Framed \((2M + 3)\)-dimensional manifolds endowed with a vertical cyclic connection structure

By FILIP DEFEVER (Leuven) RADU ROSCA (Paris)

Abstract. Geometrical and structural properties are proved for a class of framed manifolds which are equipped with a vertical cyclic connection structure.

1. Introduction

Framed manifolds and \(f\)-structures have been initiated by K. YANO and M. KON and have subsequently been studied intensively, see for example [1], [19], [22], [18]. We recall that if \(M(\phi, \Omega, \xi_r, \eta^r, g)\) is a \((2m + q)\)-dimensional manifold of this kind, then the \(\xi_r\), for \((r = 2m+1, \ldots , 2m+q)\), are the Reeb vector fields (in the large sense) of the \(f\)-structure, and \(\eta^r = \xi_r^\flat\) their corresponding covectors. One has the following structure equations:

\[
\phi^2 = -\text{Id} + \sum \eta^r \otimes \xi_r, \quad \phi \xi_r = 0, \quad \eta^r \circ \phi = 0, \quad \eta^s(\xi_r) = \delta^s_r,
\]

where \(\phi\) is a \((1,1)\) tensor field. With respect to \(g\), one has the following relation

\[
g(\phi Z, Z') + g(Z, \phi Z') = 0, \quad Z, Z' \in \mathfrak{X}(M),
\]

(i.e. \(\phi\) is skew-symmetric with respect to \(g\)). The 2-form \(\Omega\) of rank \(2m\) has

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The first author is a Postdoctoral Researcher of the Research Council of the K.U. Leuven.
the following properties

\[(2) \quad \Omega(Z, Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \cdots \wedge \eta^{2m+q} \neq 0,\]

and is called the fundamental form of the framed manifold.

In the present paper we assume that \(r \in \{2m + 1, 2m + 2, 2m + 3\}\) and for the indices \(a\) and \(b\) we have the following range \(a, b \in \{1, \ldots, 2m\}\).

Under these conditions and with reference to [18], we call \(\theta^a, \theta^r,\) and \(\theta^s\) the horizontal, the transversal, and the vertical connection forms respectively. We will assume here that the \(\theta^a\) vanish and that the \(\theta^r\) are defined by a cyclic permutation of the Reeb covectors \(\eta^r,\) which means that

\[(3) \quad \theta^r_s = f_s \eta^r - f_r \eta^s, \quad \forall r, s, t\]

where the \(f_r\) are scalar fields, called the principal scalars on \(M\). In the sequel we will call

\[(4) \quad \eta^r = f_r \eta^r, \quad \text{and} \quad \eta^s = \sum f_r \xi_r,\]

the principal pfaffian and the principal vector field of \(M\) respectively. Further, let \(D^\top_p = \{e_a\}\) and \(D^\perp_p = \{\xi_r\}\) be the horizontal, respectively vertical, distribution on \(M\).

In a first step, the following properties are proved.

(i) The manifold \(M\) under consideration may be viewed as the local Riemannian product \(M = M^\top \times M^\perp,\) where \(M^\top\) is a \(2m\)-dimensional submanifold tangent to \(D^\top_p\) and \(M^\perp\) is a \(3\)-dimensional submanifold tangent to \(D^\perp_p,\) and the immersion \(x : M^\top \to M\) is totally geodesic;

(ii) the Ricci tensor field \(\mathcal{R}\) of \(M^\perp\) is expressed by

\[\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M);\]

(iii) \(\xi\) is harmonic and if \(V\) is any vertical vector which has the property to be a skew-symmetric Killing vector field having \(\xi\) as generative, then \(V\) is an exterior concurrent vector field and by Bochner’s theorem \(g(V, \xi)\) is closed;

(iv) the principal scalars \(f_r\) define an isoparametric system [20];

(v) the gradients \(df_r^\sharp = \text{grad} f_r,\) define a commutative group.
In a second step, and making use of E. Cartan’s structure equations involving the curvature 2-forms, one finds that the vertical curvature 2-forms $\Theta^s_r$ satisfy

$$\Theta^s_r = \left( \left( \|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^r + f_r f_t \eta^t \right) \wedge \eta^s - \left( \left( \|\xi\|^2 - \frac{f_t^2}{2} \right) \eta^s + f_s f_t \eta^t \right) \wedge \eta^r,$$

\forall \; r, s, t,

and consequently, following [19] the above equations prove that $M^\perp$ is a conformally flat submanifold of $M$.

Finally, the structure 2-form $\Omega$ of $M$ is presymplectic. Then, if $X$ is any horizontal vector field and $^bX = -i_X \Omega$ means the symplectic isomorphism, and in addition the 1-form $^bX$ is $\phi$-closed, it follows that $\Omega$ is invariant by $X$. In consequence of this, $X$ is a 2-covariant recurrent vector field, which in the case under consideration is expressed by

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X, \quad \lambda \in \Lambda^0 M.$$

### 2. Preliminaries

Let $(M, g)$ be a Riemannian $C^\infty$-manifold and let $\nabla$ be the covariant differential operator with respect to the metric tensor $g$. We assume that $M$ is oriented and $\nabla$ is the Levi–Civita connection of $g$. Let $\Gamma TM$ be the set of sections of the tangent bundle, and

$$^b: TM \overset{\bullet}{\longrightarrow} T^* M \quad \text{and} \quad \sharp: TM \overset{\bullet}{\longleftarrow} T^* M$$

the isomorphisms defined by $g$ (i.e. $^b$ is the index lowering operator, and $\sharp$ is the index raising operator).

Following [14], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM),$$

the set of vector valued $q$-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to $\nabla$

$$d\nabla : A^q(M, TM) \to A^{q+1}(M, TM).$$
It should be noticed that in general $d\nabla^2 = d\nabla \circ d\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of $M$, and is also called the soldering form of $M$ [4]. Since $\nabla$ is symmetric one has that $d\nabla(dp) = 0$.

A vector field $Z$ which satisfies

$$d\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M,$$

is defined to be an exterior concurrent vector field [16] (see also [13]). The 1-form $\pi$ in (6) is called the concurrence form and is defined by

$$\pi = \lambda Z^2, \quad \lambda \in \Lambda^0 M.$$

In this case, if $\mathcal{R}$ is the Ricci tensor of $\nabla$, one has

$$\mathcal{R}(Z, V) = \varepsilon(n - 1)\lambda g(Z, V)$$

($\varepsilon = \pm 1$, $V \in \Xi(M)$, $n = \dim M$).

A function $\mathbb{R}^n \to \mathbb{R}$ is isoparametric [20] if $\|\nabla f\|^2$ and $\operatorname{div}(\nabla f)$ are functions of $f$ ($\nabla f = \operatorname{grad} f$).

Let $\mathcal{O} = \{e_A \mid A = 1, \ldots, n\}$ be a local field of orthonormal frames over $M$ and let $\mathcal{O}^* = \operatorname{covect} \{\omega^A\}$ be its associated coframe. Then E. Cartan’s structure equations can be written in indexless manner as

$$\nabla e = \theta \otimes e,$$

$$d\omega = -\theta \wedge \omega,$$

$$d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations $\theta$ (resp. $\Theta$) are the local connection forms in the tangent bundle $TM$ (resp. the curvature 2-forms on $M$).

3. The main theorem

Let $M(\phi, \Omega, \xi^r, \eta^r, g)$ be a $(2m + 3)$-dimensional $C^\infty$-manifold with soldering form $dp$ and carrying an $f$-structure $\phi$ [22], that is a tensor field
of type (1.1) of rank $2m$ which satisfies

$$
\phi^3 + \phi = 0,
$$

$$
\phi^2 = -\text{Id} + \sum \eta^r \otimes \xi^r, \quad \phi \xi^r = 0, \quad \eta^r \circ \phi = 0,
$$

$$
g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z) \eta^r(Z'),
$$

where $\text{Id}$ is the identity morphism of $M$.

If in addition the fundamental 2-form $\Omega$ of $M$ satisfies

$$
\Omega(Z, Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \neq 0,
$$

then $M$ is known [22] to be a framed $f$-manifold.

With respect to the cobasis $O^\ast = \text{covect} \{\omega^a, \eta^r\}$ of $O = \text{vect} \{e_a, \xi^r\}$ $(1 \leq a \leq 2m; \ 2m + 1 \leq r \leq 2m + 3)$, the 2-form $\Omega$ is expressed by the standard form

$$
\Omega = \sum_{i=1}^m \omega^i \wedge \omega^i, \quad i^* = i + m.
$$

Making use of (9) and (13), one finds the known Kaehlerian relations

$$
\theta^i_j = \theta^i_j, \quad \theta^i_j = \theta^i_j.
$$

We recall [18] that one may split the tangent space $T_p(M)$ of $M$ at every point $p \in M$ as

$$
T_p(M) = D_p^\top \oplus D_p^\perp,
$$

where $D_p^\top = \{e_a \mid a \in \{1, \ldots, 2m\}\}$ and $D_p^\perp = \{\xi^r\}$ are two complementary orthogonal distributions, called the horizontal and the vertical distribution respectively. As a consequence of this decomposition, one may write the soldering form as

$$
dp = dp^\top \oplus dp^\perp,
$$

where $dp^\top = dp \mid_{D^\top}$ and $dp^\perp = dp \mid_{D^\perp}$. By reference to [18] (see also [12]), the connection forms $\theta^a_b$, $\theta^r_s$, and $\theta^c_a$ are called the horizontal, the vertical, and the transversal connection forms respectively. In the present
paper we assume that the $\theta^a$ vanish and that the vertical connection forms are defined by a cyclic permutation of the Reeb covectors $\eta^r$, that is:

\begin{equation}
\theta^r_s = f_s \eta^r - f_r \eta^s, \quad \forall \hat{r}, s, t \quad \text{(cyclic)}.
\end{equation}

In the above relations, the $f_r$ are scalar fields, called the principal scalars on $M$, and setting

\begin{equation}
\eta = f_r \eta^r, \quad \eta^\sharp = \xi = \sum f_r \xi_r,
\end{equation}

$\eta$ and $\xi$ are called the principal pfaffian and the principal vector field respectively. Taking into account that

\begin{equation}
\theta^a_r = 0,
\end{equation}

one derives by (10) and (20) that

\begin{equation}
d\eta^r = \eta \wedge \eta^r.
\end{equation}

This shows that the Reeb covectors are $\eta^r$ exterior recurrent forms [3]. In addition, exterior differentiation of (23) and taking into account (21), yields

\begin{equation}
df_r = f_r \eta,
\end{equation}

which expresses that $\eta$ is an exact form. Since one has that

\begin{equation}
(d\eta^r) \neq 0, \quad \eta^r \wedge d\eta^r = 0,
\end{equation}

it follows according to a known definition [6] that in the case under discussion the Reeb covectors are of class 2. Let now

\begin{equation}
\varphi^\perp = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3}
\end{equation}

and

\begin{equation}
\varphi^\top = \omega^1 \wedge \cdots \wedge \omega^{2m}
\end{equation}

be the simple unit forms which correspond to the distributions $D_p^\perp$ and $D_p^\top$ respectively. Taking the exterior derivative of (25) and (26), and in view of (20) and (22), one derives that

\begin{equation}
d\varphi^\perp = 0
\end{equation}

and

\begin{equation}
d\varphi^\top = 0.
\end{equation}
Hence, in terms of well known terminology [9], the above equations show that \( \varphi^\perp \) and \( \varphi^\top \) are integral invariants of \( D_p^\perp \) and \( D_p^\top \) respectively. Therefore, by the theorem of Frobenius, we conclude that the manifold \( M \) under consideration may be viewed as the local Riemannian product

\[
(29) \quad M = M^\top \times M^\perp,
\]
where \( M^\top \) is a \( 2m \)-dimensional manifold tangent to \( D^\top \) and \( M^\perp \) is a 3-dimensional manifold tangent to \( D^\perp (= \{ \xi_r \}) \).

**Remark 3.1.** As the tangent space \( T_p(M) \), the soldering form \( dp \) may be split as

\[
dp = dp^\top + dp^\perp,
\]
where \( dp^\top \) and \( dp^\perp \) are the horizontal and the vertical components of \( dp \) respectively. In the case under discussion, operating on \( dp^\top \) and \( dp^\perp \) by the exterior covariant derivative operator \( d^\nabla \), one finds

\[
(30) \quad d^\nabla (dp^\perp) = 0, \quad d^\nabla (dp^\top) = 0,
\]
which, since \( \nabla \) is the Levi–Civita connection, leads to

\[
d^\nabla (dp) = 0.
\]

Using (20), (21), and (22), one gets

\[
(31) \quad \nabla \xi_r = f_r dp^\perp - \eta^r \otimes \xi,
\]
and one derives

\[
(32) \quad [\xi_r, \xi_s] = f_s \xi_r - f_r \xi_s.
\]
In view of (24), the covariant differential of \( [\xi_r, \xi_s] \) can be expressed as

\[
(33) \quad \nabla [\xi_r, \xi_s] = \eta \otimes [\xi_r, \xi_s] - [\xi_r, \xi_s]^\flat \otimes \xi,
\]
with which one can check Jacobi’s identity

\[
\sum_{r,s,t} [\xi_r, [\xi_s, \xi_t]] = 0, \quad \forall \xi_r, \xi_s, \xi_t.
\]
Next, operating on (21) with $\nabla$, and using (20) and (21), one derives that

$$\nabla\xi = \|\xi\|^2 dp^\perp;$$

consequently, following a well known definition [2] one may consider $\xi$ as a concurrent vector field on $M^\perp$. This implies [15] (see also [13]) that $\xi$ is an exterior concurrent vector field on $M^\perp$. Since $\|\xi\|^2 = \sum f^2_r$, one gets at once by (24) that

$$d\|\xi\|^2 = 2\|\xi\|^2 \eta.$$

Therefore, since $d\nabla (dp^\perp) = 0$, operating on (34) by $d\nabla$ yields

$$d\nabla (\nabla\xi) = \nabla^2\xi = 2\|\xi\|^2 \eta \wedge dp^\perp.$$

Hence, by reference to [13], the Ricci tensor field $\mathcal{R}$ of $M^\perp$ is expressed by

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M).$$

Next, by (24) one may write

$$\left((df_r)^2 = f_r \xi_r, \quad (df_r)^2 = \text{grad} f_r, \right.$$

and after further elaboration, one derives that

$$\left[ (df_r)^2, (df_s)^2 \right] = 0, \quad \forall \, r, s, t.$$

Accordingly we may say that the vector fields $(df_r)^2$, $(df_s)^2$, and $(df_t)^2$ define a commutative group.

Next, by (24) one has that

$$\|\text{grad} f_r\|^2 = \|\xi\|^2 f^2_r,$$

and since

$$\text{div} Z = \text{tr} \nabla Z, \quad Z \in \Xi(M),$$

one derives that

$$\text{div} \text{grad} f_r = f^3_r + \|\xi\|^2 f^2_r, \quad \|\xi\|^2 = \sum f^2_r.$$

Hence, noticing that $[\text{grad} f_r, \text{grad} f_t] = 0$ and on behalf of [20], we conclude from the above relations that the scalars $f_r$ define an isoparametric system.
In another perspective, we recall that the star operator $*$ on an oriented $n$-dimensional Riemannian manifold $(M, g)$ is an isometric bundle isomorphism between $\Lambda T^* M$ and itself, and maps $\Lambda^q T^* M$ isomorphically to $\Lambda^{n-q} T^* M$ (see also [14]).

Coming back to the case under consideration, one has

\begin{equation}
\Lambda^q T^* M \rightarrow \Lambda^{2m+3-q} TM.
\end{equation}

With the usual notation, we denote the codifferential of a $p$-form by $\delta = (-1)^p \ast^{-1} d\ast$, where $\ast^{-1} = (-1)^{n(n-p)}$ ($p$ is the degree of the form, $n$ is the dimension of the manifold, thus $\delta \omega$ is of degree $p - 1$; see also [14]). Then, in the case under consideration, one deduces that

\begin{equation}
d\delta \eta = 0.
\end{equation}

Since $\eta$ is a closed pfaffian, there follows at once that

\begin{equation}
\Delta \eta = 0.
\end{equation}

This shows that $\eta$ is a harmonic pfaffian (and consequently $\eta^\#$ is a harmonic vector field). Finally, consider the immersion $x : M^\top \rightarrow M$. As it is well known, the second quadratic forms $l_r$ associated with $x$ are defined by

\begin{equation}
l_r = -\langle dp^\top, \nabla \xi_r \rangle.
\end{equation}

Then, by reference to (31), it can be seen that the $l_r$ vanish, and consequently the immersion $x : M^\top \rightarrow M$ is totally geodesic.

Summarizing, we can formulate the following

**Theorem 3.1.** Let $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ be a $(2m + 3)$-dimensional manifold endowed with a vertical cyclic connection structure and with vanishing transversal connection forms. Let $\eta, \xi (= \eta^\#)$, and $f$, be the principal pfaffian, the principal vector field, and the principal scalars on $M$; and let $D_p^\top$ and $D_p^\perp = \{\xi_r\}$ be the horizontal and the vertical distributions respectively on $M$.

Then any such manifold may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is a $2m$-dimensional presymplectic submanifold tangent to $D_p^\top$ and $M^\perp$ is a 3-dimensional submanifold tangent to $D_p^\perp$.

The following properties are proved.
(i) The immersion $x : M^\perp \to M$ is totally geodesic;
(ii) the principal vector field $\xi$ is an exterior concurrent vector field on $M^\perp$, i.e.
$$\nabla^2 \xi = 2\|\xi\|^2 \eta \wedge dp^\perp,$$
and this implies
$$R(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \quad Z \in \Xi(M),$$
where $R$ denotes the Ricci tensor field of $M^\perp$;
(iii) the principal pfaffian $\eta$ is harmonic;
(iv) the vector fields $df_r$ define a commutative group, and the scalars $f_r$ define an isoparametric system.

4. Corollaries

Making use of E. Cartan’s structure equations, involving the curvature 2-forms (11), one derives by (20), (23), and (24) that the vertical curvature forms $\Theta^*_r$ satisfy

$$\Theta^*_r = \left(\|\xi\|^2 - \frac{f_t^2}{2}\right) \eta^r + f_r f_t \eta^l \wedge \eta^s - \left(\|\xi\|^2 - \frac{f_t^2}{2}\right) \eta^s + f_s f_t \eta^l \wedge \eta^r, \quad \forall r, s, l.$$

Then, by reference to [19], the above expressions for $\Theta^*_r$ affirm that the vertical submanifold $M^\perp$ of $M$ is a conformally flat submanifold of $M$.

In another perspective, let

$$V = V^\tau \xi_r, \quad r \in \{2m + 1, 2m + 2, 2m + 3\},$$

be any vertical vector field on $M^\perp$, and assume that $V$ is a skew-symmetric Killing vector field, having $\xi$ as generative [16] (see also [12]), thus

$$\nabla V = V \wedge \xi,$$

where $\wedge$ denotes the wedge product of vector fields

$$V \wedge \xi = \eta \otimes V - V^\flat \otimes \xi.$$
Since by (31) one gets

\[ \nabla V = dV^r \otimes \xi_r + g(V, \xi) dp^\perp - V^b \otimes \xi, \]

then comparison of (46) and (47) gives

\[ dV^b = \eta \wedge V^b, \]

which by (48) is in agreement by Rosca’s lemma [16], [17] (see also [12]). Moreover, since \( V \) is a Killing vector field and the vector field \( \xi (= \eta^b) \), is harmonic, one finds by (21) that

\[ dg(V, \xi) = 0, \]

and (49) is in agreement with Bochner’s theorem [21], and thus yields a confirmation for the correctness of our computations. In addition, by (34) and (46), one calculates that

\[ [V, \xi] = g(V, \xi) \xi, \]

and the above equation means that \( V \) defines an infinitesimal conformal transformation of \( \xi \). Operating now on (46) by the operator \( d\nabla \) and in view of (34), one gets

\[ d\nabla (\nabla V) = \nabla^2 V = ||\xi||^2 V^b \wedge dp^\perp, \]

which shows that \( V \) is an exterior concurrent vector field on \( M^\perp \) with \( ||\xi||^2 \) as concurrent scalar, and by (6) one may write

\[ \mathcal{R}(V, Z) = -2||\xi||^2 g(V, Z). \]

On the other hand, by (17) and (22), one finds that

\[ d\Omega = 0. \]

Since \( \Omega \) has constant rank, this means that \( \Omega \) is a presymplectic form on \( M \). We notice that in this case \( \text{Ker}(\Omega) \) coincides with the vertical distribution \( D_p^\perp = \{ \xi_r \} \) of \( M \), which is also called the characteristic distribution of \( \Omega \). Denote now with the usual notation

\[ \Omega^b : \quad TM \rightarrow T^*M : \quad Z \rightarrow -i_Z \Omega = ^b Z, \]
the symplectic isomorphism defined by $\Omega$ [8]. Since $\Omega$ is closed, any vector field $X$ with the property that $\flat X$ is closed, defines an infinitesimal automorphism of $\Omega$, i.e.

$$\mathcal{L}_X \Omega = 0.$$  

Assume that $X$ is a horizontal vector field on $M$, i.e.

$$X = X^a e_a, \quad a \in \{1, \ldots, 2m\}.$$  

Then, by (52) one has

$$\flat X = \sum (X^{i^*} \omega^i - X^i \omega^{i^*}), \quad i \in \{1, \ldots, m\}, \quad i^* = i + m,$$

and by the structure equations (10) one gets by exterior differentiation of $\flat X$

$$d \flat X = -(dX^{i^*} + X^a \theta^i_a) \wedge \omega^i - (dX^i + X^a \theta^i_a) \wedge \omega^{i^*}.$$  

Hence, in order for $\flat X$ to be a $\phi$-closed form [16], one must write

$$\begin{cases} dX^i + X^a \theta^i_a = -\lambda \omega^{i^*}, \\ dX^{i^*} + X^a \theta^{i^*}_a = \lambda \omega^i, \end{cases}$$

where $\lambda$ is a scalar. Taking now the covariant differential of the vector field $X$, one deduces by (56) and the structure equations (9) that

$$\nabla X = \lambda \phi dp.$$  

This shows that $X$ is a $\phi$-concurrent vector field. Further, operating on the vector valued 1-form $\phi dp$ by the operator $d\nabla$, one calculates that

$$d\nabla (\phi dp) = 0,$$

and therefore it follows from (57) that

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X.$$  

Hence, the above equation proves that the vector field $X$ is, according to well known terminology [10], a 2-covariant recurrent vector field with closed recurrence form.

Summarizing, we proved the following
Theorem 4.1. The vertical submanifold $M^\perp$ of the manifold $M$ under consideration is conformally flat, and the vertical skew-symmetric Killing vector field $V$ is an exterior concurrent vector field which moreover also defines an infinitesimal conformal transformation of the principal vector field $\xi$. The structure 2-form $\Omega$ of $M$ is presymplectic, and if $X$ is any horizontal vector field for which in addition $\nabla^b X(= -i_X \Omega)$ is $\phi$-closed, then $\Omega$ is invariant by $X$, i.e. $\mathcal{L}_X \Omega = 0$; moreover, $X$ also has the following 2 properties:

a) $X$ is a $\phi$-concurrent vector field, i.e.

$$\nabla X = \lambda \phi dp;$$

b) $X$ is a 2-covariant recurrent vector field with closed recurrence form, i.e.

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X.$$

References


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