An optimal control problem on the Lie group $\text{SE}(2, \mathbb{R})$

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Abstract. An optimal control problem on the Lie group $\text{SE}(2, \mathbb{R})$ is discussed and some of its properties are pointed out.

1. Introduction

Recent work in nonlinear control has drawn attention to drift-free systems with fewer controls than state variables. These arise in problems of motion planning for wheeled robots subject to nonholonomic controls [6], [7], models of kinematic drift effects in space systems subject to appendage vibrations or articulations [3], [4], models of self-propulsion of paramecia at low Reynolds number [12], kinematic model of an automobile [9] and kinematic model of an automobile with $(n - 3)$-trailers [11].

The goal of our paper is to discuss a similar problem for the Lie group $\text{SE}(2, \mathbb{R})$ which is in fact the phase space of the laser-matter dynamics and which appears naturally in the study of the 3-dimensional real valued Maxwell–Bloch equations.

1. An optimal problem for the Lie group $\text{SE}(2, \mathbb{R})$

Let $\text{SE}(2, \mathbb{R})$ be the special Euclidean group of the plane, i.e.

$$\text{SE}(2, \mathbb{R}) = \text{SO}(2, \mathbb{R}) \times \mathbb{R}^2,$$

Mathematics Subject Classification: 70Q05.
Key words and phrases: control problem, Lie group, dynamics, integration.
with the group operation given by:

\[(A, a) \cdot (B, b) = (AB, Ab + a),\]

for each \((A, a), (B, b) \in SO(2, \mathbb{R}) \times \mathbb{R}^2\). It is easy to see that via the map \(\phi\) given by

\[\phi : SE(2, \mathbb{R}) \to GL(3, \mathbb{R})\]

\[\phi(A, a) = \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix},\]

it is a closed subgroup of \(GL(3, \mathbb{R})\) and so it is a Lie group. Its Lie algebra \(\mathcal{L} SE(2, \mathbb{R})\) can be canonically identified with \(se(2, \mathbb{R})\), where

\[se(2, \mathbb{R}) = \left\{ \begin{bmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix} \mid a \in \mathbb{R}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \right\}.\]

Let

\[A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\]

be the canonical basis of \(se(2, \mathbb{R})\) with the bracket operation \([\cdot, \cdot]\) given by:

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<tr>
<th>([\cdot, \cdot])</th>
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and let us consider the following left-invariant controlled system on the matrix Lie group \(SE(2, \mathbb{R})\):

\[\dot{X} = X(A_1 u_1 + A_2 u_2).\]

Then an easy computation leads us to
Theorem 2.1. The system (2.1) is controllable.

Let $J$ be the cost function given as usual by

$$J(u_1, u_2) = \frac{1}{2} \int_0^{t_f} [c_1 u_1^2(t) + c_2 u_2^2(t)] \, dt, \quad c_1 > 0, \ c_2 > 0.$$

Theorem 2.2. The controls which minimize $J$ and steer the system (2.1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by

$$u_1 = \frac{1}{c_1} P_1, \quad u_2 = \frac{1}{c_2} P_2,$$

where $P_i$’s are solutions of

$$\begin{cases}
\dot{P}_1 = -\frac{1}{c_2} P_2 P_3 \\
\dot{P}_2 = \frac{1}{c_1} P_1 P_3 \\
\dot{P}_3 = -\frac{1}{c_1} P_1 P_2
\end{cases} \tag{2.3}$$

Proof. Let us apply Krishnaprasad’s theorem [2] to this special case. Then the extremal controls are given by (2.2), where $P_i$’s are solutions of the reduced Hamilton’s equations from $T^*\SE(2, \mathbb{R})$ to $((\text{se}(2, \mathbb{R}))^*) \simeq \mathbb{R}^3$. Here $(\text{se}(2, \mathbb{R}))^*$ means $(\text{se}(2, \mathbb{R}))^*$ together with the minus-Lie–Poisson structure $\{\cdot, \cdot\}_-$, i.e. the Poisson structure generated by the matrix

$$\Pi_- = \begin{bmatrix}
0 & -P_3 & P_2 \\
P_3 & 0 & 0 \\
-P_2 & 0 & 0
\end{bmatrix}.$$ 

Therefore,

$$\begin{bmatrix}
\dot{P}_1 \\
\dot{P}_2 \\
\dot{P}_3
\end{bmatrix} = \begin{bmatrix}
0 & -P_3 & P_2 \\
P_3 & 0 & 0 \\
-P_2 & 0 & 0
\end{bmatrix} \cdot \nabla H,$$

where $H$ is given by

$$H(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2 + \frac{1}{2c_2} P_2^2 \tag{2.4}$$
or equivalently,

\[
\begin{align*}
\dot{P}_1 &= -\frac{1}{c_2}P_2P_3 \\
\dot{P}_2 &= \frac{1}{c_1}P_1P_3 \\
\dot{P}_3 &= -\frac{1}{c_1}P_1P_2,
\end{align*}
\]

as required. □

**Remark 2.1.** The same result can be obtained using Lagrangian reduction [5].

**Remark 2.2.** The function \( C \) given by

\[
C(P_1, P_2, P_3) = \frac{1}{2}(P_2^2 + P_3^2),
\]

is a Casimir of our configuration \( ((\text{se}(2, \mathbb{R}))^*, \{\cdot, \cdot\}_-) \simeq (\mathbb{R}^3, \{\cdot, \cdot\}_-) \), i.e.

\[
\{C, f\}_- = 0,
\]

for each \( f \in C^\infty(\mathbb{R}^3, \mathbb{R}) \).

**Remark 2.3.** The integral curves of the system (2.3) are intersections of the cylinders:

\[
\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = 2H
\]

and

\[
P_2^2 + P_3^2 = 2C.
\]

3. Dynamical and geometrical properties of the equations (2.3)

In this section we want to point out some geometrical and dynamical properties of the equations (2.3).

**Theorem 3.1.** The dynamics (2.3) is equivalent to the pendulum dynamics.

**Proof.** Indeed, \( C \) is a constant of motion, so

\[
P_2^2 + P_3^2 = 2C = \text{constant}.
\]
Let us take now
\[
\begin{align*}
    P_2 &= \sqrt{2C} \cos \theta \\
    P_3 &= \sqrt{2C} \sin \theta.
\end{align*}
\]
Then
\[
\dot{P}_2 = -P_3 \dot{\theta},
\]
or equivalently,
\[
\dot{\theta} = -\frac{\dot{P}_2}{P_3} = -\frac{1}{c_1} \frac{P_1 P_3}{P_3} = -\frac{1}{c_1} P_1.
\]
Differentiating again, we get
\[
\ddot{\theta} = -\frac{1}{c_1} \dot{P}_1 = -\frac{1}{c_1 c_2} P_2 P_3 = -\frac{C}{c_1 c_2} \sin 2\theta,
\]
hence pendulum dynamics. \qed

Remark 3.1. A similar result is proved in [1] for the free rigid body.

Theorem 3.2. The system (2.3) may be realized as an Hamilton–Poisson system in an infinite number of different ways, i.e. there exists infinitely many different (in general nonisomorphic) Poisson structures on \( \mathbb{R}^3 \) such that the system (2.3) is induced by an appropriate Hamiltonian.

Proof. An easy computation shows us that the triples:
\[
(\mathbb{R}^3, \{ \cdot, \cdot \}_{ab}, H_{cd}),
\]
where
\[
\begin{align*}
    \{ f, g \}_{ab} &= -\nabla C_{ab} \cdot (\nabla f \times \nabla g), \quad (\forall) f, g \in C^\infty(\mathbb{R}^3, \mathbb{R}); \\
    C_{a,b} &= aC + bH; \\
    H_{cd} &= cC + dH; \\
    C &= \frac{1}{2} (P_2^2 + P_3^2); \\
    H &= \frac{1}{2c_1} P_1^2 + \frac{1}{2c_2} P_2^2; \\
    a, b, c, d &\in \mathbb{R}, \quad ad - bc = 1
\end{align*}
\]
are Hamilton–Poisson realizations of the system (2.3). \qed

An easy computation leads us to:
**Theorem 3.3.** The equations (2.3) may be explicitely integrated by elliptic functions.

It is easy to see that the equilibrium states of our system (2.2) are:

\[ e_1 = (M, 0, 0); \quad e_2 = (0, M, 0); \quad e_3 = (0, 0, M), \quad M \in \mathbb{R}. \]

Now we shall discuss their nonlinear stability. Recall that an equilibrium state \( P_e \) is nonlinear stable if trajectories starting close to \( P_e \) stay close to \( P_e \), or in other words, a neighborhood of \( P_e \) must be flow invariant. We have the following result:

**Theorem 3.4.** The equilibrium states \( e_1, e_2, e_3 \) have the following behaviour:

(i) The equilibrium states \((M, 0, 0), M \in \mathbb{R}, M \neq 0\) are nonlinear stable.

(ii) The equilibrium states \((0, M, 0), M \in \mathbb{R}, M \neq 0\) are unstable.

(iii) The equilibrium states \((0, 0, M), M \in \mathbb{R}, M \neq 0\) are nonlinear stable.

(iv) The equilibrium state \((0, 0, 0)\) is nonlinear stable.

4. **Numerical integration of the equations (2.3)**

In this section we shall discuss the numerical integration of the system (2.3) via the Lie–Trotter integrator and we shall point out some of its properties.

To begin with, let us observe that the Hamiltonian vector field \( X_H \) splits as follows:

\[ X_H = X_{H_1} + X_{H_2}, \]

where

\[ H_1(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2 \]

and

\[ H_2(P_1, P_2, P_3) = \frac{1}{2c_2} P_2^2. \]

The integral curves of \( X_{H_1} \) and \( X_{H_2} \) are given by:

\[ P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_1(t, P(0)) \]
An optimal control problem on the Lie group $\text{SE}(2,\mathbb{R})$

and respectively,

$$P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_2(t, P(0)).$$

Now following [8], [10], [13] the Lie–Trotter formula gives rise to an explicit integrator of the equations (2.3), namely:

$$\begin{align*}
P_{1n+1} &= P_{1n} - \frac{P_{2n(0)}}{c_2} P_{3n} t \\
P_{2n+1} &= P_{2n} \cos \frac{P_{1n(0)}}{c_1} t + P_{3n} \sin \frac{P_{1n(0)}}{c_1} t \\
P_{3n+1} &= -P_{2n} \sin \frac{P_{1n(0)}}{c_1} t + P_{3n} \cos \frac{P_{1n(0)}}{c_1} t.
\end{align*}$$

Some of its properties are sketched in the following theorem:

**Theorem 4.1.** The numerical integrator (4.1) has the following properties:

(i) It preserves the Poisson structure $\{\cdot, \cdot\}$. 

(ii) Its restriction to the coadjoint orbits $(O_k, \omega_k)$, where

$$O_k = \{(P_1, P_2, P_3) \in \mathbb{R}^3 \mid P_2^2 + P_3^2 = k^2\}$$

and

$$\omega_k = \frac{1}{k}(P_3 dP_1 \wedge dP_2 - P_2 dP_1 \wedge dP_3),$$

gives rise to a symplectic integrator.

(iii) It does not preserve the Hamiltonian (2.4).

**Proof.** The items (i) and (ii) hold because $\phi_1$ and $\phi_2$ are flows of some Hamiltonian vector fields, hence they are Poisson.

The item (iii) is a consequence of the fact that:

$$\frac{1}{2c_1}(P_{1n+1})^2 + \frac{1}{2c_2}(P_{2n+1})^2 \neq \frac{1}{2c_1}(P_{1n})^2 + \frac{1}{2c_2}(P_{2n})^2.$$

References


