Note on metric spaces and continuous functions

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Abstract. W. RING, P. SCHÖPF and J. SCHWAIGER showed in [RSS] that if $E$ is a finite dimensional normed space then a function $f : E \to \mathbb{R}$ is continuous iff $f \circ \gamma$ is continuous for every regular curve $\gamma : [0, 1] \to E$.

We investigate a similar problem for metric spaces and the class of Lipschitz curves.

1. Introduction

W. RING, P. SCHÖPF and J. SCHWAIGER constructed in [RSS] an example of a not continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \gamma$ is continuous for every analytic curve $\gamma : (-1, 1) \to \mathbb{R}^2$. They also showed that if instead of analytic we take regular curves, such a function does not exist. In view of the above results the following general problem appears:

Problem 1. Let $X, T$ be metric spaces, let $\Gamma$ be a family of functions from $T$ into $X$. We assume that $f : X \to \mathbb{R}$ is such that $f \circ \gamma$ is continuous for every $\gamma \in \Gamma$. Does this imply that $f$ is continuous?

In this paper we investigate the above problem in few cases.

Let us first consider as an illustration the case when $X$ is an arbitrary metric space, $T$ denotes the set $\{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$, and $\Gamma$ denotes the space of all continuous functions from $T$ into $X$.

Let $f : X \to \mathbb{R}$ be arbitrary. We assume that $f \circ \gamma$ is continuous for every $\gamma \in \Gamma$. We show that then $f$ is continuous. For an indirect proof, let us assume that this is not the case. Then there exists an $x_0 \in X$ and a

Mathematics Subject Classification: 46B20, 33Exx.

Key words and phrases: continuity, Lipschitz functions.
sequence \( \{x_n\} \) convergent to \( x_0 \) such that the sequence \( \{f(x_n)\} \) does not converge to \( f(x_0) \). We define \( \gamma \in \Gamma \) by

\[
\gamma(0) = x_0, \quad \gamma \left( \frac{1}{n} \right) = x_n.
\]

One can now easily notice that \( f \circ \gamma \) is not continuous, a contradiction.

Let us now consider a situation when \( T = [0, 1] \) and \( \Gamma \) is the space of all continuous functions from \( T \) into \( X \). Under no additional assumption on \( X \) the answer to Problem 1 is negative. It is sufficient to put \( X = \{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \). Then every \( \gamma \in \Gamma \) is constant, which means that \( f \circ \gamma \) is continuous for every \( f : X \to \mathbb{R} \). However, there exist non-continuous functions on \( X \).

As shows the following result, under reasonable assumption on \( X \) the answer to Problem 1 is positive.

**Theorem 1.** Let \( X \) be a locally arcwise connected metric space, and let \( T = [0, 1] \). Let \( f : X \to \mathbb{R} \). If \( f \circ \gamma \) is continuous for every continuous function \( \gamma : T \to X \) then \( f \) is continuous.

**Proof.** For an indirect proof let us assume that there exists an \( x_0 \in X \) such that \( f \) is not continuous at \( x_0 \).

Since \( X \) is locally arcwise connected for every \( n \in \mathbb{N} \) there exists \( r_n < \frac{1}{n} \) such that each two points from \( B(x_0, r_n) \) can be connected by an arc contained in \( B(x_0, \frac{1}{n}) \). Without loss of generality we may assume that \( \{r_n\} \) is a decreasing sequence.

Since \( f \) is not continuous at \( x_0 \) there exists a sequence \( \{x_n\} \) convergent to \( x_0 \) such that \( x_n \in B(x_0, r_n) \) and

\[
\liminf_{n \to \infty} |f(x_n) - f(x_0)| > 0.
\]

Then for every \( n \in \mathbb{N} \) there exists a continuous curve \( \gamma_n : [0, 1] \to B(x_0, \frac{1}{n}) \) such that \( \gamma_n(0) = x_{n+1}, \gamma_n(1) = x_n \). We define a continuous function \( \gamma : [0, 1] \to X \) by

\[
\gamma(t) := \begin{cases} 
\gamma_n(2^n t - 1) & \text{for } t \in \left[ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right], \ n \in \mathbb{N}, \\
x_0 & \text{for } t = 0.
\end{cases}
\]

We obtain a contradiction since \( f \circ \gamma \) is not continuous at \( 0 \). \( \square \)
Now let us consider as $\Gamma$ the set of all Lipschitz mappings from $T = [0, 1]$ into $X$. Then the assumption that $X$ is locally arcwise connected does not guarantee a positive solution to Problem 1. As an example one can take as $X$ the graph of an arbitrary continuous nowhere differentiable function $f : [0, 1] \to \mathbb{R}$. Then $X$ is locally connected. As there are no non-constant Lipschitz functions $\gamma : [0, 1] \to X$, $g \circ \gamma$ is continuous for every function $g : X \to \mathbb{R}$. This suggests that the assumption that $X$ is a locally arcwise connected is too weak, since there may not exist nontrivial Lipschitz functions from $[0, 1]$ into $X$. The following definition is an analogue of the definition of locally arcwise connected spaces for Lipschitz curves.

**Definition 1.** Let $X$ be a metric space. We say that $X$ is **locally Lipschitz connected** if for every point $x \in X$ and $R > 0$ there exists an $r > 0$ such that each points from $B(x, r)$ can be connected by a Lipschitz arc in $B(x, R)$.

It occurs that even this property is too weak to guarantee the positive solution to Problem 1. We have the following result.

**Theorem 2.** There exists a compact locally Lipschitz connected metric space $X \subset \mathbb{R}^2$ and a not continuous function $f : X \to \mathbb{R}$ such that $f \circ \gamma$ is continuous for every Lipschitz function $\gamma : [0, 1] \to X$.

**Proof.** We put $r(x) := |x - \text{round}(x)|$, where $\text{round}(x)$ denotes the nearest integer to $x$. For $n \in \mathbb{N}$ we define the function $g_n : [0, \frac{1}{2^n}] \to \mathbb{R}^2$ by

$$g_n(x) := \left(\frac{1}{2^n} + \frac{1}{2^n} \sqrt{1 - \frac{1}{4^n} r(4^n x)}, x\right)$$

and put

$$X_n := g_n \left(\left[0, \frac{1}{2^n}\right]\right), \quad Y := \{(x, 0) : x \in [0, 1]\}.$$

One can easily check that the $g_n$ is chosen so that the length of the curve $g_n$ is exactly 1. We put $X = \bigcup_{n \geq 0} X_n \cup Y$ (see picture below).

Clearly $X$ is locally Lipschitz connected.

Let $f_n : X_n \to \mathbb{R}$ be defined by

$$f_n(g_n(x)) = 2^n x \quad \text{for} \ x \in \left[0, \frac{1}{2^n}\right].$$
We also define $f_0 : Y \to \mathbb{R}$ by $f_0 \equiv 0$. Let $f = \bigcup_{n \geq 0} f_n$. Then $f : X \to \mathbb{R}$ is clearly not continuous at $(0, 0)$.

Let $\gamma : [0, 1] \to X$ be a Lipschitz function. We show that $f \circ \gamma$ is continuous. The function $f \circ \gamma$ is obviously continuous in the neighborhood of every $t \in [0, 1]$ such that $\gamma(t) \neq (0, 0)$. We check what happens in the neighborhood of $(0, 0)$.

Let

$$k_n := \sup \left\{ x \in \left[0, \frac{1}{2^n}\right] : g_n(x) \in \gamma([0, 1]) \right\}.$$ 

By the definition of $g_n$ the length of the part of $\gamma$ contained in $X_n$ is greater then $2^n k_n$, which implies that the length of $\gamma$ is greater then $\sum_n 2^n k_n$. Since length of $\gamma$ is finite this yields that $2^n k_n$ converges to zero. By (1) this yields that the function $f$ restricted to the set

$$X_\gamma = \bigcup_{n \geq 0} \{ g_n(x) : x \in [0, k_n] \} \cup Y$$

is continuous. As $\gamma([0, 1]) \subset X_\gamma$, this implies $f \circ \gamma$ is continuous. □

The reason why such an example can be constructed is that although $(0, 0)$ can be connected with every point $x$ of $X$ by a Lipschitz curve $\gamma_x$, the Lipschitz constant of $\gamma_x$ (as a function of $x$) is not bounded from above. This leads to the following definition.

**Definition 2.** Let $X$ be a metric space. We say that $X$ is uniformly locally Lipschitz connected if for every point $x \in X$ and $R > 0$ there exist $r > 0$, $L > 0$ such that each points from $B(x, r)$ can be connected by a Lipschitz arc in $B(x, R)$ with Lipschitz constant smaller then $L$. 

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*Figure 1: Set $X$*
We omit the proof of the following result since it is analogous to that of Theorem 1.

**Theorem 3.** Let $X$ be a uniformly locally Lipschitz connected metric space, and let $T = [0,1]$. Let $f : X \to \mathbb{R}$. If $f \circ \gamma$ is continuous for every Lipschitz function $\gamma : T \to X$ then $f$ is continuous.

**Acknowledgement.** I would like to thank my father for valuable remarks.

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*(Received July 15, 2001, revised October 9, 2001; file arrived December 19, 2001)*