Some global properties of quasi Einstein manifolds

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Abstract. In this paper some global properties of a compact, orientable quasi Einstein manifold without boundary are obtained.

Introduction

The notion of a quasi Einstein manifold was introduced in a recent paper [1] by the first author M. C. CHAKI and R. K. MAITY. According to them a non-flat Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be a quasi Einstein manifold if its Ricci tensor \(S\) of type \((0,2)\) is not identically zero and satisfies the condition

\[
S(X, Y) = a g(X, Y) + bA(X)A(Y), \quad X, Y, \cdots \in \mathfrak{X}(M)
\]

where \(a, b\) are scalars of which \(b \neq 0\) and \(A\) is a non-zero 1-form such that

\[
g(X, U) = A(X), \quad \forall X,
\]

\(U\) being a unit vector field. In such a case, \(a, b\) are called the associated scalars, \(A\) is called the associated 1-form and \(U\) is called the generator of the manifold. Such an \(n\)-dimensional manifold is denoted by the symbol \((QE)_n\).

In this paper some global properties of a compact orientable \((QE)_n\) without boundary are obtained.

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1. Preliminaries

From (1) we get
\[ S(X, X) = a g(X, X) + b A(X) A(X) = a |X|^2 + b [g(X, U)]^2 \quad \forall X. \quad (1.1) \]

Let \( \theta \) be the angle between \( U \) and any vector \( X \). Then
\[
\cos \theta = \frac{g(X, U)}{\sqrt{g(U, U)} \sqrt{g(X, X)}} = \frac{g(X, U)}{\sqrt{g(X, X)}} \quad [\therefore \text{by hypothesis } g(U, U) = 1].
\]
Hence \([g(X, U)]^2 \leq g(X, X) = |X|^2\). Therefore
\[
|X|^2 \geq [g(X, U)]^2 \quad (1.2)
\]
Hence if \( a > 0 \), then \( a |X|^2 \geq a [g(X, U)]^2 \). Adding \( b [g(X, U)]^2 \) on both sides we get
\[
a |X|^2 + b [g(X, U)]^2 \geq (a + b) [g(X, U)]^2.
\]
Hence \( S(X, X) \geq (a + b) [g(X, U)]^2 \). Again if \( b > 0 \), then from (1.2) we get
\[
b |X|^2 \geq b [g(X, U)]^2.\]
Adding \( a |X|^2 \) on both sides we have
\[
(a + b) |X|^2 \geq a |X|^2 + b [g(X, U)]^2 \geq S(X, X).
\]
Thus
\[
S(X, X) \geq (a + b) [g(X, U)]^2 \quad \text{when } a > 0
\]
and
\[
S(X, X) \leq (a + b) |X|^2 \quad \text{when } b > 0. \quad (1.3)
\]
Next, contracting (1) over \( X \) and \( Y \) we get
\[
r = na + b, \quad (1.4)
\]
where \( r \) is the scalar curvature.
These results will be used in the sequel.

2. Sufficient condition for a compact orientable \((QE)_n\) to be conformal to a sphere in \( E_{n+1} \)

We begin with the definition of conformality of one Riemannian manifold to another.
Let \((M, g)\) and \((M', g')\) be two \(n\)-dimensional Riemannian manifolds. If there exists a one-one differentiable mapping \((M, g) \to (M', g')\) such that the angle between any two vectors at a point \(p\) of \(M\) is always equal to that of the corresponding two vectors at the corresponding point \(p'\) of \(M'\), then \((M, g)\) is said to be conformal to \((M', g')\). Y. Watanabe [2] has given a sufficient condition of conformality of an \(n\)-dimensional Riemannian manifold to an \(n\)-dimensional sphere immersed in \(E_{n+1}\). Its statement is as follows:

If in an \(n\)-dimensional Riemannian manifold \(M\), there exists a non-parallel vector field \(X\) such that the condition

\[
\int_M S(X, X) dv = 1/2 \int_M |dX|^2 dv + n - 1/n \int_M (\partial X)^2 dv \quad (2.1)
\]

holds, then \(M\) is conformal to a sphere in \(E_{n+1}\), where \(dv\) is the volume element of \(M\) and \(dX\) and \(\partial X\) are the curl and divergence of \(X\) respectively.

In this section we consider a compact and orientable quasi Einstein manifold \((QE)_n = M\) without boundary having associated scalars \(a, b\) and generator \(U\). It satisfies (1) and (2). Hence

\[S(U, U) = a + b.\]

In virtue of this and by taking \(U\) for \(X\) the condition (2.1) takes the following form

\[
\int_M (a + b) dv = 1/2 \int_M |dU|^2 dv + (n - 1)/n \int_M (\partial U)^2 dv. \quad (2.2)
\]

We now suppose that \(U = \text{grad} f\). Then \(U\) cannot be parallel, for otherwise \(\nabla U = 0\) or \(\nabla \text{grad} f = 0\) or \(\Delta f = 0\), where \(\Delta\) denotes Laplacian of \(f\), and \(\nabla\) denotes the covariant differentiation with respect to the metric of \(M\). Hence by Bochner’s lemma [3, p. 39], \(f\) is constant, which implies that \(U = 0\) which is not admissible.

Since by assumption \(U = \text{grad} f, |dU|^2 = 0\). Hence (2.2) takes the form

\[
\int_M (a + b) dU = (n - 1)/n \int_M (\partial U)^2 dv. \quad (2.3)
\]
Thus if in an $n$-dimensional compact orientable quasi Einstein manifold $M$ without boundary the generator $U$ is grad $f$, then $U$ is a non-parallel vector field. If in such a case the condition (2.3) is satisfied, then by Watanabe’s condition (2.1) $M$ is conformal to a sphere in $E_{n+1}$. We can therefore state the following.

**Theorem 1.** If in a compact, orientable quasi Einstein manifold $M = (QE)_n$ $(n \geq 3)$ without boundary having $a$, $b$ as associated scalars, the generator $U$ is the gradient of a function and satisfies the condition (2.3), then the manifold $(QE)_n$ is conformal to a sphere immersed in $E_{n+1}$.

### 3. Killing vector field in a compact orientable $(EQ)_n$ $(n \geq 3)$ without boundary

In this section we consider a compact, orientable $(QE)_n = M$ $(n \geq 3)$ without boundary with $a$, $b$ as associated scalars and $U$ as the generator.

It is known ([2] or [3] p. 43) that in such a manifold $M$ the following relation holds.

$$\int_M [S(X, X) - |\nabla X|^2 - (\text{div} X)^2]dv = 0, \quad \forall X \quad (3.1)$$

If $X$ is a Killing vector field, then $\text{div} X = 0$ ([3], p. 43). Hence (3.1) takes the form

$$\int_M [S(X, X) - |\nabla X|^2]dv = 0. \quad (3.2)$$

Let $b > 0$. Then by (1.3) $(a + b)|X|^2 \geq S(X, X)$. Therefore $(a + b)|X|^2 - |\nabla X|^2 \geq S(X, X) - |\nabla X|^2$. Consequently,

$$\int_M [(a + b)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X, X) - |\nabla X|^2]dv$$

and by (3.2)

$$\int_M (a + b)|X|^2 - |\nabla X|^2 dv \geq 0.$$

If $a + b < 0$, then

$$\int_M [(a + b)|X|^2 - |\nabla X|^2]dv = 0.$$
Therefore $X = 0$. This leads to the following result:

**Theorem 2.** If in a compact, orientable $(QE)_n$ $(n \geq 3)$ without boundary the associated scalars are such that $b > 0$ and $a + b < 0$, then there exists no nonzero Killing vector field in this manifold.

We now state the following two corollaries of the above theorem which follow easily from it.

**Corollary 1.** If in a compact, orientable $M = (QE)_n$ $(n \geq 3)$ without boundary, the generator $U$ is a Killing vector field, then the following relation holds.

$$\int_M [(a + b) - |\nabla U|^2] \, dv \geq 0.$$  

**Corollary 2.** If in a compact, orientable $M = (QE)_n$ $(n \geq 3)$ without boundary $b > 0$ and $a + b < 0$, then the generator $U$ of the manifold cannot be a Killing vector field.

4. **Killing $p$-form in a compact, orientable conformally flat $M = (QE)_n$ $(n > 3)$ without boundary**

Let $\omega$ be a $p$-form in a compact orientable conformally flat $M = (QE)_n$ $(n > 3)$ without boundary and $F_p(\omega, \omega)$ be the well known [3] quadratic form given by

$$F_p(\omega, \omega) = S_{ij} \omega^{i_1 \ldots i_{p}} \omega_{i_1 \ldots i_p} + \frac{p}{2} R_{ijkl} \omega^{i_1 j i_2 \ldots i_p} \omega^{k_1 \ell i_3 \ldots i_p}, \quad (4.1)$$

where $R_{ijkl}$ and $S_{ij}$ are the components of the curvature tensor $R$ of type $(0, 4)$ and the Ricci tensor $S$ of type $(0, 2)$ of the $(QE)_n$.

Since $(QE)_n$ is conformally flat the tensor $R$ can be expressed as follows ([4], p. 234; [5], p. 40):

$$R(X, Y, Z, W) = \frac{1}{n - 2} [g(Y, Z) S(X, W) - g(X, Z) S(Y, W)$$

$$+ g(X, W) S(Y, Z) - g(Y, W) S(X, Z)] - \frac{r}{(n - 1)(n - 2)} [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)]. \quad (4.2)$$
By (1)
\[ R(X, Y, Z, W) = a'[g(Y, Z)g(X, W) - g(Y, W)g(X, Z)] \]
\[ + b'[g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W)] \]
\[ + g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)] \]
where
\[ a' = \frac{a}{n-1} - \frac{b}{(n-1)(n-2)}, \quad b' = \frac{b}{n-2}. \] (4.3)

In virtue of (1) and (4.2) we can express (4.1) as follows:
\[ F_p(\omega, \omega) = [(n-p)a' + b']|\omega|^2 + (n-2p)b'[U.\omega]^2, \] (4.4)
where the components of \( \omega \) are \( \omega_{ij_1\ldots ip} \), those of \( U \) are \( U_{ij} \) and \( U. \omega \) is a tensor of type \((0, p-1)\) with components \( U_{j\omega_i j_1\ldots ip-1} \) and
\[ |\omega|^2 = \omega_{ij_1\ldots ip} \omega_{ij_1\ldots ip}. \]

Using (4.3) the relation (4.4) can be expressed as follows:
\[ F_p(\omega, \omega) = \frac{(n-p)(n-2)a + b(p-1)}{(n-1)(n-2)}|\omega|^2 + (n-2p)\frac{b}{n-2}[U.\omega]^2 \] (4.5)

We now suppose that \( \omega \) is a Killing \( p \)-form. Then it is known [3] that
\[ \int_{(QE)_n} [F_p(\omega, \omega) - |\nabla \omega|^2] dv = 0. \] (4.6)

In virtue of (4.5) we can express (4.6) as follows:
\[ \int_{(QE)_n} \left[ \frac{(n-p)(n-2)a + b(p-1)}{(n-1)(n-2)}|\omega|^2 + \frac{n-2p}{n-2}b[U.\omega]^2 - |\nabla \omega|^2 \right] dv = 0. \] (4.7)

If \((n-2p)b < 0\) and \((n-p)(n-2)a + b(p-1) < 0\) then from (4.7) it follows that \( \omega = 0 \). This leads to the following result.

**Theorem 3.** If in a compact, orientable conformally flat \((QE)_n\) \((n>3)\) without boundary, \((n-2p)b < 0\) and \((n-p)(n-2)a + b(p-1) < 0\), where \( p > 1 \) but \( < n \), then there exists no nonzero Killing \( p \)-form in such a manifold.
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References


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