Note on a Jensen type functional equation

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Abstract. We look for solutions \( f : M \to S \) and examine the stability of the functional equation

\[
3f \left( \frac{x + y + z}{3} \right) + f(x) + f(y) + f(z)
= 2 \left[ f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right],
\]

where \( M \) is an Abelian semigroup in which the division by 2 and 3 is performable and \( S \) is an abstract convex cone. Some applications to a multivalued version of this equation are given.

1. Introduction

Let \((S, +)\) be an Abelian semigroup, written additively. Suppose that \( S \) contains the identity element 0 and for each \( \lambda \geq 0 \) and \( s \in S \), an element \( \lambda s \) in \( S \) is defined, for which the following axioms hold

\[
1s = s, \quad \lambda(\mu s) = (\lambda \mu)s, \quad \lambda(s + t) = \lambda s + \lambda t, \quad (\lambda + \mu)s = \lambda s + \mu s,
\]

where \( s, t \in S \) and \( \lambda, \mu \geq 0 \). Then \( S \) is said to be an abstract convex cone.

If \( s, t, t' \in S, t + s = t' + s \) always implies that \( t = t' \), then \( S \) is said to satisfy the cancellation law.

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Suppose that an invariant with respect to translations and positively homogeneous metric \( \varrho \) is given in \( S \), i.e.,

\[
\varrho(t + s, t' + s) = \varrho(t, t')
\]

and

\[
\varrho(\lambda s, \lambda t) = \lambda \varrho(s, t)
\]

for \( \lambda > 0 \) and \( s, t, t' \in S \).

It is easy to see that the mappings \([0, \infty) \times S \ni (\lambda, s) \mapsto \lambda s \in S \) and \( S \times S \ni (s, t) \mapsto s + t \in S \) are continuous in the metric topology.

Let \((M, +)\) be an Abelian semigroup with the identity element 0 in which the division by 2 and 3 is performable.

We are going to look for all solutions \( f : M \to S \) of the functional equation

\[
3f\left(\frac{x + y + z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right)\right].
\]

The inequality

\[
3f\left(\frac{x + y + z}{3}\right) + f(x) + f(y) + f(z) \geq 2\left[f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right)\right]
\]

appeared in T. Popoviciu’s paper [3] in connection with the following theorem: The real continuous function \( f \) defined on an interval \( I \) is convex (i.e. the second divided differences of \( f \) are non-negative) if and only if the above inequality holds true for every triples \( x, y, z \) in \( I \).

Tiberiu Trif [7] solved equation (1) in the class of functions \( f : X \to Y \), where \( X, Y \) are real vector spaces. His considerationes cannot be applied in our reality as subtraction in vector spaces was used.

The main objective of this note is to find all solutions \( f : M \to S \) of (1) and to examine its stability. The natural range of equation (1) is a commutative semigroup. If we consider its stability then the semigroup ought to be endowed with a metric. Restrictions of the second part of
the paper (the range of $f$ is an abstract convex cone with the cancellation law and endowed with a complete metric invariant under translations and positive homogeneous) enable us to prove Theorem 1. The family $clb(X)$ of all non-empty convex closed and bounded subsets of a real Banach space fulfills these conditions and we can apply Theorem 1 to study equation (1) in the multivalued case. The similar results associated with the Jensen and Pexider functional equations were obtained in [5].

2. Solutions and Hyers–Ulam stability of (1)

We shall assume that

(i) $M$ is a commutative semigroup with zero in which the division by 2 and 3 is performable;
(ii) $S$ is an abstract cone satisfying the cancellation law;
(iii) $(S, \rho)$ is a complete metric space and $\rho$ is invariant with respect to translations and positively homogeneous.

Let $a : M \to S$ be an additive function which means $a(x + y) = a(x) + a(y)$ for all $x, y \in M$. It is easily seen that for every $b \in S$, the function $f(x) = a(x) + b, x \in M$, satisfies (1). The converse follows from the following

**Theorem 1.** Assume that conditions (i)–(iii) are fulfilled. If $\varepsilon \geq 0$ and if $f : M \to S$ satisfies

$$
\rho \left( 3f \left( \frac{x + y + z}{3} \right) + f(x) + f(y) + f(z), 2 \left[ f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right] \right) \leq \varepsilon
$$

(2)

for all $x, y, z \in M$, then there exists a unique additive function $a : M \to S$ such that

$$
\rho \left( f(x), a(x) + f(0) \right) \leq \varepsilon
$$

(3)

for $x \in M$.

**Proof.** Setting in (2) $y = x$ and $z = 0$ we obtain

$$
\rho \left( 3f \left( \frac{2}{3} x \right) + 2f(x) + f(0), 2f(x) + 4f \left( \frac{1}{2} x \right) \right) \leq \varepsilon.
$$
Since the metric $\varrho$ is invariant with respect to translation and positively homogeneous, we have

$$\varrho \left( \frac{3}{4} f \left( \frac{4}{3} x \right) + \frac{1}{4} f(0), f(x) \right) \leq \frac{1}{4} \varepsilon. \quad (4)$$

Replacing $x$ by $\frac{3}{4} x$, multiplying by $\frac{3}{4}$ both the sides of (4) we infer

$$\varrho \left( \left( \frac{3}{4} \right)^2 f \left( \left( \frac{4}{3} \right)^2 x \right) + \frac{1}{4} f(0), \frac{3}{4} f \left( \frac{4}{3} x \right) \right) \leq \frac{3}{4} \varepsilon,$$

whence

$$\varrho \left( \left( \frac{3}{4} \right)^2 f \left( \left( \frac{4}{3} \right)^2 x \right) + \frac{1}{4} \left( 1 + \frac{3}{4} \right) f(0), f(x) \right) \leq \frac{3}{4} \varepsilon.$$

Hence in virtue of (4) follows

$$\varrho \left( \left( \frac{3}{4} \right)^n f \left( \left( \frac{4}{3} \right)^n x \right) + \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{n-1} \right] f(0), f(x) \right) \leq \frac{1}{4} \left( 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{n-1} \right) \varepsilon \quad (5)$$

holds. Write

$$f_n(x) := \left( \frac{3}{4} \right)^n f \left( \left( \frac{4}{3} \right)^n x \right), \quad x \in M, \ n \in \mathbb{N}. \quad (6)$$

It follows by (5) that for arbitrary $n, m \in \mathbb{N}$, and $x \in M$ we have

$$\varrho \left( f_{n+m}(x), f_n(x) \right) = \varrho \left( \left( \frac{3}{4} \right)^{n+m} f \left( \left( \frac{4}{3} \right)^{n+m} x \right), \left( \frac{3}{4} \right)^n f \left( \left( \frac{4}{3} \right)^n x \right) \right)$$

$$= \left( \frac{3}{4} \right)^n \varrho \left( \left( \frac{3}{4} \right)^m f \left( \left( \frac{4}{3} \right)^m x \right) + \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{m-1} \right] f(0), \right)$$
\[
\begin{align*}
f \left( \left( \frac{4}{3} \right)^n x \right) + & \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{m-1} \right] f(0) \\
\leq & \left( \frac{3}{4} \right)^n \phi \left( \left( \frac{3}{4} \right)^m \left( \frac{4}{3} \right)^n x \right) \\
+ & \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{m-1} \right] f(0), f \left( \left( \frac{4}{3} \right)^n x \right) \\
+ & \phi \left( f \left( \left( \frac{4}{3} \right)^n x \right), f \left( \left( \frac{4}{3} \right)^n x \right) \right) \\
\leq & \left( \frac{3}{4} \right)^n \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{m-1} \right] \epsilon \\
+ & \left( \frac{3}{4} \right)^n \frac{1}{4} \left[ 1 + \frac{3}{4} + \cdots + \left( \frac{3}{4} \right)^{m-1} \right] \phi(0, f(0)) \\
< & \left( \frac{3}{4} \right)^n \left[ \epsilon + \phi(0, f(0)) \right].
\end{align*}
\]

Thus for every \( x \in M \), \( (f_n(x))_{n \in \mathbb{N}} \) is a Cauchy sequence. Let
\[
a(x) = \lim_{n \to \infty} f_n(x), \quad x \in M. \tag{7}
\]

Letting \( n \to \infty \), we obtain by (5)
\[
\phi(a(x) + f(0), f(x)) \leq \epsilon.
\]

We have by (2)
\[
\begin{align*}
\phi & \left( \frac{3}{4} \right)^n \left[ f \left( \frac{1}{3} \left[ \left( \frac{4}{3} \right)^n x + \left( \frac{4}{3} \right)^n y + \left( \frac{4}{3} \right)^n z \right] \right) \\
+ & \left( \frac{3}{4} \right)^n \left[ f \left( \left( \frac{4}{3} \right)^n x \right) + f \left( \left( \frac{4}{3} \right)^n y \right) + f \left( \left( \frac{4}{3} \right)^n z \right) \right] \\
+ & \left( \frac{3}{4} \right)^n \left[ f \left( \left( \frac{4}{3} \right)^n \frac{x+y}{2} \right) + f \left( \frac{4}{3} \right)^n y + f \left( \frac{4}{3} \right)^n z \right] \\
+ & \left( \frac{3}{4} \right)^n \left[ f \left( \frac{4}{3} \right)^n \frac{z+x}{2} \right] \right) \leq \left( \frac{3}{4} \right)^n \epsilon,
\end{align*}
\]
i.e.,
\[
\varrho \left( 3f_n \left( \frac{x+y+z}{3} \right) + f_n(x) + f_n(y) + f_n(z),
\right.
\]
\[
2 \left[ f_n \left( \frac{x+y}{2} \right) + f_n \left( \frac{y+z}{2} \right) + f_n \left( \frac{z+x}{2} \right) \right] \leq \left( \frac{3}{4} \right)^n \varepsilon.
\]

Passing to the limit as \( n \to \infty \) we get
\[
3a \left( \frac{x+y+z}{3} \right) + a(x) + a(y) + a(z)
\]
\[
= 2 \left[ a \left( \frac{x+y}{2} \right) + a \left( \frac{y+z}{2} \right) + a \left( \frac{z+x}{2} \right) \right]
\]
for \( x, y, z \in M \), which means that \( a \) satisfies equation (1). Since \( a(0) = \lim_{n \to \infty} (3/4)^n f(0) \) by (7) and (6), we have \( a(0) = 0 \). Now we shall prove that \( a \) is an additive function. Putting in (8) \( y = x \) we get
\[
3a \left( \frac{2x+z}{3} \right) + a(z) = 4a \left( \frac{x+z}{2} \right).
\]
If we put \( u = \frac{2x+z}{3} \), (9) turns into
\[
3a(u) + a(z) = 4a \left( \frac{3u+z}{4} \right), \quad u, z \in M.
\]
Substitute \( z = 0 \) in (10). Then \( a \left( \frac{3}{4} u \right) = \frac{3}{4} a(u) \) or
\[
a(3u) = \frac{3}{4} a(4u).
\]
Setting \( u = 0 \) in (10) we obtain \( a(z) = 4a \left( \frac{1}{4} z \right) \) or \( a(4z) = 4a(z) \). Hence and by (11), \( a(3u) = 3a(u) \). Now formula (10) may be rewritten in the form \( a(3u) + a(z) = a(3u + z) \), whence the additivity of \( a \) follows.

To end the proof we have to show the uniqueness of \( a \).

Suppose that (3) holds with an additive function \( \tilde{a} : M \to S \). We have for arbitrary \( n \in \mathbb{N} \)
\[
\varrho \left( a(x), \tilde{a}(x) \right) = \left( \frac{3}{4} \right)^n \varrho \left( \left( \frac{4}{3} \right)^n a(x), \left( \frac{4}{3} \right)^n \tilde{a}(x) \right).
\]
\[
\left(\frac{3}{4}\right)^n \varrho \left( a \left( \left(\frac{4}{3}\right)^n x \right), \tilde{a} \left( \left(\frac{4}{3}\right)^n x \right) \right) \\
= \left(\frac{3}{4}\right)^n \varrho \left( a \left( \left(\frac{4}{3}\right)^n x \right) + f(0), \tilde{a} \left( \left(\frac{4}{3}\right)^n x \right) + f(0) \right) \\
\leq \left(\frac{3}{4}\right)^n \varrho \left( a \left( \left(\frac{4}{3}\right)^n x \right) + f(0), f \left( \left(\frac{4}{3}\right)^n x \right) \right) \\
+ \left(\frac{3}{4}\right)^n \varrho \left( f \left( \left(\frac{4}{3}\right)^n x \right), \tilde{a} \left( \left(\frac{4}{3}\right)^n x \right) + f(0) \right) \leq 2 \left(\frac{3}{4}\right)^n \varepsilon,
\]

whence \( a(x) = \tilde{a}(x), \ x \in M \). \qed

Taking \( \varepsilon = 0 \) in Theorem 1 we obtain the following

**Theorem 2.** Assume that conditions (i)–(iii) are fulfilled. If \( f : M \to S \) satisfies (1), then there exists an additive function \( a : M \to S \) and \( b \in S \) such that \( f(x) = a(x) + b, \ x \in M \).

### 3. Multivalued solutions of (1)

Let \( X \) be a real Banach space and let \( \text{clb}(X) \) denote the set of all non-empty convex closed and bounded subsets of \( X \). Introduce a binary operation \( + \) in \( \text{clb}(X) \) by the formula

\[
A \ast + B = \text{cl}(A + B) = \text{cl}(\text{cl} A + \text{cl} B),
\]

where \( A + B \) denotes the usual Minkowski sum of \( A \) and \( B \) while \( \text{cl} A \) denotes the closedness of the set \( A \). The second operation in \( \text{clb}(X) \) is given by

\[
\lambda A = \{ \lambda a : a \in A \}
\]

for all \( \lambda \geq 0 \) and \( A \in \text{clb}(X) \). It is easily seen that \( \text{clb}(X) \) is an abstract convex cone with the identity element \( 0 := \{0\} \).

The proof of the following generalization of the Rådström lemma (cf. [4]) can be found in [6].

**Lemma 1.** If a set \( B \subset X \) is a non-empty and bounded and \( C \subset X \) is convex and closed, then for every \( A \subset X \),

\[
A + B \subset C \ast + B \implies A \subset C.
\]
From Lemma 1, we derive that the cancellation law holds in the abstract convex cone $\text{clb}(X)$.

The set $\text{clb}(X)$ is a metric space with the Hausdorff distance $h$ defined as follows

$$h(A, B) = \max \{ \sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\} \},$$

where $d(a, B) = \inf \{\|a - b\| : b \in B\}$. The metric space $(\text{clb}(X), h)$ is complete (cf. e.g. [1]).

**Lemma 2.** If $A, B, C \in \text{clb}(X)$ and $\lambda \geq 0$, then

$$h(A^* + B, C^* + B) = h(A + B, C + B) = h(A, C), \quad (12)$$

and

$$h(\lambda A, \lambda B) = \lambda h(A, B). \quad (13)$$

The first equality in (12) is easy to verify, the proof of the second one can be found in [2]. Formula (13) is well known. Thus the abstract cone clb$(X)$ satisfy assumptions (ii) and (iii).

A multifunction $F_0 : M \rightarrow \text{clb}(X)$ is said to be $(\ast)$-additive if

$$F_0(x + y) = F_0(x)^* + F_0(y)^*$$

for all $x, y \in M$.

From Theorem 1 we derive the following result.

**Theorem 3.** Let $(M, +)$ satisfy condition (i) and let $X$ be a real Banach space. We assume that $\varepsilon \geq 0$ and that $F : M \rightarrow \text{clb}(X)$ satisfies the inequality

$$h \left(3F \left(\frac{x + y + z}{3}\right)^* + F(x)^* + F(y)^* + F(z)^*\right),$$

$$2 \left[ F \left(\frac{x + y}{2}\right)^* + F \left(\frac{y + z}{2}\right)^* + F \left(\frac{z + x}{2}\right)^*\right] \leq \varepsilon,$$

then there exists a unique $(\ast)$-additive multifunction $F_0 : M \rightarrow \text{clb}(X)$ such that

$$h \left(F(x), F_0(x)^* + F(0)^*\right) \leq \varepsilon$$

for all $x, y \in M$. 
In particular, putting $\varepsilon = 0$, we have the following result.

**Theorem 4.** Let $(M, +)$ satisfy assumption (i) and let $X$ be a real Banach space. If $F : M \to \text{clb}(X)$ satisfies the functional equation

$$
3F\left(\frac{x + y + z}{3}\right)^* + F(x)^* + F(y)^* + F(z)^*
= 2 \left[ F\left(\frac{x + y}{2}\right)^* + F\left(\frac{y + z}{2}\right)^* + F\left(\frac{z + x}{2}\right)^* \right],
$$

(14)

then there exists a ($\ast$)-additive multifunction $F_0 : M \to \text{clb}(X)$ such that

$$
F(x) = F_0(x)^* + F(0)
$$

for all $x \in M$. Conversely, every multifunction $F(x) = F_0(x)^* + B$, where $F_0 : M \to \text{clb}(X)$ is a ($\ast$)-additive multifunction and $B \in \text{clb}(X)$ is an arbitrary set, actually satisfies (14).

**Remark 1.** Every additive multifunction $F_0 : M \to \text{clb}(X)$ which means

$$
F_0(x + y) = F_0(x) + F_0(y)
$$

for all $x, y \in M$ (15) is ($\ast$)-additive. In fact by (15), $F_0(x + y) = \text{cl}(F_0(x + y)) = \text{cl}(F_0(x) + F_0(y)) = F_0(x)^* + F_0(y)$.

**Remark 2.** A ($\ast$)-additive multifunction $F_0 : M \to \text{clb}(X)$ does not have to be additive. To see that take $A, B \in \text{clb}(X)$ such that $\text{cl}(A + B) \neq A + B$. The authoress believes that an example such sets $A, B$ is known but we will construct one below for convenience of a reader. The multifunction $F : [0, \infty)^2 \to \text{clb}(X)$ given by the formula

$$
F(t_1, t_2) = \text{cl}(t_1A + t_2B)
$$

is ($\ast$)-additive. Indeed,

$$
F((t_1, t_2) + (s_1, s_2)) = F(t_1 + s_1, t_2 + s_2) = \text{cl}[(t_1 + s_1)A + (t_2 + s_2)B]
= \text{cl}[t_1A + t_2B + s_1A + s_2B] = \text{cl}[\text{cl}(t_1A + t_2B) + \text{cl}(s_1A + s_2B)]
= F(t_1, t_2)^* + F(s_1, s_2)
$$
for all \( t_1, t_2, s_1, s_2 \in [0, \infty) \). However, \( F \) is not additive, as
\[
F(1, 0) + F(0, 1) = \text{cl} A + \text{cl} B = A + B
\]
and
\[
F((1, 0) + (0, 1)) = F(1, 1) = \text{cl}(A + B) \neq A + B = F(1, 0) + F(0, 1).
\]

The following example has been suggested by Dr ANNA KUCIA (Katowice), the authoress wish to thank her for that in this place. Let \( X = l_1 \) denote the space of all summable sequences real numbers. For each \( i \in \mathbb{N} \), let \( e_i \) be the vector in \( l_1 \) with zeros in all its coordinates except the \( i^{th} \) coordinate which is equal to one. Define
\[
A_1 = \left\{ \left( 1 + \frac{1}{i} \right) e_i : i \in \mathbb{N} \right\}, \quad B_1 = \left\{ \left( -1 + \frac{1}{i} \right) e_i : i \in \mathbb{N} \right\},
\]
and
\[
A = \overline{co} A_1, \quad B = \overline{co} B_1,
\]
where \( \overline{co} A_1 \) denotes the intersection of all convex closed sets containing \( A_1 \).

At first we observe that
\[
A_0 := \left\{ \left( 2p_1, 2p_2, \frac{4}{3}p_3, \ldots, \frac{i+1}{i} p_i, \ldots \right) : \sum_{i=1}^{\infty} p_i = 1, \quad p_i \geq 0 \right\}
\]
for every element of \( A_0 \) is a limit of some sequence of points belonging to \( \overline{co} A_1 \).

Next, we shall show that \( A \subset A_0 \). Take an arbitrary \( a = (2p_1, 2p_2, \frac{4}{3}p_3, \ldots, \frac{i+1}{i} p_i, \ldots) \in A \). Of course \( p_i \geq 0, \quad i \in \mathbb{N} \). It is enough to prove that \( \sum_{i=1}^{\infty} p_i = 1 \). We can find
\[
a^n = \left( 2p_1^n, 2p_2^n, \ldots, \frac{i+1}{i} p_i^n, \ldots \right) \in \overline{co} A_1, \quad n \in \mathbb{N}
\]
such that \( \lim_{n \to \infty} a^n = a \) and \( r^n_m := 1 - \sum_{i=1}^{m} p_i^n \to 0 \) as \( m \to \infty \) for each \( n \in \mathbb{N} \). Since
\[
\frac{i+1}{i} |p^n_i - p_i| \leq \sum_{i=1}^{\infty} \frac{i+1}{i} |p^n_i - p_i| = ||a^n - a||
\]
and \( \lim_{n \to \infty} ||a^n - a|| = 0 \),
\[
\sum_{i=1}^{m} p^n_i \to \sum_{i=1}^{m} p_i \quad \text{as} \quad n \to \infty
\]
for every $m \in \mathbb{N}$. Thus

$$\lim_{n \to \infty} r_m^n = 1 - \sum_{i=1}^{m} p_i =: r_m$$

exists for each $m \in \mathbb{N}$. Now we shall show that the sequence $(r_m^n)_{n \in \mathbb{N}}$ satisfies the Cauchy condition uniformly with respect to $m$. We have for all $m, k, n \in \mathbb{N}$,

$$|r_m^n - r_k^m| = \left| \sum_{i=1}^{m} p_i^k - \sum_{i=1}^{m} p_i^n \right| \leq \sum_{i=1}^{m} |p_i^k - p_i^n| \leq \sum_{i=1}^{\infty} |p_i^k - p_i^n|.$$

Let us fix $\varepsilon > 0$. Since $(a_n)$ is convergent, there exists a positive number $\alpha$ such that

$$\sum_{i=1}^{\infty} |p_i^k - p_i^n| < \varepsilon$$

for all $n, k > \alpha$. Hence

$$|r_m^n - r_m^k| < \varepsilon, \quad n, k > \alpha \quad \text{and} \quad m \in \mathbb{N}.$$

Now, when $k \to \infty$, the sequence $(r_m^k)_{k \in \mathbb{N}}$ tends to $r_m$ and

$$|r_m^n - r_m| \leq \varepsilon, \quad \text{for} \quad n > \alpha \quad \text{and} \quad m \in \mathbb{N}.$$

Fix arbitrarily $n > \alpha$. Since for every $m \in \mathbb{N}$,

$$0 \leq r_m \leq |r_m^n - r_m^m| + r_m^n,$$

letting $m \to \infty$, we obtain $\limsup_{n \to \infty} r_m \leq \varepsilon$. Consequently, $\lim_{m \to \infty} r_m = 0$ and by (16), $\sum_{i=1}^{\infty} p_i = 1$. We have proved that $A = A_0$.

Similarly we can show that

$$B = \left\{ \left(0, \frac{1}{2} q_2, -\frac{2}{3} q_3, \ldots, -\frac{i - 1}{i} q_i, \ldots \right) : \sum_{i=2}^{\infty} q_i = 1, \ q_i \geq 0 \right\}.$$

We observe that $0 \in \text{cl}(A + B)$. Indeed the set $A + B$ contains points of the form $2 \left(p_1, \frac{1}{2} p_2, \ldots, \frac{1}{i} p_i, \ldots \right)$, where $p_i \geq 0$ and $\sum_{i=1}^{\infty} p_i = 1$. Thus this set has elements of arbitrarily small norms. To show that $0 \not\in A + B$ we argue by contradiction: if for each $i \in \mathbb{N}$, $\frac{i+1}{i} p_i - \frac{i-1}{i} q_i = 0$ and $\sum_{i=1}^{\infty} p_i = 1$, $\sum_{i=1}^{\infty} q_i = 1$, we would have then

$$1 = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{i - 1}{i + 1} q_i = 1 - 2 \sum_{i=1}^{\infty} \frac{1}{i + 1} q_i < 1.$$
References


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