Some sufficient conditions for a map to be harmonic

By HURIYE ARIKAN (Istanbul) DEMIR N. KUPELI (Ankara)

Abstract. Some integral sufficient conditions for a map to be harmonic are obtained. In achieving this result, the divergence and Laplacian of a vector field along a map are defined and a divergence theorem for a vector field along a map (the generalized divergence theorem) is used.

1. Introduction

It is claimed in ([1], p. 9) that a map from a compact Riemannian manifold to a Riemannian manifold is harmonic if the $k^{th}$ covariant differential of its tension field vanishes. See ([5], Prop. 2.5) for the proof of this result when the (first) covariant differential of its tension field vanishes. In this paper, we generalize this result to integral inequalities involving divergence and Laplacian of the tension field which in turn also provides a proof of the above claim (see Theorem 3.1 and Remark 3.11). For this, first we define divergence of a vector field along a map. Then we give a divergence theorem for a vector field along a map, called the generalized divergence theorem (Theorem 2.2). In fact, this theorem plays the crucial role in obtaining the mentioned generalization of the result above. Also we use two complementary theorems in achieving this result (see Theorems 2.5 and 2.6). But these latter two theorems are the straightforward generalizations of the well-known results of Bochner on vector fields to vector fields along a map. Cf. ([4], p. 158) and ([3], p. 46). Finally, we

Mathematics Subject Classification: 58C35, 58E20.
Key words and phrases: harmonic map, divergence and Laplacian of a vector field along a map, divergence theorem for a vector field along a map (the generalized divergence theorem), closed geodesic.
make an application of our above mentioned result to closed geodesics on Riemannian manifolds.

The main result (Theorem 3.1) of this paper may also be considered as an application of the generalized divergence theorem to harmonic maps. Indeed, the generalized divergence theorem plays the central role in obtaining Theorem 3.1 about harmonicity of maps between Riemannian manifolds.

Throughout this paper, everything at hand is assumed to be smooth.

2. Preliminaries

Let \((V_1, g_1)\) and \((V_2, g_2)\) be real inner product spaces of dimensions \(n_1\) and \(n_2\) respectively, and let \(T : (V_1, g_1) \rightarrow (V_2, g_2)\) be a linear transformation. The adjoint \(\ast T\) of \(T\) is defined to be the unique linear transformation \(\ast T : (V_2, g_2) \rightarrow (V_1, g_1)\), such that for all \(x \in V_1\) and \(y \in V_2\)

\[ g_1(x, \ast T y) = g_2(T x, y). \]

The adjoint linear transformation enables us to define an inner product \(\langle \ , \ \rangle\) in the space \(L(V_1; V_2)\) of linear transformations from \(V_1\) to \(V_2\) by

\[ \langle T, S \rangle = \text{trace} \ast T \circ S. \]

Note that if \(\{x_1, \ldots, x_{n_1}\}\) is an orthonormal basis for \((V_1, g_1)\) then

\[ \langle T, S \rangle = \sum_{i=1}^{n_1} g_2(Sx_i, Tx_i). \]

Also, let \(\| \|\) be the square norm on \(L(V_1; V_2)\) induced by \(\langle \ , \ \rangle\), that is,

\[ \|T\|^2 = \langle T, T \rangle. \]

Now let \((M_1, g_1)\) and \((M_2, g_2)\) be Riemannian manifolds of dimensions \(n_1\) and \(n_2\) with Levi–Civita connections \(\nabla^1\) and \(\nabla^2\), respectively. Let \(f : (M_1, g_1) \rightarrow (M_2, g_2)\) be a map. We denote the set of vector fields on \(M_1\) by \(\Gamma TM_1\) and the set of vector fields along \(f\) by \(\Gamma_f TM_2\). We also denote the pullback of \(\nabla^2\) along \(f\) by \(\nabla^2\). Recall that the map

\[ \nabla f : \Gamma TM_1 \times \Gamma TM_1 \rightarrow \Gamma_f TM_2, \]
Some sufficient conditions for a map to be harmonic defined by

\[(\nabla f_*) (X, Y) = 2 \nabla_X f_* Y - f_* \left( \nabla_X Y \right)\]

is called the second fundamental form of \(f\). The trace \(\tau(f)\) of \(\nabla f_*\) is called the tension field of \(f\). That is,

\[\tau(f) = \text{trace} \nabla f_* = \sum_{i=1}^{n_1} (\nabla f_*)(X_i, X_i),\]

where \(\{X_1, \ldots, X_{n_1}\}\) is a local orthonormal frame for \(TM_1\). If \(\tau(f) = 0\) then \(f\) is called harmonic.

Let \(f: (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). For a given \(Z \in \Gamma_f TM_2\), define a bundle homomorphism

\[\ast f_* \nabla^2 Z : TM_1 \to TM_1\]

by

\[\left( \ast f_* \nabla^2 Z \right) x = \ast f_{* p_1} \nabla_x Z,\]

where \(x \in T_{p_1} M_1\) and \(\ast f_{* p_1}\) is the adjoint of \(f_{* p_1}\).

**Definition 2.1.** Let \(f: (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). Then the divergence of \(Z \in \Gamma_f TM_2\) is defined by

\[\text{div } Z = \text{trace} \ast f_* \nabla Z.\]

Note that if \(\{X_1, \ldots, X_{n_1}\}\) is a local orthonormal frame for \(TM_1\), then

\[\text{div } Z = \text{trace} \ast f_* \nabla Z = \sum_{i=1}^{n_1} g_1 \left( \left( \ast f_* \nabla Z \right) X_i, X_i \right) = \sum_{i=1}^{n_1} g_2 \left( \nabla_{X_i} Z, f_* X_i \right).\]

A motivation for the definition of the divergence of a vector field along a map may be found in [2]. Also in [2], a generalization of the divergence theorem to vector fields along a map was obtained. Since this generalization is not well-known and plays a crucial role in the proof of the main theorem of this paper, we give this theorem with its proof here.
**Theorem 2.2** (The Generalized Divergence Theorem). Let \((M_1, g_1)\) be an oriented Riemannian manifold with boundary \(\partial M_1\) (possibly \(\partial M_1 = \emptyset\)) and Riemannian volume form \(\mu_1\), and let \((M_2, g_2)\) be a Riemannian manifold. Let \(f : (M_1, g_1) \rightarrow (M_2, g_2)\) be a map and \(Z \in \Gamma_f T M_2\) with compact support. Then

\[
\int_{M_1} (\text{div} \, Z) \mu_1 + \int_{M_1} g_2(Z, \tau(f)) \mu_1 = \int_{\partial M_1} g_2(Z, f_\ast N_1) \mu_{1_{\partial M_1}},
\]

where \(N_1\) is the unit outward normal vector field to \(\partial M_1\) and \(\mu_{1_{\partial M_1}}\) is the induced Riemannian volume on \(\partial M_1\).

**Proof.** Let \(*f_\ast Z\) be a vector field on \(M_1\) defined by

\[
(*f_\ast Z)(p_1) = *f_{*p_1}Z(p_1)
\]

at each \(p_1 \in M_1\), where \(*f_{*p_1}\) is the adjoint of \(f_{*p_1}\). Now, if \(\{X_1, \ldots, X_{n_1}\}\) is an adapted moving frame for \(TM_1\) near \(p_1\), that is, \(\{X_1, \ldots, X_{n_1}\}\) is a local orthonormal frame for \(TM_1\) with \(\frac{1}{2} \nabla X_i(p_1) = 0\) for \(i = 1, 2, \ldots, n_1\) (cf. [4], pp. 151–152), then we have at \(p_1\),

\[
\text{div} *f_\ast Z = \sum_{i=1}^{n_1} g_1\left(\frac{1}{2} \nabla X_i(*f_\ast Z), X_i\right)
\]

\[
= \sum_{i=1}^{n_1} X_i g_1(*f_\ast Z, X_i) = \sum_{i=1}^{n_1} X_i g_2(Z, f_\ast X_i)
\]

\[
= \sum_{i=1}^{n_1} g_2\left(\frac{1}{2} \nabla X_i, f_\ast X_i\right) + \sum_{i=1}^{n_1} g_2\left(Z, \frac{1}{2} \nabla X_i, f_\ast X_i\right)
\]

\[
= \sum_{i=1}^{n_1} g_2\left(\frac{1}{2} \nabla X_i, f_\ast X_i\right) + \sum_{i=1}^{n_1} g_2(Z, \nabla f_\ast)(X_i, X_i)
\]

\[
= \text{div} Z + g_2(Z, \tau(f)).
\]

Thus

\[
\text{div} *f_\ast Z = \text{div} Z + g_2(Z, \tau(f)).
\]

Now, by applying the usual divergence theorem to \(*f_\ast Z\), we obtain

\[
\int_{M_1} (\text{div} *f_\ast Z) \mu_1 = \int_{\partial M_1} g_1(*f_\ast Z, N_1) \mu_{1_{\partial M_1}} = \int_{\partial M_1} g_2(Z, f_\ast N_1) \mu_{1_{\partial M_1}}.
\]
Hence it follows that
\[ \int_{M_1} (\text{div } Z) \mu_1 + \int_{M_1} g_2(Z, \tau(f)) \mu_1 = \int_{\partial M_1} g_2(Z, f_\ast N_1) \mu_{1|\partial M_1}. \]

\[ \square \]

Note that if \((M_1, g_1) = (M_2, g_2)\) and \(f = \text{id}\), then the generalized divergence theorem reduces to the usual divergence theorem.

Let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), and let \(Z \in \Gamma_f TM_2\). Recall that the map
\[ \nabla^2 \nabla Z : \Gamma TM_1 \times \Gamma TM_1 \to \Gamma_f TM_2 \]
defined by
\[ \left( \nabla^2 \nabla Z \right)(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \]
is called the second covariant differential of \(Z\).

**Definition 2.3.** Let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). Then the Laplacian of \(Z \in \Gamma_f TM_2\) is defined by
\[ \Delta Z = \text{trace } \nabla^2 \nabla Z. \]

Note that, if \(\{X_1, \ldots, X_{n_1}\}\) is a local orthonormal frame for \(TM_1\), then
\[ \Delta Z = \text{trace } \nabla^2 \nabla Z = \sum_{i=1}^{n_1} \left( \nabla^2 \nabla Z \right)(X_i, X_i). \]

**Lemma 2.4.** Let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). If \(Z \in \Gamma_f TM_2\) then
\[ -\frac{1}{2} \Delta g_2(Z, Z) = g_2(\Delta Z, Z) + \left\| \nabla Z \right\|^2, \]
where \(\Delta\) is the Laplacian on \((M_1, g_1)\).

**Proof** (Following [4], p. 158.). Let \(\nabla g_2(Z, Z)\) denote the gradient of \(g_2(Z, Z)\) on \((M_1, g_1)\). First note that, for any \(X \in \Gamma TM_1\),
\[ g_1 \left( \nabla g_2(Z, Z), X \right) = X g_2(Z, Z) = 2 g_2 \left( \nabla_X Z, Z \right). \]
Now let \( \{X_1, \ldots, X_{n_1}\} \) be an adapted moving frame for \( TM_1 \) near \( p_1 \in M_1 \). Then at \( p_1 \),
\[
-\frac{1}{2} \Delta g_2(Z, Z) = \frac{1}{2} \sum_{i=1}^{n_1} g_1(\nabla_{X_i} \nabla g_2(Z, Z), X_i)
\]
\[
= \frac{1}{2} \sum_{i=1}^{n_1} X_i g_1(\nabla g_2(Z, Z), X_i) = \sum_{i=1}^{n_1} X_i g_2(\nabla X, Z, Z)
\]
\[
= \sum_{i=1}^{n_1} g_2(\nabla X, \nabla X, Z, Z) + \sum_{i=1}^{n_1} g_2(\nabla X, \nabla X, Z, Z)
\]
\[
= g_2(\Delta Z, Z) + \|\nabla Z\|^2.
\]
Thus
\[
-\frac{1}{2} \Delta g_2(Z, Z) = g_2(\Delta Z, Z) + \|\nabla Z\|^2. \tag{4.1}
\]

**Theorem 2.5.** Let \((M_1, g_1)\) be an oriented compact Riemannian manifold with Riemannian volume form \(\mu_1\) and let \((M_2, g_2)\) be a Riemannian manifold. Let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map and \(Z \in \Gamma_f TM_2\). If \(\int_{M_1} g_2(\Delta Z, Z) \mu_1 \geq 0\) then \(\nabla Z = 0\), that is, \(Z\) is parallel.

**Proof.** Since
\[
\int_{M_1} \left(\frac{1}{2} \Delta g_2(Z, Z)\right) \mu_1 = 0,
\]
it follows from Lemma 2.4 that
\[
\int_{M_1} g_2(\Delta Z, Z) \mu_1 + \int_{M_1} \|\nabla Z\|^2 \mu_1 = 0.
\]
Hence, since \(\int_{M_1} g_2(\Delta Z, Z) \mu_1 \geq 0\), it follows that \(\|\nabla Z\|^2 = 0\), that is, \(\nabla Z = 0\). \(\square\)

Let \((M_1, g_1)\) be an oriented compact Riemannian manifold with Riemannian volume form \(\mu_1\) and let \((M_2, g_2)\) be a Riemannian manifold. Let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map. Considering \(\Gamma_f TM_2\) as a real vector space, introduce on \(\Gamma_f TM_2\) the inner product
\[
(Y, Z) = \int_{M_1} g_2(Y, Z) \mu_1,
\]
where \( Y, Z \in \Gamma f TM_2 \). Then note that \((\Gamma f TM_2, (, ))\) is an inner product space and the Laplacian \( \Delta : \Gamma f TM_2 \to \Gamma f TM_2 \) is a linear operator. Furthermore we have the following properties of \( \Delta \) in \((\Gamma f TM_2, (, ))\):

**Theorem 2.6.** Let \((M_1, g_1)\) be an oriented compact Riemannian manifold with Riemannian volume form \( \mu_1 \) and let \((M_2, g_2)\) be a Riemannian manifold. Let \( f : (M_1, g_1) \to (M_2, g_2) \) be a map. Then the Laplacian \( \Delta : \Gamma f TM_2 \to \Gamma f TM_2 \) is a self-adjoint, negative semi-definite operator with respect to \((, )\).

**Proof** (Following [3], p. 46.). Let \( Y, Z \in \Gamma f TM_2 \) and \( \{X_1, \ldots, X_{n_1}\} \) be an oriented adapted moving frame for \( TM_1 \) near \( p_1 \in M_1 \). Then at \( p_1 \), we have

\[
\sum_{i=1}^{n_1} X_i g_2(\nabla X_i Y, Z) = \sum_{i=1}^{n_1} g_2(\nabla X_i, \nabla X_i Y, Z) + \sum_{i=1}^{n_1} g_2(\nabla X_i Y, \nabla X_i Z) = g_2(\Delta Y, Z) + \langle \nabla Y, \nabla Z \rangle.
\]

If we now define on \( M_1 \) a 1-form \( \omega \) by setting

\[
\omega(X) = g_2(\nabla X Y, Z),
\]

then it is not difficult to show that the above equation tells us

\[
d * \omega = \left( g_2(\Delta Y, Z) + \langle \nabla Y, \nabla Z \rangle \right) \mu_1,
\]

where \(*\) is the Hodge star operator. Integrating by using Stokes’ theorem, since \( \partial M_1 = \emptyset \), we get

\[
\int_{M_1} g_2(\Delta Y, Z) \mu_1 = -\int_{M_1} \langle \nabla Y, \nabla Z \rangle \mu_1.
\]

Hence the result now follows immediately. \( \square \)

3. Sufficient conditions for harmonicity

Let \( \Delta^k \) denote the \( k \) th power of the Laplacian \( \Delta : \Gamma f TM_2 \to \Gamma f TM_2 \) and define \( \Delta^0 = \text{id} \), that is, \( \Delta^0 = \text{id} \) and \( \Delta^k = \Delta \cdots \Delta \) \((k \geq 1 \text{ times})\).

Now we are ready to state the main theorem of this paper.
Theorem 3.1. Let \((M_1, g_1)\) be an oriented compact Riemannian manifold with Riemannian volume form \(\mu_1\) and let \((M_2, g_2)\) be a Riemannian manifold. A map \(f : (M_1, g_1) \to (M_2, g_2)\) is harmonic if it satisfies one of the conditions below for some integer \(k \geq 0\).

a) \((-1)^k \int_{M_1} (\text{div} \, \Delta^k \tau(f)) \mu_1 \geq 0\)

b) \(\int_{M_1} g_2(\Delta^{k+1} \tau(f), \Delta^k \tau(f)) \mu_1 \geq 0\).

We prove this theorem by induction. In the lemmas below, first we show that Theorem 3.1 is true for \(k = 0, 1, 2, 3\).

Throughout the lemmas, let \(f : (M_1, g_1) \to (M_2, g_2)\) be a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), where \((M_1, g_1)\) is oriented and compact with Riemannian volume form \(\mu_1\).

Lemma 3.2. If \(\int_{M_1} (\text{div} \, \tau(f)) \mu_1 \geq 0\) then \(f\) is harmonic.

Proof. Since \(\partial M_1 = \emptyset\), by Theorem 2.2,

\[
\int_{M_1} (\text{div} \, \tau(f)) \mu_1 + \int_{M_1} g_2(\tau(f), \tau(f)) \mu_1 = 0.
\]

Hence by \(\int_{M_1} (\text{div} \, \tau(f)) \mu_1 \geq 0\) it follows that \(g_2(\tau(f), \tau(f)) = 0\), that is, \(\tau(f) = 0\). \(\square\)

Lemma 3.3. If \(\int_{M_1} g_2(\Delta \tau(f), \tau(f)) \mu_1 \geq 0\) then \(f\) is harmonic.

Proof. By Theorem 2.5, \(\nabla^2 \tau(f) = 0\). Thus \(\text{div} \, \tau(f) = 0\) and it follows from Lemma 3.2 that \(f\) is harmonic. \(\square\)

Lemma 3.4. If \(\int_{M_1} (\text{div} \, \Delta \tau(f)) \mu_1 \leq 0\) then \(f\) is harmonic.

Proof. Since \(\partial M_1 = \emptyset\), by Theorem 2.2,

\[
\int_{M_1} (\text{div} \, \Delta \tau(f)) \mu_1 + \int_{M_1} g_2(\Delta \tau(f), \tau(f)) \mu_1 = 0.
\]

Hence by \(\int_{M_1} (\text{div} \, \Delta \tau(f)) \mu_1 \leq 0\), we have \(\int_{M_1} g_2(\Delta \tau(f), \tau(f)) \mu_1 \geq 0\) and it follows from Lemma 3.3 that \(f\) is harmonic. \(\square\)

Lemma 3.5. If \(\int_{M_1} g_2(\Delta^2 \tau(f), \Delta \tau(f)) \mu_1 \geq 0\) then \(f\) is harmonic.

Proof. By Theorem 2.5, \(\nabla^2 \Delta \tau(f) = 0\). Thus \(\text{div} \, \Delta \tau(f) = 0\) and it follows from Lemma 3.4 that \(f\) is harmonic. \(\square\)
Lemma 3.6. If \( \int_{M_1} (\text{div} \Delta^2 \tau(f)) \mu_1 \geq 0 \) then \( f \) is harmonic.

Proof. Since \( \partial M_1 = \emptyset \), by Theorem 2.2,

\[
\int_{M_1} (\text{div} \Delta^2 \tau(f)) \mu_1 + \int_{M_1} g_2(\Delta^2 \tau(f), \tau(f)) \mu_1 = 0.
\]

But by Theorem 2.6,

\[
\int_{M_1} g_2(\Delta^2 \tau(f), \tau(f)) \mu_1 = \int_{M_1} g_2(\Delta \tau(f), \Delta \tau(f)) \mu_1.
\]

Hence by \( \int_{M_1} (\text{div} \Delta^2 \tau(f)) \mu_1 \geq 0 \), it follows that \( g_2(\Delta \tau(f), \Delta \tau(f)) = 0 \), that is, \( \Delta \tau(f) = 0 \). Thus either of Lemmas 3.4 or 3.5 implies that \( f \) is harmonic. \( \square \)

Lemma 3.7. If \( \int_{M_1} g_2(\Delta^3 \tau(f), \Delta^2 \tau(f)) \mu_1 \geq 0 \) then \( f \) is harmonic.

Proof. By Theorem 2.5, \( \nabla \Delta^2 \tau(f) = 0 \). Thus \( \text{div} \Delta^2 \tau(f) = 0 \) and it follows from Lemma 3.6 that \( f \) is harmonic. \( \square \)

Lemma 3.8. If \( \int_{M_1} (\text{div} \Delta^3 \tau(f)) \mu_1 \leq 0 \) then \( f \) is harmonic.

Proof. Since \( \partial M_1 = \emptyset \), by Theorem 2.2,

\[
\int_{M_1} (\text{div} \Delta^3 \tau(f)) \mu_1 + \int_{M_1} g_2(\Delta^3 \tau(f), \tau(f)) \mu_1 = 0.
\]

But by Theorem 2.6,

\[
\int_{M_1} g_2(\Delta^3 \tau(f), \tau(f)) \mu_1 = \int_{M_1} g_2(\Delta^2 \tau(f), \Delta \tau(f)) \mu_1.
\]

Hence by \( \int_{M_1} (\text{div} \Delta^3 \tau(f)) \mu_1 \leq 0 \), we have \( \int_{M_1} g_2(\Delta^2 \tau(f), \Delta \tau(f)) \geq 0 \), and it follows from Lemma 3.5 that \( f \) is harmonic. \( \square \)

Lemma 3.9. If \( \int_{M_1} g_2(\Delta^4 \tau(f), \Delta^3 \tau(f)) \mu_1 \geq 0 \) then \( f \) is harmonic.

Proof. By Theorem 2.5, \( \nabla \Delta^3 \tau(f) = 0 \). Thus \( \text{div} \Delta^3 \tau(f) = 0 \) and it follows from Lemma 3.8 that \( f \) is harmonic. \( \square \)
Proof of Theorem 3.1. The above lemmas show that Theorem 3.1 is true for \( k = 0, 1, 2, 3 \). Now suppose the theorem is true for \( k = 0, 1, 2, 3, \ldots, 2m, 2m + 1 \), where \( m \geq 1 \). We show that the theorem is true for \( k = 2m + 2 \) and \( k = 2m + 3 \).

Let \( \int_{M_1} \left( \text{div} \Delta^{2m+2} \tau(f) \right) \mu_1 \geq 0 \). Then, since \( \partial M_1 = \emptyset \), by Theorem 2.2,

\[
\int_{M_1} \left( \text{div} \Delta^{2m+2} \tau(f) \right) \mu_1 + \int_{M_1} g_2(\Delta^{2m+2} \tau(f), \tau(f)) \mu_1 = 0.
\]

But by Theorem 2.6,

\[
\int_{M_1} g_2(\Delta^{2m+2} \tau(f), \tau(f)) \mu_1 = \int_{M_1} g_2(\Delta^{m+1} \tau(f), \Delta^{m+1} \tau(f)) \mu_1.
\]

Hence by \( \int_{M_1} \left( \text{div} \Delta^{2m+2} \tau(f) \right) \mu_1 \geq 0 \), it follows that \( g_2(\Delta^{m+1} \tau(f), \Delta^{m+1} \tau(f)) = 0 \), that is, \( \Delta^{m+1} \tau(f) = 0 \). Thus, by the induction hypothesis, either of (a) or (b) implies that \( f \) is harmonic.

Now let \( \int_{M_1} g_2(\Delta^{2m+3} \tau(f), \Delta^{2m+2} \tau(f)) \mu_1 \geq 0 \). Then by Theorem 2.5,

\[
\nabla^2 \Delta^{2m+2} \tau(f) = 0. \text{ Thus div } \Delta^{2m+2} \tau(f) = 0 \text{ and it follows from the above case that } f \text{ is harmonic. Consequently we showed that the theorem is true for } k = 2m + 2. \text{ Now we show that the Theorem is true for } k = 2m + 3.
\]

Let \( \int_{M_1} \left( \text{div} \Delta^{2m+3} \tau(f) \right) \mu_1 \leq 0 \). Then, since \( \partial M_1 = \emptyset \), by Theorem 2.2,

\[
\int_{M_1} \left( \text{div} \Delta^{2m+3} \tau(f) \right) \mu_1 + \int_{M_1} g_2(\Delta^{2m+3} \tau(f), \tau(f)) \mu_1 = 0.
\]

But by Theorem 2.6,

\[
\int_{M_1} g_2(\Delta^{2m+3} \tau(f), \tau(f)) \mu_1 = \int_{M_1} g_2(\Delta^{m+2} \tau(f), \Delta^{m+1} \tau(f)) \mu_1.
\]

Hence by \( \int_{M_1} \left( \text{div} \Delta^{2m+3} \tau(f) \right) \mu_1 \leq 0 \), we have \( \int_{M_1} g_2(\Delta^{m+2} \tau(f), \Delta^{m+1} \tau(f)) \mu_1 \geq 0 \), and it follows by the induction hypothesis that (b) implies \( f \) is harmonic.

Now let \( \int_{M_1} g_2(\Delta^{2m+4} \tau(f), \Delta^{2m+3} \tau(f)) \mu_1 \geq 0 \). Then, by Theorem 2.5,

\[
\nabla^2 \Delta^{2m+3} \tau(f) = 0. \text{ Thus div } \Delta^{2m+3} \tau(f) = 0 \text{ and it follows from}
the above case that \( f \) is harmonic. This completes the proof of the theorem. \( \square \)

Finally we make an application of Theorem 3.1 to closed geodesics on Riemannian manifolds. Let \((M_1, g_1) = (S^1, d\theta^2)\), where \( \theta \) is the polar coordinate on \( S^1 \), and orient \( S^1 \) by \( \frac{\partial}{\partial \theta} \). Then note that \( d\theta \) is the Riemannian volume form of \((S^1, d\theta^2)\). Also, let \((M_2, g_2) = (M, g)\) be a Riemannian manifold with Levi–Civita connection \( \nabla \). Let \( \gamma : (S^1, d\theta^2) \to (M, g) \) be a (curve) map. Define the velocity vector field of \( \gamma \) by \( \dot{\gamma} = \gamma^* \frac{\partial}{\partial \theta} \). Now, if we set \( \nabla^k \frac{\partial}{\partial \theta} = \nabla \frac{\partial}{\partial \theta} \cdots \nabla \frac{\partial}{\partial \theta} \) \((k \geq 1 \text{ times})\), then it can be easily seen that \( \tau(\gamma) = \nabla \frac{\partial}{\partial \theta} \dot{\gamma}, \ \text{div} \Delta^k \tau(\gamma) = g(\nabla^2 \Delta^k \dot{\gamma}, \nabla^2 \Delta^k \dot{\gamma}) \), where \( k \geq 0 \). Thus by Theorem 3.1, if either

\[
(-1)^{k+1} \int_0^{2\pi} g(\nabla^{2k+3} \dot{\gamma}, \dot{\gamma}) d\theta \geq 0 \quad \text{or} \quad \int_0^{2\pi} g(\nabla^{2k+1} \dot{\gamma}, \nabla^{2k-1} \dot{\gamma}) d\theta \geq 0
\]

for some integer \( k \geq 1 \), then \( \gamma \) is harmonic and hence a geodesic of \((M, g)\), that is \( \nabla \frac{\partial}{\partial \theta} \dot{\gamma} = 0 \).

**Remark 3.10.** It is easy to observe that Theorem 3.1 remains valid if \((M_1, g_1)\) is not orientable. In this case, by passing to the Riemannian orientation covering \((\tilde{M}_1, \tilde{g}_1)\) of \((M_1, g_1)\), since the Riemannian covering map \( \chi : (\tilde{M}_1, \tilde{g}_1) \to (M_1, g_1) \) is a local isometry, the integral inequalities in the statement of Theorem 3.1 hold on \((M_1, g_1)\) if and only if the corresponding integral inequalities hold on \((\tilde{M}_1, \tilde{g}_1)\) for the lift of \( f \) to \((\tilde{M}_1, \tilde{g}_1)\), that is \( f \circ \chi \). (In fact the mentioned integrals on \((\tilde{M}_1, \tilde{g}_1)\) are the twice of the corresponding ones on \((M_1, g_1)\).) Thus the same conclusion of Theorem 3.1 follows from the fact that \( f \circ \chi \) is harmonic if and only if \( f \) is harmonic, since \( \chi \) is a local isometry. (See [1], p. 15.)

**Remark 3.11.** Note that if \( f : (M_1, g_1) \to (M_2, g_2) \) is a map between Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), where \((M_1, g_1)\) is compact, then the vanishing \( k^{th} \) covariant differential of \( \tau(f) \) implies either (a) or (b) of Theorem 3.1. In fact, the vanishing odd powers of the covariant differential of \( \tau(f) \) implies (a) of Theorem 3.1 and the vanishing even powers of the covariant differential of \( \tau(f) \) implies (b) of Theorem 3.1. Hence this also proves the claim in ([1], p. 9) mentioned in the Introduction.

**Acknowledgement.** The Authors are grateful to the Department of Geometry and Topology of the University of Santiago de Compostela for their kind invitation and support during preparation of this paper.
References


