A rigidity theorem for the three dimensional critical point equation

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Abstract. On a compact 3-dimensional manifold $M^3$, a critical point of the total scalar curvature functional, restricted to the space of metrics with constant scalar curvature of volume 1, satisfies the critical point equation (CPE), given by $U^*_g(f) = z_g$. It has been conjectured that a solution $(g, f)$ of CPE is Einstein. In this paper, we prove that, if CPE has two distinct solution functions and Weyl–Schouten tensor vanishes on a certain hypersurface of $M^3$, $(M^3, g)$ is isometric to a standard 3-sphere.

1. Introduction

Let $(M^3, g)$ be a 3-dimensional compact manifold and $\mathcal{M}_1$ the set of smooth Riemannian structures on $M^3$ of volume 1. The total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \rightarrow \mathbb{R}$ is defined by

$$\mathcal{S}(g) = \int_{M^3} s_g dv_g$$

where $dv_g$ is the volume form determined by the metric and $s_g$ the scalar curvature of the metric $g$. It is well known that a critical point of this functional is Einstein. On the other hand, there has been a conjecture (Conjecture A) that a critical point of this functional $\mathcal{S}$ restricted to $\mathcal{C}$ is Einstein [1, Chp 4, F], where $\mathcal{C}$ is the set of constant scalar curvature metrics given by

$$\mathcal{C} = \{g \in \mathcal{M}_1 \mid s_g \text{ constant}\}$$

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This paper is concerned with a study of a sufficient condition that Conjecture A holds. The Euler–Lagrange equations for a critical point $g$ of $\mathcal{S}$ restricted to $\mathcal{C}$ may be represented as the following critical point equation (CPE, hereafter):

$$U_g^*(f) = z_g \equiv r_g - \frac{s_g}{3}g$$

where $f$ is a function on $M^n$. Since

$$U_g^*(f) \equiv D_g df - (\Delta_g f)g - fr_g$$

and $\Delta_g f = -\frac{s_g}{2} f$, CPE may also be given as

$$(1 + f)z_g = D_g df + \frac{s_g f}{6}g.$$ 

J. Lafontaine showed that Conjecture A holds if a solution metric $g$ of CPE is conformally flat [7]. The author showed that Conjecture A holds if a solution function $f$ of CPE is greater than or equal to $-1$ [4]. Furthermore, Conjecture A holds if CPE has two distinct solution functions and certain two surfaces of $M^3$ are disjoint [5]. M. Obata showed that, if the solution metric of CPE is Einstein, it is isometric to a standard 3-sphere [9]. This paper is partially motivated by considering a generalization of the results of authors mentioned above, Lafontaine [7] and Hwang [5]. More precisely, we prove the following theorem in the present paper:

**Main Theorem.** Let CPE have two distinct non-trivial solutions $f_1$ and $f_2$ on $(M^3, g)$. Assume that Weyl–Schouten tensor vanishes on $\Gamma$ where $\Gamma = \phi^{-1}(0)$ for $\phi = f_1 - f_2$. Then $(M^3, g)$ is isometric to a 3-sphere.

**Remark 1.** (i) Fisher and Marsden showed that $\Gamma$ in our Main Theorem is an embedded surface of $M^3$ [2]. It is easy to see that $\Gamma$ exists in $M^3$, since we have

$$\int_{M^3} \varphi = -\frac{2}{s_g} \int_{M^3} \Delta_g \varphi = 0$$

where the first equation follows from the equation $\Delta_g \varphi = -\frac{s_g}{2} \varphi$, which may be obtained by taking the trace of the equation (3) in Section 2. For the significance of the surface $\Gamma$, see [5].

(ii) Our Main Theorem is a partial answer to the conjecture of [5] which states that $(M^3, g)$ is isometric to a standard 3-sphere if CPE has two distinct non-trivial solutions $f_1$ and $f_2$ on $(M^3, g)$. 

II. The proof of Main Theorem

This section is devoted to the proof of our Main Theorem. Throughout the present paper, we assume that there exist two distinct solutions \( f_1 \) and \( f_2 \) of CPE on \((M^3, g)\). Then, it is easy to see that \( U_g^*(\varphi) = 0 \) in virtue of (1), or equivalently we have

\[
0 = D_g d\varphi - (\Delta_g \varphi) g - \varphi r_g.
\]  

Definition. For a given 3-dimensional manifold \((M^3, g)\), let \( H = r_g - \frac{s_g}{4} g \) and \( d^D H \) be the Weyl–Schouten tensor field defined by the differential operator from \( C^\infty(S^2(M)) \) into \( \Lambda^2 M \otimes T^* M \), given by

\[
d^D H(x, y, z) = D_x H(y, z) - D_y H(x, z).
\]

Remark 2. When \( n = 3 \), we note that a metric \( g \) is conformally flat if and only if Weyl–Schouten tensor \( d^D H \) vanishes identically on \( M^3 \). Therefore, our metric \( g \) is Einstein if Weyl–Schouten tensor \( d^D H \) vanishes identically on \( M^3 \), in virtue of the result in the introduction of Lafontaine [7]. Our Main Theorem states that our metric \( g \) is Einstein, if there are two distinct solutions of CPE and \( d^D H \) vanishes on the hypersurface \( \Gamma \) of \( M^3 \).

The proof of Main Theorem consists of several lemmas and corollaries. It may be sketched as follows. In the first, we derive three relations involving \(|z|^2\), which hold on \( M^3 \) or \( \Gamma \) (Lemmas 1 and 2). Lemma 2 implies that \( z \) is diagonalized on \( \Gamma \) as in (15). In the second, we prove Lemma 3, in virtue of which we may choose a solution function \( f_\alpha \) of CPE such that its gradient \( df_\alpha \) is tangent to a connected component \( \Gamma_\alpha \) of \( \Gamma \). Using this fact and (15), it may be shown that \(|z|^2\) and \( d^D H \) are related as in (16) on \( \Gamma_\alpha \) (Corollary 2). Finally, using (16) and Lemma 4 we first show that \( z_g = 0 \) on \( \Gamma \), and then prove that \( z_g = 0 \) on \( M^3 = M_{0,\varphi} \cup \Gamma \cup M^0_\varphi \) (Proof of Main Theorem).

For \( \varphi \in \text{Ker } U_g^* \), let \( \Phi = |d\varphi|^2 \) and \( N_\varphi = \Phi^{-1/2} d\varphi \). For any solution \( f \) of CPE, let \( W = |df|^2 \), \( N_f = W^{-1/2} df \), and \( h = 1 + f \). In the following lemma, we prove that the equation (5) holds for any solution function \( f \) of CPE.
Lemma 1. The following equations hold on \((M^3, g)\):

\[ 8W h^2 |z|^2 = h^4 |d^D H|^2 + 3 \left| dW + \frac{sf}{3} df \right|^2 \]  \hspace{1cm} (5)

\[ 8\varphi^2 |z|^2 = \varphi^4 |d^D H|^2 + 3 \left| d\Phi + \frac{s \varphi}{3} d\varphi \right|^2. \]  \hspace{1cm} (6)

Sketch of proof. The full detailed proof of (5) is given in \([3]\). In a given coordinate system \(\{e_a\}_{a=1,2,3}\), (4) can be rewritten as

\[ d^D H_{cba} = r_{abc} - r_{ac;b} - \frac{1}{4} (g_{ab}s_{;c} - g_{ac}s_{;b}) = r_{abc} - r_{ac;b} \]

since \(s\) is constant. In virtue of the relation

\[ hr_{ab} = f_{;ab} + \frac{2 + 3f}{6} sg_{ab} \]

obtained from (2) and the Ricci identity \(f_{;abc} - f_{;acb} = R_{bcla} f^{;d}\) with

\[ R_{ijkl} = -\frac{s}{2} (g_{ik}g_{jl} - g_{il}g_{jk}) + (r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il}) \]

for \(n = 3\), we may conclude that

\[ h^4 |d^D H|^2 = -s^2 f^2 W - 2sf \langle df, dW \rangle - 3|dW|^2 + 8|D_g df|^2 W. \]  \hspace{1cm} (7)

Since (2) gives

\[ h^2 |z|^2 = |D_g df|^2 - \frac{s^2 f^2}{12} \]  \hspace{1cm} (8)

the equation (5) follows from (7) and (8). The proof of the equation (6) is similar.

On \(\Gamma\) the equation (6) is reduced to (9) as in the following lemma:

Lemma 2. On \(\Gamma = \varphi^{-1}(0)\) we have

\[ |z|^2 = \frac{3}{2} z(N_{\varphi}, N_{\varphi})^2. \]  \hspace{1cm} (9)

Proof. Since the relation

\[ d\Phi + \frac{s \varphi}{3} d\varphi = 2D_{d\varphi} d\varphi + \frac{s \varphi}{3} d\varphi = 2\varphi z(d\varphi, \cdot) \]  \hspace{1cm} (10)
holds in virtue of (3), on \( \varphi^{-1}(c) \) with any constant \( c \) we have

\[
(11) \quad \left| d\Phi + \frac{s\varphi}{3} d\varphi \right|^2 = 4\varphi^2 \sum_{i=1}^{3} z(d\varphi, e_i)^2
\]

in a given orthonormal basis \( \{e_a\}_{i=1,2,3} \) with \( e_3 = N_\varphi \). Substitution of (11) into (6) gives

\[
(12) \quad 8\Phi |z|^2 = 12 \sum_{i=1}^{2} z(d\varphi, e_i)^2 + 12 z(d\varphi, N_\varphi)^2 \quad \text{on } \Gamma.
\]

In what follows we claim that the first term of the right-hand side of (12) vanishes. This implies that (12) is reduced to (9), completing the proof of our lemma. In virtue of (3), we have in a neighborhood of \( \Gamma \)

\[
(13) \quad \varphi z(d\varphi, X) = \langle DX d\varphi, d\varphi \rangle = \frac{1}{2} \langle d\Phi, X \rangle
\]

for any vector field \( X \) tangent to \( \Gamma \). Taking the Lie derivative of (13) with respect to \( N_\varphi \) on \( \Gamma \), we have

\[
(14) \quad \Phi^{1/2} z(d\varphi, e_i) = \frac{1}{2} \langle D_{N_\varphi} d\Phi, e_i \rangle + \frac{1}{2} \langle d\Phi, D_{N_\varphi} e_i \rangle.
\]

On the other hand, we note that

\[
\langle D_{N_\varphi} d\Phi, e_i \rangle = \langle D_{e_i} d\Phi, N_\varphi \rangle = e_i \langle d\Phi, N_\varphi \rangle - \langle d\Phi, D_{e_i} N_\varphi \rangle = 0
\]

and \( d\Phi = \frac{1}{2} \langle D_g d\varphi, d\varphi \rangle = 0 \) on \( \Gamma \). Hence, the right-hand side of (14) vanishes, completing the proof of our claim. \( \square \)

As a consequence of Lemma 2, we have the following result:

**Corollary 1.** For any vector field \( X \) tangent to \( \Gamma \), \( z(X, d\varphi) = 0 \) on \( \Gamma \).

**Proof.** With respect to a given orthonormal basis \( \{e_a\}_{i=1,2,3} \) with \( e_3 = N_\varphi \), Lemma 2 implies that on \( \Gamma \)

\[
(15) \quad z(e_1, e_1) = z(e_2, e_2) = -\frac{1}{2} z(N_\varphi, N_\varphi) \quad \text{and} \quad z(e_i, e_j) = 0 \quad \text{for } i \neq j
\]
since for \( z_{ij} = z(e_i, e_j) \) we have

\[
0 = \left| z \right|^2 - \frac{3}{2} z(N_\varphi, N_\varphi)^2 = \sum_{i,j} z(e_i, e_j)^2 - \frac{3}{2} z(N_\varphi, N_\varphi)^2
\]

\[
= z_{11}^2 + z_{22}^2 - \frac{1}{2} z_{33}^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)
\]

\[
= z_{11}^2 + z_{22}^2 - \frac{1}{2} (z_{11} + z_{22})^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)
\]

\[
= \frac{1}{2} (z_{11} - z_{22})^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)
\]

where use of the fact that \( z_{33} = -(z_{11} + z_{22}) \) is made in the fourth equality. Hence, the proof of the corollary follows from (15). \( \square \)

Let \( \Gamma_\alpha \) be a connected component of \( \Gamma \). Then one can find a solution of CPE so that its gradient is tangent to \( \Gamma_\alpha \), as shown in the following lemma. In fact, the proof of Lemma 3 is given in [5]. However, we include its proof here again for the sake of completeness.

**Lemma 3.** There exists a solution \( f_\alpha \) of CPE such that \( \langle df_\alpha, d\varphi \rangle = 0 \) on \( \Gamma_\alpha \).

**Proof.** First we claim that both \( \Phi = |d\varphi|^2 \) and \( \eta = \langle df, d\varphi \rangle \) are constant along \( \Gamma_\alpha \), where \( f \) is \( f_1 \) or \( f_2 \). The first statement follows from the fact that we have in virtue of (3)

\[
\xi(\Phi) = 2 \langle D_\xi d\varphi, d\varphi \rangle = -s_g \langle \xi, d\varphi \rangle \varphi + 2 \varphi r_g(\xi, d\varphi) = 0
\]

for any tangent vector \( \xi \) to \( \Gamma \). The second statement follows from the fact that for any tangent vector field \( X \) to \( \Gamma_\alpha \) we have

\[
X(\eta) = \langle DX df, d\varphi \rangle + \langle df, DX d\varphi \rangle = \langle DX df, d\varphi \rangle = (1 + f)z(X, d\varphi) = 0
\]

which are the results of (2) and Corollary 1.

Now, let \( f_\alpha = f - \Phi_\alpha^{-1} \eta_\alpha \varphi \), where \( \Phi_\alpha = \Phi|_{\Gamma_\alpha} \) and \( \eta_\alpha = \eta|_{\Gamma_\alpha} \). Then, clearly \( f_\alpha \) is a solution of CPE, since \( U_\varphi^g(\varphi) = 0 \). Also, along \( \Gamma_\alpha \) we have

\[
\langle df_\alpha, d\varphi \rangle = \langle df, d\varphi \rangle - \Phi_\alpha^{-1} \eta_\alpha \langle d\varphi, d\varphi \rangle = \eta_\alpha - \eta_\alpha = 0.
\]

This implies that the gradient of \( f_\alpha \) is tangent to \( \Gamma_\alpha \), and hence the proof of Lemma 3 is completed. \( \square \)
Corollary 2. Let $\Gamma^r_\alpha = \{x \in \Gamma_\alpha \mid df_\alpha \neq 0\}$. Then we have

$$6W_\alpha |z|^2 = h_\alpha^2 |d^D H|^2 \quad \text{on } \Gamma^r_\alpha$$

where $W_\alpha = |df_\alpha|^2$ and $h_\alpha = 1 + f_\alpha$.

**Proof.** We first note that we may take $e_2 = N_{f_\alpha} = |df_\alpha|^{-1/2} df_\alpha$ on $\Gamma^r_\alpha$ in virtue of Lemma 3 and hence that (15) gives

$$z(N_{\varphi}, N_{\varphi}) = -2z(N_{f_\alpha}, N_{f_\alpha}) \quad \text{on } \Gamma^r_\alpha.$$  

In the next, we also note that

$$\left| dW_\alpha + \frac{s f_\alpha}{3} df_\alpha \right|^2 = 4h_\alpha^2 \sum_{i=1}^{3} z(df_\alpha, e_i)^2$$

since (2) implies that $dW_\alpha + \frac{s f_\alpha}{3} df_\alpha = 2Ddf_\alpha + \frac{s f_\alpha}{3} df_\alpha = 2h_\alpha z(df_\alpha, \cdot)$. Hence, in virtue of (15) and (17), (18) becomes

$$\left| dW_\alpha + \frac{s f_\alpha}{3} df_\alpha \right|^2 = 4h_\alpha^2 W_\alpha z(N_f, N_f)^2$$

$$= h_\alpha^2 W_\alpha z(N_{\varphi}, N_{\varphi})^2 = \frac{2}{3} h_\alpha^2 W_\alpha |z|^2 \quad \text{on } \Gamma^r_\alpha$$

where the last equality comes from (9). Consequently, substitution of (19) into (5) gives (16). \(\square\)

In Corollary 2 the behavior of $|z|^2$ on $\Gamma^r_\alpha$ was studied. In the following final lemma we investigate the behavior of $|z|^2$ on the set $\Gamma^c_\alpha = \Gamma_\alpha \setminus \Gamma^r_\alpha$ when $\Gamma^c_\alpha$ is not of measure zero. Note that if $\Gamma^c_\alpha$ is not of measure zero, there exists an open subset $\Omega$ of $\Gamma^c_\alpha$.

**Lemma 4.** If the set $\Gamma^c_\alpha$ is not of measure zero, $|z|^2$ is constant on any open subset $\Omega$ of $\Gamma^c_\alpha$.

**Proof.** Since $df_\alpha = 0$ on $\Omega$, $f_\alpha$ is constant on $\Omega$ and $\langle df_\alpha, \xi \rangle = 0$ on $\Omega$ for any tangent vector field $\xi \in T\Omega \subset T\Gamma_\alpha$. Thus, for $p \in \Omega$ and $\eta \in T_p \Omega \subset T_p \Gamma_\alpha$, it follows that $\eta \langle df_\alpha, \xi \rangle(p) = 0$, and hence

$$\langle D_\eta df_\alpha, \xi \rangle_p = \eta \langle df_\alpha, \xi \rangle(p) - \langle df_\alpha, D_\eta \xi \rangle_p = 0.$$
Since $p \in \Omega$ is arbitrary, we may conclude that $D_g df_\alpha(e_i, e_i) = 0$, where $e_i \in T\Omega$ for $i = 1, 2$. Thus, in virtue of (15), we have $h_\alpha z(e_i, e_i) = D_g df_\alpha(e_i, e_i) + \frac{s f_\alpha}{6} = \frac{s f_\alpha}{6}$ and

\begin{equation}
(20) \quad h_\alpha z(N_\varphi, N_\varphi) = -2h_\alpha z(e_i, e_i) = -\frac{s f_\alpha}{3}
\end{equation}

on $\Omega$. Note that $h_\alpha = 1 + f_\alpha \neq 0$ on $\Omega$ in virtue of (20). Therefore on $\Omega$ we have

\begin{equation}
(21) \quad |z|^2 = \frac{3}{2} z(N_\varphi, N_\varphi)^2 = \frac{s^2 f^2_\alpha}{6h^2_\alpha}
\end{equation}

which is a constant. \hfill \Box

Now we are ready to prove our Main Theorem.

**Proof of Main Theorem.** Since an Einstein solution metric $g$ is isometric to a 3-sphere due to the result in the introduction of Obata [9], it suffices to show that $z_g \equiv 0$ on $M^3 = M_{0,\varphi} \cup \Gamma \cup M^0_{\varphi}$.

We first show that $z_g = 0$ on each connected component $\Gamma_\alpha$ of $\Gamma = \bigcup_\alpha \Gamma_\alpha$. It follows from Lemma 3 that there exists a solution function $f_\alpha$ of CPE such that $df_\alpha \in TT_\alpha$. Since the right-hand side of (16) vanishes for this $f_\alpha$ in virtue of the assumption, we have

\begin{equation}
(22) \quad |z|^2 = 0 \quad \text{on } \Gamma^r_\alpha.
\end{equation}

Furthermore, if the set $\Gamma^c_\alpha = \Gamma_\alpha \setminus \Gamma^r_\alpha$ is of measure zero, it is clear that $|z|^2 = 0$ on $\Gamma^c_\alpha$ by the continuity of $|z|^2$. In the case that $\Gamma^c_\alpha$ is not of measure zero, Lemma 4 gives that $|z|^2$ is constant on any connected component of $\Gamma^c_\alpha$. Hence, by the continuity of $|z|^2$, we also have $|z|^2 = 0$ on $\Gamma^c_\alpha$ even when $\Gamma^c_\alpha$ is not of measure zero. This proves that regardless of the measure of $\Gamma^c_\alpha$ we have $z_g = 0$ on $\Gamma_\alpha = \Gamma^c_\alpha \cup \Gamma^r_\alpha$ in virtue of (21). Therefore, $z_g = 0$ on $\Gamma = \bigcup_\alpha \Gamma_\alpha$.

In order to show that $z_g = 0$ on all of $M^3$, we next prove that $z_g = 0$ on both

\[ M_{0,\varphi} = \{ x \in M^3 \mid \varphi(x) < 0 \} \quad \text{and} \quad M^0_{\varphi} = \{ x \in M^3 \mid \varphi(x) > 0 \}. \]
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Let $M'_0, \varphi$ be a connected component of $M_{0,\varphi}$ with $\partial M'_0, \varphi = \bigcup \Gamma_\beta$ for some $\{\beta\} \subset \{\alpha\}$. Integration by parts gives

$$\int_{M'_0, \varphi} \varphi |z|^2 = \int_{M'_0, \varphi} \left(D_g d\varphi + \frac{s_\varphi}{6} g\right)_{ij} z^{ij} = \int_{M'_0, \varphi} (D_g d\varphi)_{ij} z^{ij} = \int_{M'_0, \varphi} \text{div}(z(\varphi, \cdot)) - \int_{M'_0, \varphi} (d\varphi)_i (\delta z)^{ik}$$

$$= \int_{\partial M'_0, \varphi} z(\varphi, N_\varphi) = \sum_{\beta} \int_{\Gamma_\beta} z(\varphi, N_\varphi) = 0$$

where $\delta$ is the codifferential. Here, the fourth equality comes from the Stokes theorem and the fact that

$$\delta z_g = \delta \left(r_g - \frac{s_g}{3} g\right) = \delta r_g + d \left(\frac{s_g}{3}\right) = \delta r_g = -\frac{1}{2} d(s_g) = 0.$$ 

Therefore $z_g = 0$ on each connected component $M'_0, \varphi$ of $M_{0,\varphi}$, since $\varphi < 0$ on $M'_0, \varphi$. Thus $z_g = 0$ on all of $M_{0,\varphi}$. Applying the similar arguments to $M^0_\varphi$, we have $z_g = 0$ on each connected component of $M^0_\varphi$, and so on all of $M^0_\varphi$. Consequently, we may conclude that $z_g = 0$ on $M^3 = M_{0,\varphi} \cup \Gamma \cup M^0_\varphi$, or equivalently that $g$ is Einstein.

Now that we have proved that $z_g \equiv 0$ on $M^3$, the proof of our Main Theorem is now completed. \hfill $\square$

References


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