An existence theorem for the commutative neutrix product of distributions

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Abstract. In this paper we prove that the commutative neutrix product of the distributions $x_+^{-r}$ and $x_+^{-s}$ exists for $r, s = 1, 2, \ldots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distribution $x_+^{-r}$ is defined by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \ldots$ and not as in Gel’fand and Shilov [6]. If we denote Gel’fand and Shilov’s definition of $x_+^{-r}$ by $F(x_+,-r)$, it was proved in [4] that

$$x_+^{-r} = F(x_+,-r) + \frac{(-1)^r \phi(r-1)}{(r-1)!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \ldots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

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Our definition of $x^{-r}$ is more convenient to use because it satisfies the equation

$$(x^{-r})' = -rx^{-r-1}$$

for $r = 1, 2, \ldots$.

Further, the distribution $x^{-1} \ln x$ is defined by

$$x^{-1} \ln x = \frac{1}{2}(\ln^2 x)'$$

and in general, the distribution $x^{-r} \ln x$ is defined inductively by the equation

$$x^{-r} \ln x = \frac{x^{-r} - (x^{-r+1} \ln x)'}{r-1}$$

for $r = 2, 3, \ldots$.

Now let $\rho(x)$ be a function in $D$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now $f$ is an arbitrary distribution in $D'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative neutrix product of two distributions was given in [3].

**Definition 1.** Let $f$ and $g$ be distributions in $D'$ and let $f_n(x) = (f * \delta_n)(x)$, $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \, \square \, g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a, b)$ if

$$\lim_{n \to \infty} \langle f_n(x)g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$
for all functions $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions
\[ n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \ldots \]
and all functions which converge to zero in the normal sense as $n$ tends to infinity. Further, if
\[
\lim_{n \to \infty} \langle f_n(x)g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle
\]
we simply say that the product $f.g$ exists and equals $h$, see [2].

Before proving our main result, we note the following lemmas which are easily proved by induction.

**Lemma 1.** If $\varphi$ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$, then
\[
\langle x_{+}^{-r}, \varphi(x) \rangle = \int_{0}^{1} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \varphi^{(i)}(0) \right] dx
\]
\[ - \sum_{i=0}^{r-2} \frac{x^i \varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0), \]
for $r = 1, 2, \ldots$.

**Lemma 2.**
\[
\int_{-1}^{1} v^i \rho^{(r)}(v) dv = \begin{cases} 0, & 0 \leq i < r, \\ (-1)^r r!, & i = r \end{cases}
\]
for $r = 0, 1, 2, \ldots$.

The following theorem was proved in [5].

**Theorem 1.** The neutrix product $x^{-r} \Box x^{-s}$ exists and
\[ x^{-r} \Box x^{-s} = x^{-r-s} \]
for $r, s = 1, 2, \ldots$.

The limits involved in the proof of Theorem 1 were easily evaluated. However, in the following, we are going to consider the neutrix product
\( x_+^{-r} \square x_+^{-s} \). For this neutrix product, the limits are more complicated and so we only prove the existence of the limits and thus the existence of the neutrix product \( x_+^{-r} \square x_+^{-s} \).

We now prove the following theorem.

**Theorem 2.** The neutrix product \( x_+^{-r} \square x_+^{-s} \) exists for \( r, s = 1, 2, \ldots \).

**Proof.** We first of all consider the case \( s = 1 \) and put

\[
(x_+^{-r})_n = x_+^{-r} * \delta_n(x) = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{1/n} \ln(x-t) + \delta_n^{(r)}(t) \, dt,
\]

for \( r = 1, 2, \ldots \). Then

\[
(-1)^{r-1}(r-1)! \int_{-1}^{1} (x_+^{-r})_n x^k \, dx
\]

\[
= \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta_n^{(r)}(s) \int_{-1/n}^{1/n} x^k \ln(x-t) \ln(x-s) \, dx \, ds \, dt
\]

\[
+ \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta_n^{(r)}(s) \int_{-1/n}^{1/n} x^k \ln(x-t) \ln(x-s) \, dx \, ds \, dt
\]

\[
+ \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta_n^{(r)}(s) \int_{-1/n}^{1/n} x^k \ln(x-t) \ln(x-s) \, dx \, ds \, dt
\]

\[
= n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{1} \rho'(u)
\]

\[
\times \int_{u}^{1} w^k \ln[(w-v)/n] \ln[(w-u)/n] \, dw \, du \, dv
\]

\[
+ n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{1} \rho'(u)
\]

\[
\times \int_{v}^{1} w^k \ln[(w-v)/n] \ln[(w-u)/n] \, dw \, du \, dv
\]

\[
+ n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{1} \rho'(u)
\]

\[
\times \int_{v}^{1} w^k \ln[(w-v)/n] \ln[(w-u)/n] \, dw \, du \, dv = I_1 + I_2 + I_3,
\]

where the substitutions \( ns = u, nt = v \) and \( nx = w \) have been made.
It follows immediately that

\[ (4) \lim_{n \to \infty} I_1 = \lim_{n \to \infty} I_2 = 0, \]

for \( k = 0, 1, 2, \ldots, r - 1. \)

Now

\[
\int_1^n w^k \ln[(w - v)/n] \ln[(w - u)/n] \, dw \\
= \int_1^n w^k \ln(w - v) - \ln n \ln(w - u) - \ln n \, dw \\
= \ln^2 n \int_1^n w^k \, dw - 2 \ln n \int_1^n w^k \ln(w - v) \, dw \\
+ \int_1^n w^k \ln(w - v) \ln(w - u) \, dw
\]

and it follows immediately that

\[ (6) \lim_{n \to \infty} n^{r-k} \ln^2 n \int_1^n w^k \, dw = 0 \]

for \( k = 0, 1, 2, \ldots. \)

Further, by expanding \( \ln(w - v) \) in powers of \( v/w \), it also follows that

\[ (7) \lim_{n \to \infty} n^{r-k} \ln n \int_1^n w^k \ln(w - v) \, dw = 0 \]

for \( k = 0, 1, 2, \ldots. \)

Finally, we have

\[
\int_1^n w^k \ln(w - v) \ln(w - u) \, dw \\
= \int_1^n w^k \left[ \ln w - \sum_{i=1}^{\infty} \frac{v^i}{iw^i} \right] \left[ \ln w - \sum_{j=1}^{\infty} \frac{u^j}{jw^j} \right] \, dw \\
= \int_1^n w^k \ln^2 w \, dw - 2 \sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{k-i} \ln w \, dw \\
+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v^i u^j}{ij} \int_1^n w^{k-i-j} \, dw
\]
and it follows that

\[
N\lim_{n \to \infty} n^{r-k} \int_1^n w^k \ln(w - v) \ln(w - u) \, dw = - \sum_{j=1}^r \frac{v^{r-j+1}u^j}{j(r-k)(r-j+1)}.
\]

Hence

\[
N\lim_{n \to \infty} n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{1} \rho'(u) \int_{1}^{n} w^k \ln(w - v) \ln(w - u) \, dw \, du \, dv = \left(\frac{-1}{r-1}\right)! \frac{(r-1)!}{r-k},
\]

for \( k = 0, 1, 2, \ldots, r - 1 \) on using equation (2). It follows from equations (5), (6), (7) and (9) that

\[
N\lim_{n \to \infty} I_3 = \left(\frac{-1}{r-1}\right)! \frac{(r-1)!}{r-k}.
\]

It now follows from equations (3), (4) and (10) that

\[
N\lim_{n \to \infty} \int_{-1}^{1} (x_+^r)_n(x_-^{r-1})_n x^k \, dx = - \frac{1}{r-k},
\]

for \( k = 0, 1, 2, \ldots, r - 1 \).

We now deal with the case \( k = r \). Equation (3) still holds but this time it follows that

\[
N\lim_{n \to \infty} I_1 = \int_{-1}^{1} \rho^{(r)}(v) \int_{v}^{1} \rho'(u) \int_{u}^{1} w^r \ln|w-v| \ln|(w-u)| \, dw \, du \, dv,
\]

\[
N\lim_{n \to \infty} I_2 = \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{v} \rho'(u) \int_{u}^{1} w^r \ln|w-v| \ln|(w-u)| \, dw \, du \, dv.
\]
Further, equation (8) is replaced by the equation

\[
\int_1^n w^r \ln(w - v) \ln(w - u) \, dw = \int_1^n w^r \ln^2 w \, dw
\]

\[
-2 \sum_{i=1}^{\infty} \frac{v_i}{i} \int_1^n w^{r-i} \ln w \, dw + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v_i u_j}{ij} \int_1^n w^{r-i-j} \, dw.
\]

It follows that

\[
\text{N–lim}_{n \to \infty} \int_1^n w^r \ln(w - v) \ln(w - u) \, dw = g_r(u, v),
\]

say and so

\[
\text{(14)} \quad \text{N–lim}_{n \to \infty} I_3 = \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) g_r(u, v) \, du \, dv.
\]

We therefore see from equations (3), (12), (13) and (14) that

\[
\text{N–lim}_{n \to \infty} \int_{-1}^1 (x^r - r)_{\pm} (x^r - 1)_{\pm} n x^r \, dx
\]

exists and we put

\[
\text{(15)} \quad \text{N–lim}_{n \to \infty} \int_{-1}^1 (x^r - r)_{\pm} (x^r - 1)_{\pm} n x^r \, dx = L_{r,1}.
\]

When \( k = r + 1 \), it follows as for equation (3) that for any continuous function \( \psi \)

\[
\int_{-1/n}^{1/n} (x^r - r)_{\pm} (x^r - 1)_{\pm} x^{r+1} \psi(x) \, dx
\]

\[
= n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \int_{-1}^1 w^{r+1} \psi(w/n) \ln |(w - v)/n| \times \ln |(w - u)/n| \, dw \, du \, dv
\]

\[
+ n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \int_{-1}^1 w^{r+1} \psi(w/n) \ln |(w - v)/n| \times \ln |(w - u)/n| \, dw \, du \, dv
\]
and it follows that

\[
\lim_{n \to \infty} \int_{-1/n}^{1/n} (x_+^{-r})_n(x_+^{-1})_n x^{r+1} \psi(x) \, dx = 0.
\]

Next, when \( x \geq 1/n \), we have

\[
(-1)^{r-1}(r-1)! (x_+^{-r})_n \int_{-1/n}^{1/n} \ln | x - t | \delta_n^r(t) \, dt
\]

\[
= n^r \int_{-1}^{1} \ln | x - v/n \rho^r(v) | \, dv
\]

\[
= n^r \int_{-1}^{1} \left[ \ln x - \sum_{i=1}^{\infty} \frac{v^i}{i!} \right] \rho^r(v) \, dv
\]

\[
= - \sum_{i=r}^{\infty} \int_{-1}^{1} \frac{v^i}{i! x^i} \rho^r(v) \, dv.
\]

It follows that

\[
|(r-1)! (x_+^{-r})_n| \leq \sum_{i=r}^{\infty} \int_{-1}^{1} \frac{|v|^i}{i! x^i} |\rho^r(v)| \, dv \leq \sum_{i=r}^{\infty} \frac{K_r}{i! x^i},
\]

where

\[
K_r = \int_{-1}^{1} |\rho^r(v)| \, dv
\]

for \( r = 1, 2, \ldots \).

If now \( n^{-1} < \eta < 1 \), then

\[
(r-1)! \int_{1/n}^{\eta} |(x_+^{-r})_n(x_+^{-1})_n x^{r+1}| | \, dx
\]

\[
\leq K_1 K_r \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_{1/n}^{\eta} \frac{n^{r-i-j+1} x^{r-i-j-1}}{ij} \, dx
\]

\[
= K_1 K_r \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_{1}^{\eta n} \frac{w^{r-i-j+1}}{ij} \, dw
\]
An existence theorem for the commutative neutrix product

\[
\begin{align*}
K_1K_r &= \frac{\ln w}{r+1} + \frac{\ln w}{2r} + \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \frac{w^{r-i-j+2}}{ij(r-i-j+2)} \eta_n
\end{align*}
\]

and it follows that

\[
\lim_{n \to \infty} \int_{1/n}^{\eta} |(x_+^-)^r_n(x_+^-)^{m+r}_nx^{m+r}dx| \leq \frac{K_1K_r\eta}{r!}
\]

for \( r = 1, 2, \ldots \).

Thus, if \( \psi \) is a continuous function

\[
\begin{align*}
(17) \quad \lim_{n \to \infty} \left| \int_{1/n}^{\eta} (x_+^-)^r_n(x_+^-)^{m+r}_nx^{m+r+1}\psi(x)dx \right| = O(\eta)
\end{align*}
\]

for \( r = 1, 2, \ldots \).

Now let \( \varphi \) be an arbitrary function in \( \mathcal{D} \) with support contained in the interval \([-1, 1]\). By Taylor’s Theorem we have

\[
\varphi(x) = \sum_{k=0}^{r} \frac{x^k\varphi^{(k)}(0)}{k!} + \frac{x^{r+1}\varphi^{(r+1)}(\xi x)}{(r+1)!},
\]

where \( 0 < \xi < 1 \). Thus

\[
\begin{align*}
\langle (x_+^-)^r_n(x_+^-)^{m+r}_n, \varphi(x) \rangle &= \int_{-1}^{1} \frac{(x_+^-)^r_n(x_+^-)^{m+r}_n\varphi(x)}{x^{m+r+1}}dx
\end{align*}
\]

\[
= \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} \frac{(x_+^-)^r_n(x_+^-)^{m+r}_nx^k}{x^{m+r+1}}dx
\]

\[
+ \int_{-1/n}^{1/n} \frac{(x_+^-)^r_n(x_+^-)^{m+r}_nx^{r+1}\varphi^{(r+1)}(\xi x)}{(r+1)!}dx
\]

\[
+ \int_{1/n}^{\eta} \frac{(x_+^-)^r_n(x_+^-)^{m+r}_nx^{r+1}\varphi^{(r+1)}(\xi x)}{(r+1)!}dx
\]

\[
+ \int_{\eta}^{1} \frac{(x_+^-)^r_n(x_+^-)^{m+r}_nx^{r+1}\varphi^{(r+1)}(\xi x)}{(r+1)!}dx.
\]

On using the equations (11), (15), (16) and (17) and noting that the sequence \( \{(x_+^-)^r_n(x_+^-)^{m+r}_n\} \) converges uniformly to \( x^{-r-1} \) on the interval
\[ [\eta, 1], \text{it follows that} \]

\[
\lim_{n \to \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle = -\sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_0^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} \, dx + O(\eta),
\]

but since \( \eta \) can be made arbitrarily small, it follows that

\[
\lim_{n \to \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle = -\sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_0^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} \, dx + O(\eta),
\]

on using equation (1).

The neutrix product \( x_+^{-r} \square x_+^{-1} \) therefore exists and

\[
x_+^{-r} \square x_+^{-1} = x_+^{-r-1} + \frac{(-1)^r}{r!} [L_{r,1} + \phi(r)] \delta^{(r)}(x) \]

on the interval \([-1,1]\). However, the product \( x_+^{-r} . x_+^{-1} \) obviously exists on any interval not containing the origin, and so the neutrix product \( x_+^{-r} \square x_+^{-1} \) exists on the real line for \( r = 1, 2, \ldots \).

Suppose now that \( x_+^{-r} \square x_+^{-s} \) exists and is of the form

\[
x_+^{-r} \square x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x)
\]

for \( r = 1, 2, \ldots \) and for some positive integer \( s \). Then the derivative of \( x_+^{-r} \square x_+^{-s} \) exists, and

\[
(x_+^{-r} \square x_+^{-s})' = -(r+s)x_+^{-r-s-1} + a_{r,s} \delta^{(r+s)}(x)
\]

\[
= -sx_+^{-r} \square x_+^{-s-1} - rx_+^{-r-1} \square x_+^{-s}
\]

\[
= -sx_+^{-r} \square x_+^{-s-1} - rx_+^{-r-s-1} - ra_{r+1,s} \delta^{(r+s)}(x).
\]
The product \( x_+^{-r} \Box x_+^{-s-1} \) therefore exists and
\[
x_+^{-r} \Box x_+^{-s-1} = x_+^{-r-s-1} - \frac{ra_{r+1,s} + ar,s \delta(r+s)}{s}(x) = x_+^{-r-s-1} + ar,s+1 \delta(r+s)(x).
\]

It follows by induction that the product \( x_+^{-r} \Box x_+^{-s} \) exists for \( r, s = 1, 2, \ldots \).

Defining the distribution \( x_+^{-r} \) by
\[
x_+^{-r} = (-x)_+^{-r}
\]
for \( r = 1, 2, \ldots \), we have \( \Box \).

**Corollary 2.1.** The neutrix product \( x_+^{-r} \Box x_+^{-s} \) exists for \( r, s = 1, 2, \ldots \).

**Proof.** With the above notation we have
\[
(18) \quad x_+^{-r} \Box x_+^{-s} = x_+^{-r-s} + ar,s \delta(r+s-1)(x).
\]
Replacing \( x \) in this equation by \( -x \), we get
\[
(19) \quad x_+^{-r} \Box x_+^{-s} = x_+^{-r-s} - (-1)^{r+s}ar,s \delta(r+s-1)(x),
\]
proving the existence of neutrix product \( x_+^{-r} \Box x_+^{-s} \).

**Corollary 2.2.**
\[
(20) \quad x_+^{-r} \Box x_+^{-s} + (-1)^{r+s}x_+^{-r} \Box x_+^{-s} = x_+^{-r-s}
\]
for \( r, s = 1, 2, \ldots \).

**Proof.** Equation (20) follows immediately from equations (18) and (19).

**Theorem 3.** The neutrix product \( x_+^{-r} \Box \ln x_+ \) exists for \( r = 1, 2, \ldots \). In particular, the product \( x_+^{-1} \Box \ln x_+ \) exists and
\[
(21) \quad x_+^{-1} \Box \ln x_+ = x_+^{-1} \ln x_+.
\]
PROOF. We put
\[(\ln x_+)_n = \ln x_+ \ast \delta_n(x) = \int_{-1/n}^{1/n} \ln(x - t)_+ \delta_n(t) \, dt\]
and
\[(x_+^{-1})_n = x_+^{-1} \ast \delta_n(x) = \int_{-1/n}^{1/n} \ln(x - t)_+ \delta'_n(t) \, dt.\]

Since \(\ln x_+\) and \(\ln^2 x_+\) are locally summable functions, it follows that
\[
\lim_{n \to \infty} (\ln x_+)_n = \ln x_+.
\]
Thus, for arbitrary \(\varphi\) in \(D\), we have
\[
\lim_{n \to \infty} \langle ([\ln x_+^2]_n)', \varphi(x) \rangle = 2 \lim_{n \to \infty} \langle ([\ln x_+^{1}]_n), \varphi(x) \rangle = \langle ([\ln x_+^2]', \varphi(x) \rangle = 2 \langle [x_+^{-1} \ln x_+], \varphi(x) \rangle.
\]
and equation (21) follows.

Now suppose that the neutrix product \(x_+^{-r} \square \ln x_+\) exists and is of the form
\[
x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x)
\]
for some positive integer \(r\). Then the derivative of \(x_+^{-r} \square \ln x_+\) exists and
\[
(x_+^{-r} \square \ln x_+)' = -rx_+^{-r-1} \ln x_+ + x_+^{-r-1} + a_{r,0} \delta^{(r)}(x)
\]
\[
= -rx_+^{-r-1} \square \ln x_+ + x_+^{-r} \square x_+^{-1}
\]
\[
= -rx_+^{-r-1} \square \ln x_+ + x_+^{-r-1} + a_{r,1} \delta^{(r)}(x).
\]

The product \(x_+^{-r-1} \square \ln x_+\) therefore exists and
\[
x_+^{-r-1} \square \ln x_+ = x_+^{-r-1} \ln x_+ + \frac{a_{r,1} - a_{r,0}}{r} \delta^{(r)}(x)
\]
\[
= x_+^{-r-1} \ln x_+ + a_{r+1,1} \delta^{(r)}(x).
\]
It follows by induction that the product \(x_+^{-r} \square \ln x_+\) exists for \(r = 1, 2, \ldots\). \(\Box\)
Corollary 3.1. The neutrix product $x^{-r} \square \ln x_-$ exists for $r = 1, 2, \ldots$. In particular, the product $x^{-1} \ln x_-$ exists and
\begin{equation}
(22) \quad x^{-1} \ln x_- = x^{-1} \ln x_-.
\end{equation}

Proof. With the above notation we have
\begin{equation}
(23) \quad x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x).
\end{equation}
Replacing $x$ in this equation by $-x$, we get
\begin{equation}
(24) \quad x_-^{-r} \square \ln x_- = x_-^{-r} \ln x_- - (-1)^r a_{r,0} \delta^{(r-1)}(x),
\end{equation}
proving the existence of neutrix product $x_-^{-r} \square \ln x_-$. The particular case $r = 1$ of course reduces to the product
\begin{equation*}
(25) \quad x_-^{-1} \ln x_- = x_-^{-1} \ln x_-.
\end{equation*}

Corollary 3.2.
\begin{equation}
(25) \quad x_+^{-r} \square \ln x_+ + (-1)^r x_-^{-r} \square \ln x_- = x^{-r} \ln |x|
\end{equation}
for $r = 1, 2, \ldots$.

Proof. Equation (25) follows immediately from equations (23) and (24). \hfill \Box

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References


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