On varieties defined by pseudocomplemented nondistributive lattices

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Abstract. Lattices with 1, where for each element \( a \in L \) the interval \([a, 1]\) is pseudocomplemented, can be equipped with a binary operation “\( \circ \)” similar to the operation of relative pseudocomplementation. These algebras \((L, \wedge, \vee, \circ, 1)\) form an arithmetical and 1-regular variety. We investigate the subvarieties and the congruence kernels in this variety. It is shown that all algebras \((L, \wedge, \vee, \circ, 1)\) where \(L\) is a finite sublattice of a free lattice can be characterized by a particular identity.

1. Introduction

A bounded lattice \( L \) is called pseudocomplemented if for any \( x \in L \) there exists an element \( x^* \in L \) with the property that

\[
y \wedge x = 0 \quad \text{if and only if} \quad y \leq x^*.
\]

In [4] were characterized lattices with greatest element 1 where for each element \( a \in L \) the interval \([a, 1]\) is pseudocomplemented. It was shown that they can be equipped with a binary operation “\( \circ \)” having similar properties as the operation of relative pseudocomplementation and that the class \( \mathcal{P} \) of all these algebras \((L, \wedge, \vee, \circ, 1)\) is equational. Although lattices with relative pseudocomplementation are always distributive (see e.g. [2]), the

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above mentioned operation $\circ$ can be defined for nondistributive lattices, too. These preliminary results are discussed in Section 2.

An important subclass of the class of relatively pseudocomplemented lattices are relative $(L_n)$-lattices introduced in [11]. In Section 3, by the mean of the operation $\circ$, this notion is successfully extended to the nondistributive case. As a result we obtain new subvarieties of the variety $\mathcal{P}$. In Section 4 we apply our results to finite sublattices of free lattices. In Section 5 the congruence properties of $\mathcal{P}$ are investigated and we characterize the congruence kernels in the algebras of $\mathcal{P}$.

2. Preliminaries

Let $L$ be a lattice with 1 and $x, y \in L$. The pseudocomplement of $x \lor y$ in the interval $[y, 1]$ (if it exists) is denoted by $x \circ y$ (see [4]), and it is called the section pseudocomplement of $x$ with respect to $y$.

**Lemma 2.1.** The following conditions are equivalent for a lattice $L$ with 1.

(a) For any $x, y \in L$ the section pseudocomplement $x \circ y$ exists in $L$.

(b) Any principal filter $[a, 1]$ of $L$ is a pseudocomplemented lattice.

(c) For any $a \leq b$ the interval $[a, b]$ is a pseudocomplemented lattice.

**Proof.** The equivalence of (a) and (b) was proved in [4] and (c) $\implies$ (b) is clear. As any principal ideal of a pseudocomplemented lattice is also a pseudocomplemented lattice, (b) $\implies$ (c) is obvious. \(\square\)

A lattice $L$ with 1 is called *sectionally pseudocomplemented*, if the section pseudocomplement $x \circ y$ exists for each $x, y \in L$.

**Remark 2.2.** We recall that $(L, \land, \lor, \ast, 1)$ is a Brouwerian algebra (or a relatively pseudocomplemented lattice) if $(L, \land, \lor, 1)$ is a lattice with 1 and having the property that for any $a, b, x \in L$,

$$a \land x \leq b \iff x \leq a \ast b.$$  

The operation $\circ$ can be considered as an extension of $\ast$, since for $x \in [y, 1]$ we have $x \ast y = x \circ y$, whenever $x \ast y$ exists (see [4]). If $L$ is a distributive lattice, then the operations $\circ$ and $\ast$ coincide (see [1]).
Example 2.3. The lattice $N_5$ (see Figure 1) is sectionally pseudocomplemented but not relatively pseudocomplemented (see [4]).

![Figure 1](image)

E.g. the relative pseudocomplement $y \circ x$ does not exist, however the section pseudocomplement $y \triangleleft x$ exists and equals to $x$.

A lattice $L$ is $\wedge$-semidistributive if $a \wedge b_1 = a \wedge b_2$ implies $a \wedge b_1 = a \wedge (b_1 \vee b_2)$ for any $a, b_1, b_2 \in L$. A complete lattice $L$ is called completely $\wedge$-semidistributive if for any $b_i \in L$, $i \in I$ and $a \in L$ the relations $a \wedge b_i = y$, $i \in I$ imply $a \wedge (\bigvee \{b_i \mid i \in I\}) = y$. In view of [4], for any complete lattice $L$ the conditions of the Lemma 2.1 are equivalent to the condition

(d) $L$ is completely $\wedge$-semidistributive.

A lattice $L$ with 0 is called 0-distributive if for any elements $a, b, c \in L$ $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. Clearly, any principal filter of a $\wedge$-semidistributive lattice is 0-distributive. According to [14; Theorem 1], an algebraic lattice is pseudocomplemented if and only if it is 0-distributive. These results leads us to the following

**Proposition 2.4.** If $L$ is an algebraic lattice, then the following conditions are equivalent:

(i) $L$ is sectionally pseudocomplemented.
(ii) $L$ is completely $\wedge$-semidistributive.
(iii) $L$ is $\wedge$-semidistributive.
(iv) Any principal filter $[a, 1]$ of $L$ is a 0-distributive lattice.

**Proof.** The implications (ii) $\implies$ (iii) $\implies$ (iv) are obvious and the equivalence (i) $\iff$ (ii) was established in [4]. As any principal filter
\[ a, 1 \] of an algebraic lattice is also an algebraic lattice, the above mentioned result of [14] gives (iv) \( \implies \) (b), whence using Lemma 2.1 we get (iv) \( \implies \) (i).

\[ \square \]

We note that in the rest of the paper we deal with arbitrary sectionally pseudocomplemented lattices and in general we do not assume that they are algebraic or complete.

Let \( P \) denote the class of all algebras \((L, \wedge, \vee, \circ, 1)\), defined on sectionally pseudocomplemented lattices \((L, \wedge, \vee, 1)\). In [4] it was shown that the class \( P \) is determined by identities in signature \( \{\wedge, \vee, \circ, 1\} \), namely by the lattice axioms and by the identities

\begin{align*}
(1) \quad & x \circ x = 1, \ 1 \circ x = x \\
(2) \quad & ((x \circ y) \circ y) \wedge (x \vee y) = x \vee y \\
(3) \quad & (x \vee y) \circ y = x \circ y, \ y \vee (x \circ y) = x \circ y \\
(4) \quad & (((x \vee z) \wedge (y \vee z)) \circ z) \wedge ((x \vee z) \wedge (y \circ z)) \circ z = x \circ z
\end{align*}

Thus \( P \) is a variety and, according to Remark 2.3, \( P \) contains as a subvariety the variety \( B \) of all Brouwerian algebras.

\section*{3. Hereditary weakly \( L_n \)-lattices}

\textit{Definition 3.1.}  (i) Let \( L \) be a pseudocomplemented lattice and \( n \geq 1 \). We say that \( L \) is a weakly \( L_n \)-lattice, if it satisfies the equation:

\[ (x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n^*) \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1. \]  \((L_n)\)

If in addition \( L \) is distributive, then it is called an \((L_n)\)-lattice [11].

(ii) \( L \) is called a hereditary weakly \((L_n)\)-lattice if any principal filter \([a, 1]\) of it is a weakly \((L_n)\)-lattice.

Notice, that for \( n = 1 \) Definition 3.1(i) gives \( x^* \vee x^{**} = 1 \) and we say that the lattice \( L \) is weakly Stonean. If \( L \) is a distributive lattice, then Definition 3.1(ii) implies that any interval \([a, b] \subseteq L\) is also an \((L_n)\)-lattice. Lattices with this property were called in [11] relative \((L_n)\)-lattices. (Relative \((L_1)\)-lattices are known also under the name relative Stone lattices, see e.g. [10].)
In [11] M. HAVIAR and T. KATŘIŇÁK established that any (distributive) relative \((L_n)\)-lattice is characterized by the equation

\[(x_1 \land \ldots \land x_n) \lor y \lor (x_1 \land x_2 \lor \ldots \land x_n) \lor y \lor \ldots \lor (x_1 \land x_2 \lor \ldots \land x_n) \lor y = 1. \quad (L_n)'
\]

We shall deduce a similar equation with \(\circ\) for hereditary weakly \((L_n)\)-lattices. The lemma below follows directly from the definition of a section pseudocomplemented lattice:

**Lemma 3.2.** If \(L\) is a sectionally pseudocomplemented lattice, then \(x \leq y \implies x \circ y \geq z \circ y\).

**Theorem 3.3.** Let \(L\) be a lattice with 1. Then the following assertions are equivalent:

(i) \(L\) is a hereditary weakly \((L_n)\)-lattice.

(ii) \(x \circ y\) is defined for all \(x, y \in L\) and the algebra \((L, \land, \lor, \circ, 1)\) satisfies the equation:

\[(x_1 \land \ldots \land x_n) \circ y \lor (x_1 \circ y \land \ldots \land x_n) \circ y \lor \ldots \lor (x_1 \land x_2 \lor \ldots \land x_n) \circ y = 1. \quad (P_n)\]

**Proof.** (i) \(\implies\) (ii): As \(L\) is a hereditary weakly \((L_n)\)-lattice, any principal filter \([y]\) of it is pseudocomplemented, therefore, in virtue of Lemma 2.1, \(x \circ y\) is defined for all \(x, y \in L\). Since \(x_1 \land \ldots \land x_n \leq (x_1 \lor y) \land \ldots \land (x_n \lor y)\), Lemma 3.2 gives \((x_1 \land \ldots \land x_n) \circ y \geq [(x_1 \lor y) \land \ldots \land (x_n \lor y)] \circ y\).

On the other hand, using the identity \((x \lor y) \circ y = x \circ y\) we get

\[x_1 \circ y \land \ldots \land x_n \leq (x_1 \lor y) \circ y \land \ldots \land (x_n \lor y),\]

\[x_1 \land \ldots \land x_n \circ y \leq (x_1 \lor y) \land \ldots \land (x_n \lor y) \circ y.\]

By applying Lemma 3.2 again, we deduce

\[(x_1 \land \ldots \land x_n) \circ y \lor (x_1 \circ y \land \ldots \land x_n) \circ y \lor \ldots \lor (x_1 \land x_2 \lor \ldots \land x_n) \circ y \circ y \geq [(x_1 \lor y) \land \ldots \land (x_n \lor y)] \circ y \lor \ldots \lor [(x_1 \lor y) \land \ldots \land (x_n \lor y)] \circ y \circ y.

Since \(x_1 \lor y \geq y, \ldots, x_n \lor y \geq y\), and \((x_1 \lor y) \circ y \geq y, \ldots, (x_n \lor y) \circ y \geq y\), all these elements belong to the pseudocomplemented lattice \([y, 1]\). Let \(u^y\)
denote the pseudocomplement of an element \( u \geq y \) in the lattice \([y, 1]\). Now, in view of definition of the operation \( \circ \) we obtain:

\[
[(x_1 \lor y) \land \ldots \land (x_n \lor y)] \circ y = [(x_1 \lor y) \land \ldots \land (x_n \lor y)]^y,
\]

\[
[(x_1 \lor y) \circ y \land \ldots \land (x_n \lor y)] \circ y = [(x_1 \lor y)^y \land \ldots \land (x_n \lor y)]^y,
\]

\[
(\ldots) \circ y = [(x_1 \lor y) \land \ldots \land (x_n \lor y)]^y.
\]

Summarizing the above results, and taking in consideration that by assumption the lattice \([y, 1]\) satisfies the identity \((L_n)\) we obtain:

\[
(x_1 \land \ldots \land x_n) \circ y \lor (x_1 \circ y \land \ldots \land x_n) \circ y \lor \ldots \lor (x_1 \land \ldots \land x_n \circ y) \circ y \geq [(x_1 \lor y) \land \ldots \land (x_n \lor y)]^y \lor [(x_1 \lor y)^y \land \ldots \land (x_n \lor y)]^y \lor \ldots
\]

\[
\lor [(x_1 \lor y) \land \ldots \land (x_n \lor y)]^y = 1.
\]

Hence \((P_n)\) holds in \(L\) and this proves (ii).

(ii) \(\implies\) (i): Assume that \((P_n)\) holds in the algebra \((L, \land, \lor, \circ, 1)\) and take an \(a \in L\). As \(\circ\) is defined for all \(x, y \in L\), in view of Lemma 2.2 the interval \([a, 1]\) is a pseudocomplemented lattice. Let \(x_1, \ldots, x_n \geq a\). Since for any \(x \in [a, 1]\) we have \(x \circ a = x^a\), we get

\[
(x_1 \land \ldots \land x_n)^a \lor (x_1^a \land \ldots \land x_n^a) \lor \ldots \lor (x_1 \land \ldots \land x_n^a)
\]

\[
= (x_1 \land \ldots \land x_n) \circ a \lor (x_1 \circ a \land \ldots \land x_n) \circ a \lor \ldots
\]

\[
\lor (x_1 \land \ldots \land x_n \circ a) \circ a = 1.
\]

This equation shows that for any \(a \in L\), the principal filter \([a]\) is an \((L_n)\)-lattice. Thus \(L\) is a hereditary weakly \((L_n)\)-lattice.

**Example 3.4.** The algebra \((N_5, \land, \lor, \circ, 1)\) satisfies the identity \((P_1)\), i.e. \(x \circ y \lor (x \circ y) \circ y = 1\).

Indeed, it is not hard to see that any principal filter \([a]\) of \(N_5\) satisfies the equality \(x^a \lor (x^a)^a = 1\), therefore \(N_5\) is a hereditary weakly \((L_1)\)-lattice. In view of Theorem 3.3, \((N_5, \land, \lor, \circ, 1)\) satisfies \((P_1)\), too.

Let \(\mathcal{P}_n\) denote the class of all algebras \((L, \land, \lor, \circ, 1)\) corresponding to hereditary weakly \((L_n)\)-lattices. Because any \(\mathcal{P}_n\) is a subclass of \(\mathcal{P}\) determined by the identity \((P_n)\), any \(\mathcal{P}_n\) is a subvariety of \(\mathcal{P}\). Since in
the variety $\mathcal{B}$ of Brouwerian algebras the identity $(P_n)$ is the same as $(L_n)'$, the subvarieties $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}_n$ of $\mathcal{B}$ consist of algebras $(L, \wedge, \vee, *, 1)$ corresponding to relative $(L_n)$-lattices. As in view of [12] relative $(L_n)$-lattices form a proper subclass of the class of relative $(L_{n+1})$-lattices, we have $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. Let $\mathcal{B}_{-1}$ and $\mathcal{B}_0$ be the class of all trivial Brouwerian algebras and the subclass of $\mathcal{B}$ determined by the identity $x \vee y \vee (x \ast y) = 1$, respectively.

Clearly, $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ corresponds to relative $(L_n)$-lattices. As in view of [12] relative $(L_n)$-lattices form a proper subclass of the class of relative $(L_{n+1})$-lattices, we have $\mathcal{L} \subset \mathcal{L}_{n+1}$. Let $\mathcal{L}_{-1}$ and $\mathcal{L}_{0}$ be the class of all trivial lattices and the subclass of $\mathcal{L}$ determined by the identity $x \vee y \vee (x \ast y) = 1$, respectively.

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Since any finite sublattice \( L \) of a free lattice is an algebraic \( \land \)-semi-distributive lattice (see e.g. Theorem 2.4 in [6]), according to Proposition 2.2 any principal filter of such a lattice \( L \) is a pseudocomplemented lattice. Moreover, we prove:

**Theorem 4.1.** Any finite sublattice of a free lattice is a hereditary weakly \((L_4)\)-lattice.

**Proof.** As any principal filter \([a]\) of a finite sublattice of a free lattice \( F \) is also a finite sublattice of \( F \), it is enough to prove that each finite sublattice \( L \) of \( F \) satisfies the identity \((L_4)\).

In view of [13, Corollary 3.9], a lattice \( L \) satisfying the descending chain condition is a weakly \((L_n)\)-lattice whenever under each join-irreducible element of \( L \) are at most \( n \) atoms. In [7] is proved that any finite sublattice \( L \) of a free lattice has a breadth at most 4 (see also [6, Corollary 5.5]), i.e. for any \( n = 4 \) and any finite set \( \{a_1, a_2, \ldots, a_n\} \) there exist \( a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in \{a_1, a_2, \ldots, a_n\} \) such that \( \bigvee_{i=1}^{n} a_i = a_{i_1} \lor a_{i_2} \lor a_{i_3} \lor a_{i_4} \).

Now, let \( L \) be a finite sublattice of a free lattice and \( p \) a join-irreducible element of \( L \). We show that under \( p \) are at most 4 atoms.

Indeed, let us denote by \( a_1, a_2, \ldots, a_n \) the atoms of \( L \) which are under \( p \). Then there are at most four atoms \( a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in [0, p] \) such that \( \bigvee_{i=1}^{n} a_i = a_{i_1} \lor a_{i_2} \lor a_{i_3} \lor a_{i_4} \). If \( n > 4 \), then there exist an atom \( a_{i_0}, i_0 \in \{1, \ldots, n\} \) such that \( a_{i_0} \notin \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\} \). As \( L \) is \( \land \)-semidistributive, the relations \( a_{i_1} \land a_{i_0} = 0, a_{i_2} \land a_{i_0} = 0, a_{i_3} \land a_{i_0} = 0 \) and \( a_{i_4} \land a_{i_0} = 0 \) imply \( a_{i_0} = (a_{i_1} \lor a_{i_2} \lor a_{i_3} \lor a_{i_4}) \land a_{i_0} = 0 \), - a contradiction.

Hence \( L \) satisfies the identity \((L_4)\). \( \square \)

Using the above result and applying Theorem 3.3 we obtain:

**Corollary 4.2.** If \( L \) is a finite sublattice of a free lattice, then \( L \) is sectionally pseudocomplemented and the algebra \((L, \land, \lor, \circ, 1)\) satisfies the equation \((P_4)\).

**5. On the congruence properties of the variety \( \mathcal{P} \)**

In this section we shall study the congruence properties of algebras \( L = (L, \land, \lor, \circ, 1) \) from the variety \( \mathcal{P} \).
Of course, the variety $\mathcal{P}$ is congruence distributive because its reduct to the signature $\{\land, \lor\}$ is a class of lattices. Moreover, $\mathcal{P}$ is also congruence permutable. Indeed, one can deduce that a Mal’cev term of $\mathcal{P}$ can be e.g.

$$p(x, y, z) = (y \circ x) \land (x \lor z) \land (y \circ z).$$

We recall that a variety is \textit{arithmetical} if it is congruence distributive and congruence permutable at the same time. Thus we have:

**Theorem 5.1.** The variety $\mathcal{P}$ is arithmetical.

Let $\Theta \in \text{Con} \mathbb{L}$. The set $[1]_\Theta = \{x \in L \mid (1, x) \in \Theta\}$ is called the \textit{kernel} of $\Theta$. We say that the set $K \subseteq L$ is a \textit{congruence kernel} if $K = [1]_\Theta$ for some $\Theta \in \text{Con} \mathbb{L}$.

Recall that $\mathbb{L}$ is \textit{1-regular} if $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for each $\Theta, \Phi \in \text{Con} \mathbb{L}$. The following result was given by B. Csákány in [5]:

**Proposition 5.2.** A variety $\mathcal{V}$ with the constant 1 is 1-regular if and only if there exist $n \in \mathbb{N}$ and binary terms $b_1(x, y), \ldots, b_n(x, y)$ such that $\mathcal{V}$ satisfies the equivalence

$$b_1(x, y) = \ldots = b_n(x, y) = 1 \iff x = y.$$

By using this proposition we can prove:

**Theorem 5.3.** The variety $\mathcal{P}$ is 1-regular.

**Proof.** Take $n = 2$ and $b_1(x, y) = x \circ y$, $b_2(x, y) = y \circ x$. Of course, $b_1(x, x) = b_2(x, x) = 1$. Conversely, suppose $b_1(x, y) = b_2(x, y) = 1$.

Then $x \circ y = 1$ implies $(x \lor y)^\circ = 1$, i.e. $x \lor y = y$ and $y \circ x = 1$ implies $(x \lor y)^\circ = 1$, i.e. $x \lor y = x$, thus we get $x = y$. \hfill \square

**Remark 5.4.** (i) We can get also the Pixley term for arithmecity of $\mathcal{P}$, which is $t(x, y, z) = [(x \circ y) \circ z] \land [(z \circ y) \circ x] \land (x \lor z)$.

(ii) Let us note, as shown in [3], that a variety $\mathcal{V}$ is 1-regular and permutable if and only if there exist $n \in \mathbb{N}$, binary terms $s_1(x, y), \ldots, s_n(x, y)$ and a $(2 + n)$-ary term $q$ such that $\mathcal{V}$ satisfies the identities

$$s_i(x, x) = 1, \quad \text{for } i = 1, \ldots, n$$

$$x = q(x, y, s_1(x, y), \ldots, s_n(x, y)).$$
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\[ y = q(x, y, 1, \ldots, 1). \]

To verify this Mal'cev condition, one can take \( n = 2 \), \( s_1(x, y) = x \circ y \), \( s_2(x, y) = y \circ x \) and

\[ q(x, y, z, v) = (z \circ y) \land [(v \circ (y \circ x)) \circ x] \land (x \lor y). \]

This term \( q \) will be also important in the proof of the Theorem 5.7.

Since the variety \( \mathcal{P} \) is 1-regular, every congruence \( \Theta \in \text{Con} \mathbb{L} \) for \( \mathbb{L} \in \mathcal{P} \) is determined by its kernel \([1]\)\( _\Theta \). Hence, our task is to determine the congruence kernels and get an explicit description of a congruence determined by a given kernel.

Let \( K \subseteq L \) and let \( t(x_1, \ldots, x_n, y_1, \ldots, y_k) \) be a term of \( L = (L, \land, \lor, \circ, 1) \) in two sorts of variables. We say that \( K \) is \( y \)-closed \ with \ respect \ to \( t \) if \( t(a_1, \ldots, a_n, b_1, \ldots, b_k) \in K \) whenever \( b_1, \ldots, b_k \in K \) and for each \( a_1, \ldots, a_n \in L \).

For the sake of brevity, we introduce the notations:

\[ Q_1 = q(x_1, x_2, y_1, y_2) = (y_1 \circ x_2) \land [(y_2 \circ (x_2 \circ x_1)) \circ x_1] \land (x_1 \lor x_2), \]
\[ Q_2 = q(x_3, x_4, y_3, y_4) = (y_3 \circ x_4) \land [(y_4 \circ (x_4 \circ x_3)) \circ x_3] \land (x_3 \lor x_4). \]

Further, define the following terms in two sorts of variables:

\[ t_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \circ Q_2) \circ (x_2 \circ x_4), \]
\[ t_2(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \circ x_4) \circ (Q_1 \circ Q_2), \]
\[ t_3(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \land Q_2) \circ (x_2 \land x_4), \]
\[ t_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \land x_4) \circ (Q_1 \land Q_2), \]
\[ t_5(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \lor Q_2) \circ (x_2 \lor x_4), \]
\[ t_6(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \lor x_4) \circ (Q_1 \lor Q_2). \]

**Lemma 5.5.** (i) Let \( K = [1]_\Theta \) for some \( \Theta \in \text{Con} \mathbb{L} \) and \( t(x_1, \ldots, x_n, y_1, \ldots, y_k) \) be a term of \( \mathbb{L} \) such that \( t(x_1, \ldots, x_n, 1, \ldots, 1) = 1 \).

If \( y_1, \ldots, y_k \in K \) then \( t(x_1, \ldots, x_n, y_1, \ldots, y_k) \in K \).
(ii) For the terms \( t_1, \ldots, t_6 \) defined above, we have
\[
t_i(x_1, x_2, x_3, x_4, 1, 1, 1) = 1.
\]

**Proof.** The proof of (i) is elementary and hence omitted. (ii) is an easy consequence of \( q(x_1, x_2, 1, 1) = x_2 \) and \( q(x_3, x_4, 1, 1) = x_4 \). \( \square \)

**Lemma 5.6.** Let \( K \) be a subset of \( L \) with \( 1 \in K \). Define the relation \( \Phi_K \) on \( L \) by
\[
(x, y) \in \Phi_K \iff x \circ y \in K \quad \text{and} \quad y \circ x \in K.
\]

Then \( K = [1]_{\Phi_K} \).

**Proof.** If \( a \in K \), then \( 1 \circ a = a \in K \) and \( a \circ 1 = 1 \in K \), thus by (\*) \( (1, a) \in \Phi_K \) and so \( a \in [1]_{\Phi_K} \). Conversely, if \( a \in [1]_{\Phi_K} \), then (\*) gives \( a = 1 \circ a \in K \). Thus \( K = [1]_{\Phi_K} \). \( \square \)

A filter \( F \) of a lattice \( L \) is called **standard** if it is a standard element in the lattice \( \mathcal{F}(L) \) of all filters of \( L \), i.e. if the equality \([a] \land (F \lor [b]) = ([a] \land F) \lor [a \lor b] \) holds in \( \mathcal{F}(L) \).

**Theorem 5.7.** Let \( L = (L, \land, \lor, \circ, 1) \in \mathcal{P} \) and \( K \subseteq L \) with \( 1 \in K \). Then the following assertions are equivalent:

(i) \( K \) is a congruence kernel.

(ii) \( K \) is \( y \)-closed with respect to the terms \( t_1, \ldots, t_6 \).

(iii) The relation \( \Phi_K \) defined by (\*) is a congruence of \( \mathcal{L} \).

(iv) \( K \) is a standard filter of \( L \).

**Proof.** (i) \( \implies \) (ii): Assume that \( K = [1]_{\Theta} \) for some \( \Theta \in \text{Con}\mathcal{L} \). Then \( 1 \in K \) and, by Lemma 5.5, \( K \) is \( y \)-closed with respect to \( t_1, \ldots, t_6 \).

(ii) \( \implies \) (iii): Obviously, \( \Phi_K \) is reflexive. Suppose \( (a, b) \in \Phi_K \) and \( (c, d) \in \Phi_K \) for some \( a, b, c, d \in L \). Then \( a \circ b, b \circ a, c \circ d, d \circ c \in K \). Since in view of Remark 5.4(ii) we have
\[
a = q(a, b, s_1(a, b), s_2(a, b)) = q(a, b, a \circ b, b \circ a)
\]
and
\[
c = q(c, d, s_1(c, d), s_2(c, d)) = q(c, d, c \circ d, d \circ c),
\]
and since $K$ is $y$-closed with respect to $t_1$, applying the term $t_1$, we get 

\[(a \circ c) \circ (b \circ d) = (g(a, b, a \circ b, b \circ a) \circ q(c, d, c \circ d, d \circ c)) \circ (b \circ d) = t_1(a, b, c, d, a \circ b, b \circ a, c \circ d, d \circ c) \in K.\]

Analogously we can show $(b \circ d) \circ (a \circ c) \in K$ applying $t_2$ instead of $t_1$. Hence, by (*) we have also $(a \circ c, b \circ d) \in \Phi_K$.

Substituting $t_3$ and $t_4$ (instead of $t_1$ and $t_2$) in the above argument, we get $(a \land c, b \land d) \in \Phi_K$ and for $t_5, t_6$ we get $(a \lor c, b \lor d) \in \Phi_K$.

Together, $\Phi_K$ is a reflexive relation on $L$ having the substitution property with respect to all operations of $L$. As $L$ belongs to a Mal’cev variety, by the theorem of Werner [15] we obtain $\Phi_K \in \text{Con} L$.

(iii) $\implies$ (i): Since by Lemma 5.6 we have $K = [1]_{\Phi_K}$ and since by assumption $\Phi_K \in \text{Con} L$, $K$ is a congruence kernel.

(i) $\implies$ (iv): Assume that $\Theta$ is a congruence of $L$ such that $K = [1]_{\Theta}$. Since $\Theta$ is also a congruence of the lattice $(L, \land, \lor, 1)$, it is clear that $[1]_{\Theta}$ is a lattice filter. Suppose that $(x, y) \in \Theta$. Then $(x \land y, x \lor y) \in \Theta$, as $\Theta$ is a lattice congruence. Hence $(x \lor y) \circ (x \land y) \in K$, because $\Theta$ is a congruence on $L$. Since

\[x \land y = (x \lor y) \land [(x \lor y) \circ (x \land y)],\]

$K$ is a standard filter by [8, Theorem III.5].

(iv) $\implies$ (i): Assume that $K$ is a standard filter of $L$ and take $\Theta = \Theta[K]$ the smallest congruence on $L$ generated by $K$. This exists by [8, Theorem III.5] and it is easy to check that $K = [1]_{\Theta}$. We have only to show that $\Theta$ is compatible with the binary operation $\circ$. It is enough to show that the factor-lattice $L/\Theta$ is sectionally pseudocomplemented. More precisely, we claim that $[a]_{\Theta} \circ [b]_{\Theta} = [a \circ b]_{\Theta}$ for any $a, b \in L$. Really,

\[(a \circ b) \circ (a \circ b) = (a \lor b) \circ (a \lor b) = [(a \lor b) \land (a \lor b)]_{\Theta} = [b]_{\Theta} \circ [b]_{\Theta},\]

as $\Theta$ is a lattice congruence.

Now, take $[x]_{\Theta} \geq [b]_{\Theta}$ in $L/\Theta$ and suppose that

\[(a \circ b) \land [x]_{\Theta} = [(a \circ b) \land x]_{\Theta} = [b]_{\Theta}.\]

Without loss of generality we can assume $x \geq b$ in $L$. Then $(a \lor b) \land x \geq b$ and in view of [8, Theorem III.5] there exists a $g \in K$ such that $(a \lor b) \land x \land g = b$ holds in $L$. It follows that $x \land g \leq a \circ b$, as $L$ is sectionally pseudocomplemented. As $g \in K$, we have $[x \land g]_{\Theta} = [x]_{\Theta}$.  

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Finally, we obtain \([x]_\Theta \leq [a \circ b]_\Theta\) and hence \([a \circ b]_\Theta = [a]_\Theta \circ [b]_\Theta\), as claimed.

\[\Box\]

**Corollary 5.8.** For any \(\Theta \in \text{Con} L\) we have \(\Theta = \Phi_{[1]_\Theta}\).

**Proof.** Let \(\Theta \in \text{Con} L\) and take \(K = [1]_\Theta\). Then by Theorem 5.7 we have \(\Phi_K \in \text{Con} L\) and Lemma 5.6 gives \([1]_\Theta = [1]_{\Phi_K}\). As \(L\) is an algebra of a congruence 1-regular variety, we get \(\Theta = \Phi_K\), i.e. \(\Theta = \Phi_{[1]_\Theta}\). \(\Box\)

**Problems**

1) Characterize the subdirectly irreducible algebras in the varieties \(P_n\), \(n \in \mathbb{N}\).

2) Characterize the lattice of subvarieties of \(P\).

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