Filling space with cubes of two sizes

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Abstract. The problem to classify the unilateral and equitransitive tilings of the plane by squares of different sizes has been revived in the last few years ([6], [1], [8]). The analogous problem in three-dimensional space seems to be more difficult and has not been investigated so far systematically ([2], [3]). In this paper we prove that there is only one unilateral tiling of $\mathbb{R}^3$ by cubes of two sizes and that is necessarily equitransitive. Finally we describe the maximal crystallographic group the tiling is equipped with.

1. Introduction

The investigation of unilateral and equitransitive tilings of $\mathbb{R}^2$ by squares of three different sizes has been revived in the last few years. After the constructions of D. SCHATTENBERGER [4, p. 76], MARTINI, MAIKI and SOLTAN [6] and B. GRÜNBAUM [6], the classification problem was finally solved in [1] and in [8], by different methods.

A similar question in three-dimensional space has not been investigated yet. The papers [2] and [3] contain some results on whether the space can be filled with cubes so that no neighbouring cubes have the same edge-length and they shortly cite the construction of Rogers filling the space using cubes of two sizes only.

Our purpose is to describe all the possible unilateral, and then necessarily equitransitive, tilings of the three-dimensional space with cubes of just two different sizes using combinatorial and crystallographic methods.

For the purpose of classification, two tilings $(T, \Gamma)$ and $(T', \Gamma')$, with corresponding symmetry groups $\Gamma$ and $\Gamma'$, respectively, will be considered

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5 equivalent if they are topologically equivariant (homeometric). It means that there exists a homeomorphism $\psi$ that maps $T$ onto $T'$ (preserving tiles, faces, edges and vertices) such that $\psi \Gamma \psi^{-1} = \Gamma'$.

The main statement of the paper will be

**Theorem 1.** Using only cubes of two sizes and having no two cubes with a face in common, there is only one way to tile space. It is known as Rogers filling.

*Remark 1.* In the literature this type of tiling is known as unilateral ([4], [6], [1]).

*Remark 2.* We say that a tiling is equitransitive if for any two congruent tiles $S$ and $S'$ there is a congruence transform of the space which maps $S$ onto $S'$ in such a manner, that the whole tiling is mapped onto itself. The filling of Theorem 1 has this property.

Based on this result and Theorem 2 (see below) we formulate a

**Conjecture.** In every dimension $d$ there exists precisely one equitransitive unilateral tiling by $d$-dimensional cubes of two sizes.

Throughout the paper we use the following notation for squares and cubes. The sizes of the objects we distinguish denoting the smaller ones by $\lambda_1$ and the bigger ones by $\lambda_2$. Individual objects will be denoted by capital letters: $A, B, C \ldots$ Furthermore, we use the method of Dawson for identifying the faces, edges and vertices of a cube. Namely, we characterize the faces by indices

- $u$-meaning the upper
- $d$-meaning the downmost
- $l$-meaning the left
- $r$-meaning the right
- $b$-meaning the back
- $f$-meaning the front

face of the cube as e.g. $A_u, A_d \ldots$ The edges can be represented by double, the vertices by triple indices in a unique way.
2. Non-edge-to-edge planar arrangements by squares

Before the spatial case we deal with arrangements in the plane by squares of two sizes. The following assertion just repeats the well-known fact (see [4] and Figure 1) that

**Theorem 2.** In the plane there exists only one unilateral and equi-transitive tiling by squares of two different sizes.

*Figure 1*

From Figure 1 we can easily read off the unique neighbourhood of a $\lambda_1$- and of a $\lambda_2$-type square.

We can observe that if we do not require equitransitivity the $\lambda_1$-environment has to be the one above. Namely, if $X$ has a $\lambda_1$-neighbour, then the other neighbours are uniquely determined as shown in Figure 2. But the last neighbour has again to be a $\lambda_1$-square, which contradicts the unilaterality condition.

In this way, if we require non-edge-to-edge configuration just at three sides, there are three possible arrangements, shown in Figures 2 and 3.
Moreover, if unilaterality holds just at two sides, the arrangements are those in Figure 4.

3. Environments of $\lambda_1$-cubes in a unilateral tiling

In this section we focus our attention on three-dimensional unilateral tilings not necessarily equitransitive. First we prove an important lemma in three parts.

**Lemma 1.** For the mutual position of neighboring $\lambda_1$- and $\lambda_2$-cubes there is only one possibility in the unilateral tiling of cubes of two sizes shown in Figure 5.

**Proof.** We conduct the proof by excluding the other possibilities as follows.

1. There does not exist such a unilateral construction, where a smaller cube, $X$, would stand in the interior of a face of a bigger one (on $A_u$).
   Namely, suppose on the contrary that such an arrangement exists. In this case the common neighbours of $X$ and $A$ would form a non-edge-to-edge tiling of squares of two sizes locally on $A_u$. Then the environment of the $\lambda_1$-square consists of $\lambda_2$-squares only. Thus the faces of $X$ are adjacent to $\lambda_2$-cubes except for $X_u$. But the only possible neighbour at this face would be a $\lambda_1$-cube, which contradicts the unilaterality.
2. There does not exist such a unilateral tiling, where the mutual position of the cubes is that shown in Figure 6.

If there existed such a tiling, the neighbours of \( X \) would form a non-edge-to-edge environment of \( X_d \) just at three sides on \( A_u \). But again, each of the three allowed neighbourhoods would cause that at \( X_u \) we are able to border only with another \( \lambda_1 \)-cube, a contradiction.

3. There does not exist such a unilateral construction, where the mutual position of the cubes would be that in Figure 7 by projecting them parallelly e.g. with edge \( A_{rf} \).

Let us suppose the contrary. Consider first Figure 8. Here we depicted the arrangement arising when the neighbourhood of \( X_d \) is of the first type from among the three ones in Figure 4. The other
neighbours of $X$ are then strictly determined: now we have to put a $\lambda_2$-cube to $X_u$, next we are forced to put another $\lambda_2$-cube to $X_r$. But in this way we necessarily get a gap between the last cube and $A$, shown in Figure 8, right. Therefore, this construction is not permitted, neither the other two possible neighbourhoods of Figure 4 because of similar argumentation. □

Now we can formulate our observations about the environments of the cubes of different types. First let us consider a $\lambda_1$-cube.

**Lemma 2.** The environment of a cube of $\lambda_1$-type is uniquely determined up to an isometry.

**Proof.** In Figure 5 we have depicted a $\lambda_1$-cube $X$ and its $\lambda_2$-neighbour called $A$. Now we consider the common neighbours of $X$ and $A$. We have a planar arrangement of squares where unilaterality at two sides is needed. The only possibility is just the first case in Figure 4 up to an isometry. This is true because we have to set smaller squares to a corner which is obviously impossible for the other cases. In this way we are forced to build two $\lambda_2$ cubes to $X_l$ and $X_b$ respectively, and the arrangement is straightforward (see Figure 9). There is another possible solution with starting configuration which is just the reflected image of the former one that refers to the tiling on $A_u$, changing the orientation. □

For the purpose of algebraic description we introduce a coordinate system with $O$ as origo, $OI$, $OJ$ and $OK$ as axis and with the lengths $a =$
OI, \( b = OE \) (Figure 5). The coordinates of the centers of the central \( \lambda_1 \)-cube are \((\frac{a}{2}, \frac{a}{2}, \frac{a}{2})\), the centers of the surrounding \( \lambda_2 \)-cubes are \((\frac{b}{2}, \frac{b}{2}, -\frac{b}{2})\), \((a + \frac{b}{2}, a - \frac{b}{2}, \frac{b}{2})\), \((b, a + \frac{b}{2}, \frac{b}{2})\), \((-\frac{b}{2}, b, a - \frac{b}{2})\), \((a - \frac{b}{2}, -\frac{b}{2}, a - \frac{b}{2})\), \((a - \frac{b}{2}, a - \frac{b}{2}, \frac{b}{2})\).

Surprisingly, a similar statement is true for \( \lambda_2 \)-cubes:

**Lemma 3.** The environment of a cube of \( \lambda_2 \)-type is uniquely determined up to an isometry.

**Proof.** In Figure 10 we can see the previously discussed arrangement of a \( \lambda_1 \)-cube but from the opposite direction. (To simplify the figure we omitted the cube \( D \).) Our aim is to form the neighbours of \( A \). We assert that either \( A_l \) or \( A_b \) borders a \( \lambda_1 \)-square. This is necessary because we cannot cover these faces with \( \lambda_2 \)-cubes only by avoiding common faces. The smaller cube has to join either the corner \( A_{lb} \) or \( A_{ub} \).

Figure 10

Indeed, first consider the face \( A_l \). The two corners above are not proper because the gap between them and \( B_d \) allows at most a \( \lambda_1 \)-square in contradiction with the previous lemma. By a similar reasoning, the corner \( A_{ul} \) is not good either.

The only remaining possibility is \( A_{lbu} \). Indeed, we can build the neighbourhood of \( A \) beginning at this corner with a cube of \( \lambda_1 \)-type. After forming its environment, we can easily see that we have to continue on \( A_b \) at \( A_{rba} \). Building up its environment we are forced to set the next smaller cube and the other ones step by step in a similar way. The neighbours of \( A \) are uniquely determined.
If we start at \( A_b \), our only chance is indeed the corner \( A_{rbu} \). Namely, if we put \( \lambda_1 \)-cubes to the other corners, we cannot form their environments just by overlapping. But if we put a cube to \( A_{rbu} \) we can build the environment in a unique way as above. Our solution will be the same as the former one.

Of course another congruent solution comes into consideration as with the previous lemma.

□

The analytic description of the environment is the following: the center of the base \( \lambda_2 \)-cube is located in \((\frac{b}{2}, \frac{b}{2}, -\frac{b}{2})\).

The centers of the \( \lambda_1 \)-cubes are: \((\frac{a}{2}, \frac{a}{2}, \frac{a}{2})\), \((b - \frac{a}{2}, b + \frac{a}{2}, -\frac{a}{2})\), \((b + \frac{a}{2}, \frac{a}{2}, -\frac{a}{2})\), \((-b + \frac{a}{2}, -b + \frac{a}{2})\), \((b - \frac{a}{2}, b - \frac{a}{2}, -b - \frac{a}{2})\).

The centers of the \( \lambda_2 \)-cubes are: \((a + \frac{b}{2}, a - \frac{b}{2}, \frac{b}{2})\), \((-a + b, -a + b, \frac{b}{2})\), \((a - \frac{b}{2}, \frac{b}{2}, a - \frac{b}{2})\), \((-a + \frac{b}{2}, -a + \frac{b}{2}, \frac{b}{2})\), \((-a + b, -a + b, -\frac{b}{2})\), \((a + \frac{b}{2}, a - \frac{b}{2}, -\frac{b}{2})\), \((-a + b, -a + b, -\frac{b}{2})\).

Having proved the lemmas our Theorem 1 simply follows.

Proof of Theorem 1. The statement is just a simple consequence of the uniqueness of the environments above. If we start either with a cube of \( \lambda_1 \) or of \( \lambda_2 \)-type the neighbours of it are determined as the further neighbourhoods. In addition, the isometry, which may occur at the local environments, preserves topological equivariance. This tiling is known as Rogers filling. □

4. The maximal symmetry group

If we consider the configuration of cubes in the filling we see that there are many symmetries of the space which map the tiling onto itself. E.g. the translations in three directions that move the centers of small cubes to the centers of any other small cubes form a space group 1. (\( P1 \) of [5].) In order to determine the corresponding maximal symmetry group we use the analytic description.

Firstly, we take a fundamental domain and give the face identifications due to the Poincaré algorithm (for more information see [7]). In Figure 11
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Figure 11

a fundamental domain with our base configuration is drawn. The generators are the following:

\[ z_1 : LHJ \rightarrow JKL \]
\[ z_2 : EKD \rightarrow DGE \]
\[ s_1 : GJHE \rightarrow DTSK \]
\[ s_2 : HLK \rightarrow GJT \]

We can give the analytic forms of the transformations as usual in crystallography [5] by homogeneous coordinates according to Figure 5 as \( OI = i, OJ = j, OK = k \):

\[
Z_1 = \begin{bmatrix} 0 & 0 & -1 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
S_1 = \begin{bmatrix} 0 & 0 & 1 & \frac{2b}{3} \\ -1 & 0 & 0 & \frac{4b}{3} \\ 0 & -1 & 0 & -\frac{b}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & -1 & 0 & a + b \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]
The first two generators are rotoinversions (3), the last ones are screw motions of order three (3). The axes are parallel. These transformations generate the corresponding crystallographic group which is 148. (R3 in [5].) This is the maximal symmetry group of the tiling.

References


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