eta = \varepsilon^2$

$$= \left\langle \left( f * \phi_{\varepsilon^2} \right)(\varepsilon x) - \sum_{k=1}^{m} A_k (f_{\alpha k+1} * \phi_{\varepsilon^2})(\varepsilon x) \right, \alpha(x) \right\rangle$$

$$= \left\langle \left( f(\varepsilon x) - \sum_{k=1}^{m} A_k f_{\alpha k+1}(\varepsilon x) \right) \right, \frac{\check{\phi}_{\varepsilon^2}(t) \alpha(x - t/\varepsilon)}{c_m(\varepsilon)} \right\rangle$$

$$= \left\langle \left( f(\varepsilon x) - \sum_{k=1}^{m} A_k f_{\alpha k+1}(\varepsilon x) \right) \right, \psi_\varepsilon(x) \right\rangle,$$

where

$$\psi_\varepsilon(x) = \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t) \alpha(x - t/\varepsilon) dt = \int_{-\infty}^{\infty} \check{\phi}(t) \alpha(x - \varepsilon t) dt.$$

Since $\{\psi_\varepsilon; \varepsilon \in (0, 1)\}$ is a bounded set in $\mathcal{S}$ and $\psi_\varepsilon \to \alpha$ in $\mathcal{S}$, it follows

$$\lim_{\varepsilon \to 0^+} \left\langle \left( f * \phi_{\varepsilon^2} \right)(\varepsilon x) - \sum_{k=0}^{m} A_k (f_{\alpha k+1} * \phi_{\varepsilon^2})(\varepsilon x) \right, \alpha(x) \right\rangle$$

$$= \left\langle g(x) - \sum_{k=0}^{m} A_k f_{\alpha k+1}(x), \alpha(x) \right\rangle.$$

This implies the assertion in both directions.

4. Generalized Watson lemma

The Laplace transformation $\mathcal{L}_g$ for an element $G \in \mathcal{G}_t(\mathbb{R})$ supported
by \([0, \infty)\) is defined in [4] by

\[
\mathcal{L}_g(G)(p) = \left[ \int_{\mathbb{R}} e^{-px} G_\varepsilon(x) \eta(x) dx \right], \quad \text{Re} \ p > 0,
\]

where \(G_\varepsilon\) is a representative of \(G\), and \(\eta \in C_0^\infty(\mathbb{R})\) has the properties \(|\eta| \leq 1\); \(\eta = 1\) on \([-a/2, \infty)\) and \(\eta = 0\) on \((-\infty, -a)\) for some \(a > 0\).

We will use the notation \(\mathcal{L}\) for the usual distributional or classical Laplace transformation (if it exists).

It is well known that

\[
\mathcal{L}(f_{\alpha+1, \varepsilon}^*)(\lambda) = \mathcal{L}(f_{\alpha+1}^* \phi_{\varepsilon^2})(\lambda) = (-i\lambda)^{-\alpha-1} \mathcal{L}(\phi)(\varepsilon^2 \lambda), \quad \text{Re} \ \lambda > 0.
\]

We will give an Abelian-type result for the Laplace transformation in \(G_t\) by using the quasiasymptotic expansion given in Remark 2.

**Proposition 3.** Let \(\alpha_k, k \in \mathbb{N}\), be an increasing sequence of complex numbers, \(A_k, k \in \mathbb{N}\), be a sequence of real numbers and let \(c_k(\varepsilon), k \in \mathbb{N}\), be a sequence in \(K\) which satisfies (1).

Assume that \(G \in \mathcal{G}_t(\mathbb{R})\), \(\text{supp} \ G \subset [0, \infty)\) and \(G\) has a representative \(G_\varepsilon\) with the property \(\text{supp} \ G_\varepsilon \subset [-b\varepsilon, \infty), \varepsilon \leq \varepsilon_0, \) for some \(b > 0\). Let

\[
G(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k F_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to} \ \{c_k(\varepsilon), k \in \mathbb{N}\}.
\]

Then

\[
\mathcal{L}_g(G) \left( \frac{x}{\varepsilon} \right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k t^{i\alpha_k+1} \left[ t^{-\alpha_k-1} \mathcal{L}(\phi)(\varepsilon^2 t) \right] \left( \frac{x}{\varepsilon} \right) \quad \text{at} \ \infty,
\]

with respect to \(\{c_k(\varepsilon); k \in \mathbb{N}\}\).

**Proof.** We again use the fact that

\[
f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k=1}^{m} A_k f_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to} \ \{c_k(\varepsilon); k \in \Lambda\}
\]

in the sense of convergence in \(\mathcal{S}'\) if and only if this holds in the sense of convergence in \(\mathcal{D}'\).
Let \( a = 2b \) and \( \eta \) be defined as above. We have

\[
G_{\varepsilon}(x) = \eta\left(\frac{x}{\varepsilon}\right) G_{\varepsilon}(x), \quad x \in \mathbb{R}, \ \varepsilon \leq \varepsilon_0.
\]

This implies

\[
\frac{1}{c(\varepsilon)} \mathcal{L}_g(G_{\varepsilon})\left(\frac{p}{\varepsilon}\right) = \frac{1}{c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_{\varepsilon}(u) \eta\left(\frac{u}{\varepsilon}\right) \eta(u) du
\]

\[
= \frac{1}{c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_{\varepsilon}(u \varepsilon) \eta(u) \eta(u \varepsilon) du, \quad p > 0.
\]

Thus, if \( \frac{G_{\varepsilon}(ue)}{c(\varepsilon)} \) converges in \( \mathcal{S}' \) as \( \varepsilon \to 0^+ \), then by using the fact that \( \{e^{-pu} \eta(ue) \eta(u); \ \varepsilon \in (0, 1)\} \) is bounded in \( \mathcal{S} \) it follows that

\[
\mathcal{L}_g(G_{\varepsilon})\left(\frac{p}{\varepsilon}\right) \varepsilon c(\varepsilon) \int_{\mathbb{R}} e^{-pu} G_{\varepsilon}(u \varepsilon) \eta(u) \eta(u \varepsilon) du, \quad p > 0.
\]

By applying (3) and previous arguments to

\[
\frac{1}{c_N(\varepsilon)} \left[ \mathcal{L}_g(G_{\varepsilon})\left(\frac{x}{\varepsilon}\right) - \sum_{k=1}^{N} A_k \mathcal{L}_g(F_{\alpha_k+1})\left(\frac{x}{\varepsilon}\right) \right], \quad \forall N \in \mathbb{N},
\]

we obtain

\[
\mathcal{L}_g(G)\left(\frac{x}{\varepsilon}\right) \sim \sum_{k=0}^{\infty} A_k \left(-i \frac{x}{\varepsilon}\right)^{-\alpha_k-1} \left[ (\mathcal{L}(\varepsilon^3 \phi)) \left(\frac{x}{\varepsilon}\right) \right] \text{ at } \infty,
\]

with respect to \( \{c_k; \ k \in \mathbb{N}\} \).

In the next proposition we define an element of \( \mathcal{G} \backslash \mathcal{D}' \) which we call \( \delta^2 \). Note that “various squares of \( \delta \)” can be defined in this way.

**Proposition 4.** Let \( \delta^2 = \left\lfloor \frac{1}{\pi^2} \phi^2\left(\frac{t}{\varepsilon^2}\right)\right\rfloor \), where \( \phi \in C_0^\infty \), \( \int \phi = 1 \), \( \int x^m \phi(x) dx = 0 \), \( m = 1, \ldots, N, \ N \in \mathbb{N} \). Then

\[a) \quad \delta^2(\varepsilon x) \overset{q.e.c.}{\sim} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \left[ \frac{1}{\varepsilon^2} \phi(k) \left(\frac{t}{\varepsilon^2}\right) \right] (\varepsilon x)
\]

with respect to the scale \( \{\varepsilon^{-3+m}; \ m = 0, 1, \ldots, N\} \).

\[b) \quad \left(\mathcal{L}_g(\delta^2)\right)\left(\frac{x}{\varepsilon}\right) \overset{q.e.c.}{\sim} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \left[ \varepsilon^2 k^k \mathcal{L} \left(\phi\left(\frac{u}{\varepsilon^2}\right)\right) \left(t\frac{x}{\varepsilon}\right) \right] (\varepsilon x), \text{ at } \infty,
\]
with respect to the scale \( \{ \varepsilon^{-3+m}; m = 0, 1, \ldots, N \} \).

**Proof.** a) Let

\[ \mu_k = \int x^k \phi^2(x) dx, \quad k \leq N. \]

We have (as \( \varepsilon \to 0^+ \))

\[
\frac{1}{\varepsilon^{-3}} \int \left( \frac{1}{\varepsilon^4} \phi^2 \left( \frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left( \frac{\varepsilon x}{\varepsilon^2} \right) \right) \psi(x) dx \to 0,
\]

\[
\frac{1}{\varepsilon^{-2}} \int \left( \frac{1}{\varepsilon^4} \phi^2 \left( \frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left( \frac{\varepsilon x}{\varepsilon^2} \right) + \frac{\mu_1}{\varepsilon^4} \phi' \left( \frac{\varepsilon x}{\varepsilon^2} \right) \right) \psi(x) dx \to 0,
\]

\[
\frac{1}{\varepsilon^s} \int \left( \frac{1}{\varepsilon^4} \phi^2 \left( \frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left( \frac{\varepsilon x}{\varepsilon^2} \right) + \cdots + \frac{(-1)^{s+2}}{\varepsilon^4(s+3)!} \phi^{(s+3)} \left( \frac{\varepsilon x}{\varepsilon^2} \right) \right) \times \psi(x) dx \to 0, \quad s \in \mathbb{N}.
\]

Thus,

\[
\left[ \frac{1}{\varepsilon^4} \phi^2 \left( \frac{t}{\varepsilon^2} \right) \right] (\varepsilon x) \sim \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \left[ \frac{1}{\varepsilon^4} \phi^{(k)} \left( \frac{t}{\varepsilon^2} \right) \right] (\varepsilon x)
\]

with respect to the scale \( \{ \varepsilon^{-3+m}; m = 0, 1, \ldots, N \} \).

b) Let \( m \leq N \). We have

\[
(\mathcal{L}_\phi(\delta^2)) \left( \frac{x}{\varepsilon} \right) = \left[ \mathcal{L} \left( \frac{1}{\varepsilon^4} \phi^2 \left( \frac{t}{\varepsilon^2} \right) \right) \left( \frac{x}{\varepsilon} \right) \right]
\]

\[
= \left[ \frac{1}{\varepsilon^2} (\mathcal{L}(\phi^2)) (\varepsilon x) \right], \quad x > 0,
\]

\[
\mathcal{L} \left( \phi^{(k)} \left( \frac{t}{\varepsilon^2} \right) \right) (x) = \int \exp(-xt) \phi^{(k)} \left( \frac{t}{\varepsilon^2} \right) dt
\]

\[
= \varepsilon^{2k} x^k \left( \mathcal{L}_\phi \left( \frac{t}{\varepsilon^2} \right) \right) (x), \quad x > 0,
\]
and
\[
\frac{1}{\varepsilon^{-3+m}} \mathcal{L}\left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{\varepsilon t}{\varepsilon^2}\right) - \frac{1}{\varepsilon^4} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \phi^{(k)} \left(\frac{\varepsilon t}{\varepsilon^2}\right)\right)(x) \to 0,
\]
\[x > 0, \text{ as } \varepsilon \to 0^+.
\]
This implies
\[
\frac{1}{\varepsilon^{-2+m}} \mathcal{L}\left(\frac{\phi^2}{\varepsilon^2} \left(\frac{t}{\varepsilon^2}\right)\right) \left(\frac{x}{\varepsilon}\right)
- \frac{1}{\varepsilon^{-2+m}} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \mathcal{L}\left(\phi^{(k)} \left(\frac{t}{\varepsilon^2}\right)\right) \left(\frac{x}{\varepsilon}\right) \to 0,
\]
\[x > 0, \varepsilon \to 0^+, \text{ and}
\]
\[
\mathcal{L}_d(\delta^2) \left(\frac{x}{\varepsilon}\right) \overset{s.e.c.}{\sim} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \left[\varepsilon^{2k} t^k \mathcal{L}\left(\phi \left(\frac{t}{\varepsilon^2}\right)\right) \left(\frac{x}{\varepsilon}\right)\right] \text{ at } \infty,
\]
with respect to the scale \(\{\varepsilon^{-3+m}; m = 0, 1, \ldots, N\}\).

References

The quasiasymptotic expansion at zero . . .


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