Strong convergence theorems for $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$

By FERENC WEISZ (Budapest)

Abstract. Multiplier operators on the Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ are investigated and Bernstein’s inequality for multi-parameter trigonometric polynomials is verified. We prove that certain means of the partial sums of the multi-parameter trigonometric Fourier series are uniformly bounded operators from $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ to $L_p$ $(1/2 < p \leq 1)$. As a consequence we obtain strong convergence theorems concerning the partial sums. The dual inequalities are also verified and a Marcinkiewicz–Zygmund type inequalities is obtained for the $BMO(\mathbb{T} \times \cdots \times \mathbb{T})$ spaces.

1. Introduction

We introduce the d-dimensional Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ by the $L_p(\mathbb{T}^d)$ norm of the non-tangential maximal function of a distribution on $\mathbb{T}^d$. It is known that the trigonometric system is not a basis in $L_1(\mathbb{T})$. Moreover, there exist functions in $H_1(\mathbb{T})$, the partial sums of which are not bounded in $L_1(\mathbb{T})$. Smith [10] and recently Belinskii [1] proved the following strong convergence result for one-parameter trigonometric Fourier series:

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|s_k f - f\|_1}{k} = 0$$

where $f \in H_1(\mathbb{T})$ and $s_k f$ denotes the k-th partial sum of the Fourier series. This result for one-parameter Walsh–Fourier series can be found in Simon [9].

Mathematics Subject Classification: Primary: 43A75, 42B05, 42B08; Secondary: 42B30.

Key words and phrases: Hardy spaces, multiplier operators, Bernstein’s inequality, strong means.

This research was supported by OTKA No. F019633, FKFP No. 0228/1999 and by the Széchenyi Professorship.
Recently the author [12] generalized this result for two-parameter trigonometric Fourier series by taking the sum over a cone. More exactly, we verified that there exists a constant $C$ depending only on $\alpha > 0$ such that

$$
\frac{1}{\log n \log m} \sum_{2^{-\alpha} \leq k/l \leq 2^\alpha \atop (k,l) \leq (n,m)} \frac{\|s_{kl}f\|_1}{kl} \leq C\|f\|_{H_1(T^2)}.
$$

Note that the space $H_1(T^2)$ defined in [12] is different from $H_1(T \times T)$ used here. With the help of Riesz and conjugate transforms one can show that $\|f\|_{H_1(T^2)} \leq \|f\|_{H_1(T \times T)}$. We obtained also the convergence result

$$
\frac{1}{\log n \log m} \sum_{2^{-\alpha} \leq k/l \leq 2^\alpha \atop (k,l) \leq (n,m)} \frac{\|s_{kl}f - f\|_1}{kl} \to 0 \quad \text{as } n, m \to \infty
$$

where $f \in H_1(T^2)$. The analogous results for two-parameter Walsh–Fourier series can also be found in [12].

In this paper we extend these theorems to the $d$-dimensional case and prove an even stronger inequality for $f \in H_1(T \times \cdots \times T)$:

$$
\frac{1}{\prod_{i=1}^{d} \log n_i} \sum_{i=1}^{d} \sum_{k_i=1}^{n_i} \frac{\|s_{k_i}f\|_{H_1(T \times \cdots \times T)}}{\prod_{i=1}^{d} k_i} \leq C\|f\|_{H_1(T \times \cdots \times T)}
$$

where $C$ is an absolute constant. From this it follows easily that

$$
\lim_{n \to \infty} \frac{1}{\prod_{i=1}^{d} \log n_i} \sum_{i=1}^{d} \sum_{k_i=1}^{n_i} \|s_{k_i}f - f\|_{H_1(T \times \cdots \times T)} \prod_{i=1}^{d} k_i = 0
$$

whenever $f \in H_1(T \times \cdots \times T)$. We extend these results also to $p < 1$, which was unknown even in the one-parameter case.

In the proof we have to use a different method than in [12], we use the multi-parameter Hardy–Littlewood inequality (see Jawerth and Torchinsky [8]) and the fact that the maximal operator of the Cesàro means of a distribution is bounded from $H_p(T \times \cdots \times T)$ to $L_p(T^d)$ (see Weisz [13]).

Moreover, we extend Bernstein’s inequality to multi-parameter trigonometric polynomials. We investigate also multiplier operators and give
Strong convergence theorems for $H_p(T \times \cdots \times T)$

a sufficient condition for the multiplier such that the operator is bounded on the Hardy space.

I would like to thank the referee for reading the paper carefully and for his useful comments.

2. Hardy spaces and conjugate functions

For a set $X \neq \emptyset$ let $X^d$ be its Cartesian product taken with itself $d$-times, moreover, let $T := [-\pi, \pi]$ and $\lambda$ be the Lebesgue measure. We briefly write $L^p$ instead of the $L^p(T^d, \lambda)$ space while the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{T^d} |f|^p \, d\lambda)^{1/p}$ ($0 < p \leq \infty$).

For $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x = (x_1, \ldots, x_d) \in T^d$ set $n \cdot x := \sum_{i=1}^d n_i x_i$. Let $f$ be a distribution on $C^\infty(T^d)$. The $n$th Fourier coefficient is defined by $\hat{f}(n) := \langle f, e^{-\mathbf{i}n \cdot x} \rangle$ where $\mathbf{i} = \sqrt{-1}$ and $n \in \mathbb{Z}^d$. In the special case when $f$ is an integrable function then

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{T^d} f(x) e^{-\mathbf{i}n \cdot x} \, dx.$$

For a distribution $f$ and $z_i := r_i e^{ix_i}$ ($0 < r_i < 1$) let

$$u(z) = u(r_1 e^{ix_1}, \ldots, r_d e^{ix_d}) := (f * P_{r_1} \times \cdots \times P_{r_d})(x) \quad (x \in T^d)$$

where $*$ denotes the convolution and

$$P_r(y) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{iky} = \frac{1 - r^2}{1 + r^2 - 2r \cos y} \quad (y \in T)$$

is the Poisson kernel. It is easy to show that $u(z)$ is a multi-harmonic function.

Let $0 < \alpha < 1$ be an arbitrary number. We denote by $\Omega_\alpha(x) (x \in T)$ the region bounded by two tangents to the circle $|z| = \alpha$ from $e^{ix}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u^*_\alpha(x) := \sup_{z_i \in \Omega_{\alpha_i}(x_i)} |u(z)| \quad (0 < \alpha_i < 1; \ i = 1, \ldots, d).$$

The Hardy space $H_p(T \times \cdots \times T) = H_p (0 < p \leq \infty)$ consists of all distributions $f$ for which $u^*_\alpha \in L^p$ and set

$$\|f\|_{H_p} := \|u^*_{1/2,\ldots,1/2}\|_p.$$
The equivalence $\|u_\alpha^*\|_p \sim \|f\|_{H^p}$ ($0 < p \leq \infty$, $0 < \alpha_i < 1$) was proved in Fefferman, Stein [4] and Gundy, Stein [7].

For a distribution $f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n)e^{in \cdot x}$ the conjugate distributions are defined by

$$\tilde{f}(j_1, \ldots, j_d) \sim \sum_{n \in \mathbb{Z}^d} \left( \prod_{i=1}^{d} (\text{sgn } n_i)^{j_i} \right) \hat{f}(n)e^{in \cdot x} \quad (j_i = 0, 1).$$

Note that $\tilde{f}(0, \ldots, 0) := f$. Gundy and Stein [6], [7] verified that if $f \in H^p$ ($0 < p < \infty$) then all conjugate distributions are also in $H^p$ and

$$\|f\|_{H^p} = \|\tilde{f}(j_1, \ldots, j_d)\|_{H^p} \quad (j_i = 0, 1).$$

Furthermore (see also Chang and Fefferman [2], Frazier [5], Duren [3]),

$$\|f\|_{H^p} \sim \sum_{i=1}^{d} \sum_{j_i=0}^{1} \|\tilde{f}(j_1, \ldots, j_d)\|_p,$$

where $\sim$ denotes the equivalences of the spaces and norms.

For a distribution $f$ with Fourier series $f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n)e^{in \cdot x}$ let $Pf \sim \sum_{n \in \mathbb{N}^d} \hat{f}(n)e^{in \cdot x}$ be the Riesz projection. Then $f \in H^p$ if and only if $Pf \in L^p$ and

$$\|f\|_{H^p} \sim \|Pf\|_p \quad (0 < p < \infty)$$

(see Gundy and Stein [6], [7]). Moreover, it is known that $H^p \sim L^p$ ($1 < p < \infty$).

In this paper the constants $C$ are absolute constants and the constants $C_p$ are depending only on $p$ and may denote different constants in different contexts.

Jawerth and Torchinsky [8] proved the following theorem.
Theorem A. For every distribution \( f \in H_p \)

\[
\left( \sum_{i=1}^{d} \sum_{|n_i|=0}^{\infty} \frac{|\hat{f}(n)|^p}{\prod_{i=1}^{d} |n_i \vee 1|^{2-p}} \right)^{1/p} \leq C_p \| f \|_{H_p} \quad (0 < p \leq 2).
\]

Denote by \( s_n f \) the \( n \)th partial sum of the Fourier series of a distribution \( f \), namely,

\[
s_n f(x) := \sum_{i=1}^{d} \sum_{k_i=-n_i}^{n_i} \hat{f}(k)e^{ik \cdot x}.
\]

For \( n \in \mathbb{N}^d \) and a distribution \( f \) the Cesàro mean of order \( n \) of the Fourier series of \( f \) is given by

\[
\sigma_n f := \frac{1}{\prod_{i=1}^{d} (n_i + 1)} \sum_{i=1}^{d} \sum_{k_i=0}^{n_i} s_k f = f \ast (K_{n_1} \times \cdots \times K_{n_d})
\]

where

\[
K_m(t) := \sum_{|j|=0}^{m} \left( 1 - \frac{|j|}{m+1} \right) e^{ijt} \quad (m \in \mathbb{N})
\]

is the one-dimensional Fejér kernel of order \( m \). It is shown in Zygmund [14] that \( K_m \geq 0 \) and

\[
\int_{\mathbb{T}} K_m(t) \, dt = \pi \quad (m \in \mathbb{N}).
\]

The following result is due to the author [13].

Theorem B. If \( f \in H_p \), then

\[
\| \sup_{n \in \mathbb{N}^d} |\sigma_n f| \|_p \leq C_p \| f \|_{H_p} \quad (1/2 < p < \infty).
\]

3. Strong convergence results

A sequence \( (\lambda_k; k \in \mathbb{Z}^d) \) is said to be a multiplier and the multiplier operator is defined by

\[
M_\lambda f(x) := \sum_{k \in \mathbb{Z}^d} \lambda_k \hat{f}(k)e^{ik \cdot x}.
\]
Let \((\lambda_k; k \in \mathbb{Z}^d)\) be an even sequence of real numbers, i.e. \(\lambda_{\epsilon_1 k_1, \ldots, \epsilon_d k_d} = \lambda_k\) for all \(\epsilon_i = -1, 1\) and \(k \in \mathbb{Z}^d\). Suppose that there exists \(K \in \mathbb{N}^d\) such that \(\lambda_k = 0\) if \(k_j \geq K_j\) for some \(j = 1, \ldots, d\). Let

\[
\Delta^1 \lambda_k := \sum_{\epsilon_1, \ldots, \epsilon_d \in \{0, 1\}} (-1)^{\epsilon_1 + \cdots + \epsilon_d} \lambda_{k_1 + \epsilon_1, \ldots, k_d + \epsilon_d}
\]

be the first and

\[
\Delta^2 \lambda_k := \sum_{\epsilon_1, \ldots, \epsilon_d \in \{0, 1\}} (-1)^{\epsilon_1 + \cdots + \epsilon_d} \Delta^1 \lambda_{k_1 + \epsilon_1, \ldots, k_d + \epsilon_d}
\]

be the second difference of \((\lambda_k)\).

**Lemma 1.** Suppose that \((\lambda_k)\) is an even multiplier and there exists \(K \in \mathbb{N}^d\) such that \(\lambda_k = 0\) if \(k_j \geq K_j\) for some \(j = 1, \ldots, d\). If \(\Lambda := \sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d (k_i + 1)\right)|\Delta^2 \lambda_k| < \infty\) then

\[
\|M_\lambda f\|_{H_p} \leq C_p \Lambda \|f\|_{H_p} \quad (f \in H_p)
\]

for every \(1/2 < p < \infty\).

**Proof.** Applying Abel rearrangement twice and Theorem B we get that

\[
\|M_\lambda f\|_p = \left\| \sum_{k \in \mathbb{N}^d} \left(\prod_{i=1}^d (k_i + 1)\right)|\Delta^2 \lambda_k| \sigma_k f \right\|_p \leq C_p \Lambda \|f\|_{H_p} \quad (1/2 < p < \infty).
\]

This together with (1) implies that

\[
\|(M_\lambda f)^{(j_1, \ldots, j_d)}\|_p = \|M_\lambda \tilde{f}^{(j_1, \ldots, j_d)}\|_p \leq C_p \Lambda \|\tilde{f}^{(j_1, \ldots, j_d)}\|_{H_p} = C_p \Lambda \|f\|_{H_p}
\]

for \(j_i = 0, 1\) and \(1/2 < p < \infty\). The equivalence (2) proves now the lemma. \(\square\)

Let us consider the function

\[
v(t) := \begin{cases}
1 & \text{if } |t| < 1 \\
2 - |t| & \text{if } 1 \leq |t| \leq 2 \\
0 & \text{if } |t| > 2
\end{cases}
\]
and the multiplier operator $V_{2N}$ defined by

$$V_{2N} f(x) := \sum_{k \in \mathbb{N}^d} \left( \prod_{i=1}^d v \left( \frac{k_i}{N_i} \right) \right) \hat{f}(k) e^{ik \cdot x}.$$ 

**Lemma 2.** If $1/2 < p < \infty$ then

$$\|V_{2N} f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p).$$

**Proof.** Let $\lambda_k := \prod_{i=1}^d v(\frac{k_i}{N_i})$. It is easy to see that $\Delta^2 \lambda_k = \prod_{i=1}^d \Delta^2 v(\frac{k_i}{N_i})$ and so we have

$$\sum_{k \in \mathbb{N}^d} \left( \prod_{i=1}^d (k_i + 1) \right) |\Delta^2 \lambda_k| = 3^d$$

which proves the result. \[\square\]

Now we extend the well known Bernstein’s inequality from one- to multi-parameter trigonometric polynomials.

**Lemma 3.** Let $f$ be a trigonometric polynomial in the $i$-th variable of order $N_i$. If $I \subset \{1, \ldots, d\}$, then for every $1 \leq p < \infty$

$$\left\| \left( \prod_{i \in I} \partial_i \right) f \right\|_p \leq C \left( \prod_{i \in I} N_i \right) \|f\|_p.$$

**Proof.** Let us define

$$\phi_{N_i, i \in I}(y) := \prod_{i \in I} (K_{N_i-1}(y_i)(e^{iN_iy_i} + e^{-iN_iy_i})) \quad (y = (y_i, \ i \in I)).$$

Then by (4), $\|\phi_{N_i, i \in I}\|_1 = C$ and

$$\phi_{N_i, i \in I}(y) = \sum_{i \in I} \sum_{|k_i|=0}^{N_i-1} \prod_{i \in I} \left( 1 - \frac{|k_i|}{N_i} \right) \left( e^{i(k_i+N_i)y_i} + e^{i(k_i-N_i)y_i} \right).$$

It is easy to see that $\hat{\phi}_{N_i, i \in I}(k) = \prod_{i \in I} \frac{k_i}{N_i}$ for $-N_i \leq k_i \leq N_i$. Then

$$\phi_{N_i, i \in I} * f = \frac{(\prod_{i \in I} \partial_i f)}{\prod_{i \in I} N_i}$$

proves the lemma. \[\square\]
Our main result is the following

**Theorem 1.** If $f \in H_p$ and $1/2 < p \leq 1$ then

$$\sup_{N_i \geq 2} \left( \frac{1}{\prod_{i=1}^{d} \log N_i} \right)^{[p]} \sum_{i=1}^{d} \sum_{k_i=1}^{N_i} \left( \prod_{i=1}^{d} k_i^{2-p} \right) \leq C_p \| f \|_{H_p}^p$$

where $[p]$ denotes the integer part of $p$.

**Proof.** To avoid some technical difficulties, we prove the theorem for two parameters, only. By (3), it is enough to show that

$$\sup_{N,M \geq 2} \left( \frac{1}{\log N \log M} \right)^{[p]} \sum_{k=1}^{N} \sum_{l=1}^{M} \frac{\| s_{k,l}(Pf) \|_p}{(kl)^{2-p}} \leq C_p \| Pf \|_p^p$$

whenever $f \in H_p$ and $1/2 < p \leq 1$.

It is easy to see that

$$\sum_{k=1}^{N} \sum_{l=1}^{M} \frac{\| s_{k,l}(Pf) \|_p}{(kl)^{2-p}} \leq \sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{\| s_{k,l}(V_{2N,2M}(Pf)) \|_p}{(kl)^{2-p}}.$$

For fixed $x$ and $y$, the $(k,l)$-th Fourier coefficient of

$$\sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(Pf))(x,y) e^{ikt} e^{iul}$$

is $s_{k,l}(V_{2N,2M}(Pf))(x,y)$. Then we can apply Theorem A and (3) to obtain

$$\sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{| s_{k,l}(V_{2N,2M}(Pf))(x,y) |^p}{(kl)^{2-p}} \leq C_p \int_{T} \int_{T} \left| \sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(Pf))(x,y) e^{ikt} e^{iul} \right|^p dt du.$$

Using the notation

$$a_{n,m} := v \left( \frac{n}{N} \right) v \left( \frac{m}{M} \right) \hat{f}(n,m) e^{inx} e^{imy}$$
we have
\[
\sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}(V_{2N,2M}(P f))(x,y)e^{ikt}e^{ilu} = \sum_{k=1}^{2N} \sum_{l=1}^{2M} \sum_{n=1}^{k} \sum_{m=1}^{l} a_{n,m}e^{ikt}e^{ilu}
\]
\[
= \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \left( \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \frac{1 - e^{imu}}{e^{iu} - 1} \right)
\]
\[
+ \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \left( \frac{1 - e^{int}}{e^{it} - 1} \frac{e^{i(2M+1)u} - 1}{e^{iu} - 1} \right)
\]
\[
+ \sum_{n=1}^{2N} \sum_{m=1}^{2M} a_{n,m} \left( \frac{1 - e^{int}}{e^{it} - 1} \frac{1 - e^{imu}}{e^{iu} - 1} \right)
\]
\[
= (A) + (B) + (C) + (D).
\]
Recall that for the Dirichlet kernel
\[
D_N(t) := \frac{1}{2} e^{-itN} \frac{e^{i(2N+1)t} - 1}{e^{it} - 1}
\]
we have
\[
\|D_N\|_1 \sim \log N \quad \text{and} \quad |D_N(t)| \leq \frac{C}{t} \quad (N \in \mathbb{N})
\]
(see e.g. TORCHINSKY [11]). Applying this, Lemma 2 and (3) we conclude that
\[
\int_{\mathbb{T}^4} |(A)|^p \, dt \, du \, dx \, dy = \int_{\mathbb{T}^2} \left| \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \frac{e^{i(2M+1)u} - 1}{e^{iu} - 1} \right|^p \, dt \, du
\]
\[
\times \|V_{2N,2M}(P f)\|_p^p \leq \begin{cases} 
C \log N \log M \|P f\|_1 & \text{if } p = 1, \\
C_p \|P f\|_p^p & \text{if } 1/2 < p < 1.
\end{cases}
\]
For the second term we obtain

\[ \int_{T^4} |(B)|^p \, dt \, du \, dx \, dy = \int_{T} \left| \frac{e^{(2N+1)t} - 1}{e^t - 1} \right|^p \, dt \]

\[ \times \int_{T^2} \left| \frac{1}{e^{tu} - 1} \right|^p \int_{T^2} V_{2,2M}(Pf)(x, y) - V_{2,2M}(Pf)(x, y + u) \right|^p \, dx \, dy \, du, \]

which can be estimated by \( C_p \| Pf \|_p^p \) if \( 1/2 < p < 1 \) and, moreover, if \( p = 1 \) then by

\[ C \log N \int_{|u| < 1/M} \frac{1}{|u|} \int_{T^2} \left| \int_0^u \partial_2 V_{2,2M}(Pf)(x, y + w) \, dw \right| \, dx \, dy \, du \]

\[ + C \log N \int_{|u| \geq 1/M} \frac{1}{|u|} \| V_{2,2M}(Pf) \|_1 \, du =: (B_1) + (B_2). \]

It is easy to see that \( (B_2) \leq C \log N \log M \| Pf \|_1 \). By Lemma 3,

\[ (B_1) \leq C \log N \| V_{2,2M}(Pf) \|_1 \leq C \log N \| Pf \|_1. \]

The estimation of \( (C) \) is similar. Let us consider \( (D) \).

\[ \int_{T^4} |(D)|^p \, dt \, du \, dx \, dy = \int_{T^2} \left| \frac{1}{e^{tu} - 1} \right|^p \int_{T^2} V_{2,2M}(Pf)(x, y) \]

\[ - V_{2,2M}(Pf)(x, y + u) - V_{2,2M}(Pf)(x + t, y) \]

\[ + V_{2,2M}(Pf)(x + t, y + u) \right|^p \, dx \, dy \, dt \, du. \]

This can be estimated by \( C_p \| Pf \|_p^p \) if \( 1/2 < p < 1 \). In case \( p = 1 \) we split the integral with respect to \( t \) and \( u \) into the integrals over the sets \( \{|t| < 1/N, |u| < 1/M\} \), \( \{|t| < 1/N, |u| \geq 1/M\} \), \( \{|t| \geq 1/N, |u| < 1/M\} \) and \( \{|t| \geq 1/N, |u| \geq 1/M\} \) and we denote these integrals by \( (D_1) \), \( (D_2) \), \( (D_3) \) and \( (D_4) \), respectively. Applying Bernstein’s inequality we obtain

\[ (D_1) \leq C \int_{|t| < 1/N} \int_{|u| < 1/M} \frac{1}{|tu|} \]

\[ \times \int_{T^2} \int_0^u \partial_1 \partial_2 V_{2,2M}(Pf)(x + v, y + w) \, dv \, dw \, dx \, dy \, dt \, du \leq C \| Pf \|_1. \]
Similarly,

\[(D_2) \leq C \int_{|t| < 1/N} \int_{|u| \geq 1/M} \left| \frac{1}{|tu|} \int_{T^2} \partial_t V_{2N,2M}(Pf)(x + v, y) - \partial_t V_{2N,2M}(Pf)(x + v, y + u) \right| \, dv \, dy \, dt \, du \leq C \log M \|Pf\|_1.\]

\[(D_3)\) can be estimated in the same way. For \((D_4)\) we have simply

\[(D_4) \leq C \log N \log M \|Pf\|_1,

which finishes the proof of Theorem 1. \(\square\)

The set of the trigonometric polynomials is dense in \(H_p\), so by the usual density argument we can easily verify the next consequence (cf. Weisz [12]).

**Corollary 1.** If \(f \in H_p\) and \(1/2 < p \leq 1\) then

\[
\lim_{N \to \infty} \left( \frac{1}{\prod_{i=1}^d \log N_i} \right)^{|p|} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \|s_{k_i} f - f\|_{H_p}^p \prod_{i=1}^d k_i^{2-p} = 0.
\]

Since \(\| \cdot \|_p \leq \| \cdot \|_{H_p}\), we get

\[
\lim_{N \to \infty} \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \|s_{k_i} f - f\|_1 \prod_{i=1}^d k_i = 0
\]

whenever \(f \in H_1\), which was proved by Smith [10] in the one-parameter case.

We now give the dual inequality to Theorem 1, which is a Marcinkiewicz–Zygmund type inequality for the \(BMO\) space, where \(BMO\) is the dual of \(H_1\). Since the proof is similar to that of Theorem 3 in Weisz [12], we omit it.

**Theorem 2.** If \(g^k (k \in \mathbb{N}^d)\) are uniformly bounded in \(BMO\) then

\[
\sup_{N_i \geq 2} \left\| \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} s_{k_i} g^k \right\|_{BMO} \leq C \sup_{k \in \mathbb{N}^d} \|g^k\|_{BMO}.
\]

Note that the corresponding results for multi-parameter Walsh–Fourier series are still unknown.
References