Square free part of products of consecutive integers

By ANIRBAN MUKHOPADHYAY (Allahabad) and T. N. SHOREY (Mumbai)

Dedicated to Professor K. Ramachandra on his 70th birthday

Abstract. Defining $\Delta(n, k) = n(n + 1) \cdots (n + k - 1)$, it is proved that, for $k \geq 10$ and $n > k^2$, there are at least 8 distinct primes exceeding $k$ dividing $\Delta(n, k)$ to odd powers except a few explicitly given values of $n$ and $k$. We also list all the squares which can be written as a product of $k - 2$ distinct terms out of $k$ consecutive positive integers.

1. Introduction

Let $n$ and $k \geq 3$ be positive integers. For an integer $\nu > 1$, we denote by $P(\nu)$ the greatest prime factor of $\nu$ and we write $P(1) = 1$. Further we put

$$\Delta(n, k) = n(n + 1) \cdots (n + k - 1).$$

We write $G = G(n, k)$ for the set of all $i$ with $0 \leq i \leq k - 1$ such that $n + i$ is divisible by a prime $> k$ to odd power. Further we denote by $G' = G'(n, k)$ the set of prime divisors of $\Delta(n, k)$ exceeding $k$. We put $g = g(n, k) = |G|$ and $g' = g'(n, k) = |G'|$. A theorem of SYLVESTER [12]

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dating back to 1892 states that 
\[ g' > 0 \text{ if } n > k. \]

Here the assumption \( n > k \) is necessary since \( P(1 \times 2 \times \cdots \times k) \leq k. \)

Further SARADHA and SHOREY [10] showed that 
\[ g' \geq \left\lceil \frac{\pi(k)}{3} \right\rceil + 2 \text{ if } n > k \]

unless
\[
\begin{align*}
  n \in \{4, 6, 7, 8, 16\} & \quad \text{if } k = 3; \\
  n \in \{6\} & \quad \text{if } k = 4; \\
  n \in \{6, 7, 8, 9, 12, 14, 15, 16, 23, 24\} & \quad \text{if } k = 5; \\
  n \in \{7, 8, 15\} & \quad \text{if } k = 6; \\
  n \in \{8, 9, 10, 12, 14, 15, 24\} & \quad \text{if } k = 7; \\
  n \in \{9, 14\} & \quad \text{if } k = 8; \\
  n \in \{14, 15, 16, 18, 20, 21, 24\} & \quad \text{if } k = 13; \\
  n \in \{15, 20\} & \quad \text{if } k = 14; \\
  n = \{20\} & \quad \text{if } k = 17.
\end{align*}
\]

We observe that 
\[ g' = \pi(2k) - \pi(k) \text{ if } n = k + 1. \]

Therefore \( \frac{1}{3} \) cannot be replaced by a constant larger than 1. SHANTA LAISHRAM and SHOREY [6] sharpened (1) for \( k \geq 19 \) to
\[ g' \geq \left\lceil \frac{3}{4} \pi(k) \right\rceil - 1 \text{ if } n > k \geq 19 \]
unless \((n,k)\) is given by

\[
\begin{align*}
\text{if } &k = 19; \quad n \in \{20 - 22, 24\} \quad \text{if } k = 20; \\
&k = 47; \quad n \in \{48 - 50, 54\} \quad \text{if } k = 48; \\
&k = 71; \quad n \in \{74, 75\} \quad \text{if } k = 72; \\
&k = 73; \quad n \in \{74 - 76, 84\} \quad \text{if } k = 79; \\
&k = 83; \quad n \in \{84, 90, 108, 110\} \quad \text{if } k = 83; \\
&k = 89; \quad n \in \{90, 102, 104\} \quad \text{if } k = 89; \\
&k = 103; \quad n \in \{108, 110, 111, 114, 115\} \quad \text{if } k = 103; \\
&k = 107; \quad n \in \{110, 114\} \quad \text{if } k = 107; \\
&k = 109; \quad n \in \{109 - 118\} \quad \text{if } k = 109; \\
&k = 111; \quad n \in \{111 - 117\} \quad \text{if } k = 111; \\
&k = 112; \quad n \in \{113 - 115\} \quad \text{if } k = 112; \\
&k = 113; \quad n \in \{114 - 120, 138, 140, 141\} \quad \text{if } k = 113; \\
&k = 114; \quad n \in \{115 - 119, 140\} \quad \text{if } k = 114; \\
&k = 115; \quad n \in \{116 - 118\} \quad \text{if } k = 115; \\
&k = 116; \quad n \in \{117\} \quad \text{if } k = 116; \\
&k = 173; \quad n \in \{174\} \quad \text{if } k = 173; \\
&k = 181; \quad n \in \{198, 200, 201\} \quad \text{if } k = 181; \\
&k = 193; \quad n \in \{200\} \quad \text{if } k = 193; \\
&k = 197; \quad n \in \{200\} \quad \text{if } k = 197; \\
&k = 200; \quad n \in \{200\} \quad \text{if } k = 200; \\
&k = 271; \quad n \in \{200 - 202\} \quad \text{if } k = 271; \\
&k = 272; \quad n \in \{282, 284, 285\} \quad \text{if } k = 272; \\
&k = 273; \quad n \in \{284\} \quad \text{if } k = 273; \\
&k = 277; \quad n \in \{278 - 280, 282 - 286\} \quad \text{if } k = 277; \\
&k = 278; \quad n \in \{279, 282 - 285\} \quad \text{if } k = 278; \\
&k = 279; \quad n \in \{282 - 284\} \quad \text{if } k = 279; \\
&k = 280; \quad n \in \{282\} \quad \text{if } k = 280; \\
&k = 281; \quad n \in \{282 - 288\} \quad \text{if } k = 281; \\
&k = 282; \quad n \in \{283 - 287\} \quad \text{if } k = 282; \\
&k = 283; \quad n \in \{284 - 288, 294\} \quad \text{if } k = 283; \\
&k = 284; \quad n \in \{285 - 287\} \quad \text{if } k = 284; \\
&k = 285; \quad n \in \{286\} \quad \text{if } k = 285; \\
&k = 293. \quad n \in \{294\} \quad \text{if } k = 293.
\end{align*}
\]
Thus the estimate (3) always holds for \( k > 293 \). Further they derived from their result that
\[
g' \geq \min \left( \left[ \frac{3}{4} \pi(k) \right] - 1, \, \pi(2k) - \pi(k) - 1 \right) \quad \text{if } n > k. \tag{5}
\]

Now we turn to giving lower bounds for \( g \). If \( k < n \leq k^2 \), we see that \( G = G' \) implying \( g = g' \) and lower bounds for \( g' \) have already been given above. Thus we assume that \( n > k^2 \). Erdős and Selfridge [4], developing on the method of Erdős [2] and Rigge [7], proved that there exists a prime \( p \geq k \) dividing \( \Delta(n, k) \) to odd power unless \( (n, k) = (48, 3) \). Further Saradha [9] sharpened the assertion \( p \geq k \) to \( p > k \) in the preceding result. Thus
\[ g \geq 1 \quad \text{if } (n, k) \neq (48, 3). \]

Next Saradha and Shorey [10] showed that
\[ g \geq 2 \quad \text{if } k \geq 4, \, (n, k) \neq (24, 4), (47, 4), (48, 4). \tag{6} \]

In fact (6) is stated in [10] for the number of distinct prime divisors \( > k \) dividing \( \Delta(n, k) \) to odd powers but it is clear from the proof that the assertion is valid for \( g \). We sharpen (6) as follows.

**Theorem 1.** Let \( k \geq 10 \) and \( n > k^2 \). Then
\[ g \geq 8 \]

unless
\[
\]
$n \in \{122 - 126, 140, 144, 158 - 162, 165 - 168, 188 - 192, 215, 216,$
$235 - 243, 287, 288, 375, 440, 480, 719, 720, 837 - 840, 1680,$
$2880, 5036 - 5040, 6718 - 6720, 15119, 15120\}$ \text{ if } k = 11;

$n \in \{158 - 160, 165, 189, 239 - 242\}$ \text{ if } k = 12;

$n \in \{188, 189, 240\}$ \text{ if } k = 13.

Since $x^2 - 2y^2 = -1$ has infinitely many solutions in integers $x$ and $y$, it is clear that the assumption $k \geq 10$ is necessary in Theorem 1. We also observe that $g \leq 7$ for every exception stated in Theorem 1. Further we notice that the number of distinct primes $> k$ dividing $\Delta(n, k)$ to odd powers is at least $g$. Therefore Theorem 1 implies the following results immediately.

**Corollary 1.** For $k \geq 10$ and $n > k^2$, there are at least 8 distinct primes exceeding $k$ dividing $\Delta(n, k)$ to odd powers unless

$n \in \{103 - 105, 112, 116 - 126, 144, 159 - 162, 166 - 168, 188, 189, 191,$
$192, 234 - 243, 287, 288, 354 - 360, 482, 483, 672, 717 - 721,$
$837 - 841, 1444, 5039\}$ \text{ if } k = 10;

$n \in \{122 - 126, 140, 144, 158 - 162, 165 - 168, 188 - 192, 235, 236, 240,$
$242, 287, 288, 719, 720, 837 - 840, 1680\}$ \text{ if } k = 11;

$n \in \{158 - 160, 165, 189\}$ \text{ if } k = 12;

$n \in \{188, 189, 240\}$ \text{ if } k = 13.

**Corollary 2.** Let $k \geq 10$ if $n \geq 5040$ and $k \geq 14$ otherwise. Assume that $n > k^2$. Then there are at least 8 distinct primes exceeding $k$ each dividing $\Delta(n, k)$ to odd power.

We observe that the exceptions mentioned in Corollary 1 are necessary. Sharper lower bounds for $g$ have been given when $n > k^2$ and $k$ is sufficiently large. **ERDŐS** [3] showed that

$$g \geq C_1 \frac{k}{\log k}$$
where $C_1 > 0$ is an effectively computable absolute constant. This has been improved by Shorey [11] to

$$g \geq C_2 \frac{k \log \log k}{\log k}$$

where $C_2 > 0$ is an effectively computable absolute constant. The improvement depends upon a theorem of Baker [1] that a hyperelliptic equation, under necessary assumptions, has only finitely many solutions. The constants $C_1$, $C_2$ turn out to be small and therefore, the above estimates for $g$ are of interest only when $k$ is large. As an immediate consequence of the result of Baker referred above, we have

$$g \geq k - 2$$

whenever $n \geq n_0(k)$ and $n_0(k)$ is a sufficiently large number depending only on $k$.

We shall derive Theorem 1 from the following more general result which also covers smaller values $k < 10$.

**Theorem 2.** Let $2 \leq g_1 \leq 7$, $k \geq 3 + g_1$ and $n > k^2$. Then all values of $n$ and $k$ for which $g = g_1$ are given in Table 1.

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Square free part of products of consecutive integers

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| Table 1 |
The assumption $2 \leq g_1 \leq 7$ in Theorem 2 can be relaxed and the assertion $g \geq 8$ in Theorem 1 can be strengthened but this will increase the computations and the number of exceptions. We prove Theorem 2 by induction and the first step of induction is given by (6). We write $G = \{i_1, \cdots, i_g\}$ with $i_1 < i_2 < \cdots < i_g$. Then

$$\frac{\Delta(n, k)}{\prod_{i \in G} (n + i)} = by^2$$

(7)

where $b$ and $y$ are positive integers such that $b$ is square free and $P(b) \leq k$. We derive from (7) that

$$n + i = a_i x_i^2 \quad \text{for} \quad 0 \leq i \leq k - 1, \quad i \not\in G$$

(8)

where $a_i$’s are square free positive integers with $P(a_i) \leq k$. Further we see that $a_i$’s are distinct whenever $n > k^2$. We observe that the assumptions $k \geq 3 + g_1$ and $n > k^2$ in Theorem 2 are necessary otherwise (7) has infinitely many solutions.

The proof of Theorem 2 depends on elementary and combinatorial arguments of Erdős [2] and Rigge [7] as developed by Erdős and Selfridge [4]. We shall also use SIMATH for solving elliptic curves

$$X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 12, \quad P(b) \leq 7$$

in positive integers $X$ and $Y$. We shall apply some combinatorial arguments to keep a check on the number of elliptic curves and securing the ones that can be solved by SIMATH.

We conclude from Theorem 2 that $g \geq 8$ unless $(n, k)$ with $k \geq 10$ is included in Table 1. This is the assertion of Theorem 1. By omitting all exceptions $(n, k)$ in Theorem 1 for which the number of distinct primes $> k$ dividing $\Delta(n, k)$ to odd power is at least 8, we conclude Corollary 1. For Corollary 2, we observe that there are no exceptions in Corollary 1 whenever $n \geq 5040$ or $k \geq 14$. 

<table>
<thead>
<tr>
<th>$g_1$</th>
<th>$k$</th>
<th>$n$</th>
<th>$g_1$</th>
<th>$k$</th>
<th>$n$</th>
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<tr>
<td>7</td>
<td>11</td>
<td>837-840</td>
<td>5</td>
<td>12</td>
<td>158-160</td>
<td>7</td>
<td>13</td>
<td>188-189</td>
</tr>
<tr>
<td>1680</td>
<td>4</td>
<td>6718-6720</td>
<td>165</td>
<td>7</td>
<td>189</td>
<td>240</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2880</td>
<td>15</td>
<td>15119-15120</td>
<td>1680</td>
<td>6</td>
<td>16518-16520</td>
<td>189</td>
<td>240</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 (continued)
Erdős [4] and Rigge [7], independently, proved that a product of two or more consecutive positive integers is never a square. Further Saradha and Shorey [10] showed that any product of distinct \( k - 1 \) terms out of \( k \) consecutive positive integers is a square only if

\[
\frac{6!}{5} = 12^2, \quad \frac{10!}{7} = 720^2.
\]  

(9)

This confirms a conjecture of Erdős and Selfridge [4, p. 300]. We re-write (9) as

\[
\frac{6!}{1 \cdot 5} = \frac{7!}{5 \cdot 7} = 12^2, \quad \frac{10!}{1 \cdot 7} = \frac{11!}{7 \cdot 11} = 720^2.
\]  

(10)

These may be viewed as examples of squares which are products of \( k - 2 \) distinct terms out of \( k \) consecutive positive integers. There are more examples:

\[
\begin{align*}
\frac{4!}{2 \cdot 3} &= 2^2, \quad \frac{6!}{4 \cdot 5} = 6^2, \quad \frac{8!}{2 \cdot 5 \cdot 7} = 24^2, \quad \frac{10!}{2 \cdot 3 \cdot 4 \cdot 6 \cdot 7} = 60^2, \quad \frac{9!}{2 \cdot 5 \cdot 7} = 72^2, \\
\frac{10!}{2 \cdot 3 \cdot 6 \cdot 7} &= 120^2, \quad \frac{10!}{2 \cdot 7 \cdot 8} = 180^2, \quad \frac{10!}{7 \cdot 9} = 240^2, \quad \frac{10!}{4 \cdot 7} = 360^2, \\
\frac{21!}{13! \cdot 17 \cdot 19} &= 5040^2, \quad \frac{14!}{2 \cdot 3 \cdot 4 \cdot 11 \cdot 13} = 5040^2, \quad \frac{14!}{2 \cdot 3 \cdot 11 \cdot 13} = 10080^2.
\end{align*}
\]  

(11)

We derive from Theorem 3 that there are no more.

**Corollary 3.** Let \( k \geq 4 \). A product of \( k - 2 \) distinct terms out of \( k \) consecutive positive integers is a square only if it is given by (10) and (11).

It is clear that the assumption \( k \geq 4 \) is necessary in Corollary 3 otherwise there are infinitely many solutions. Let \( \kappa(t) \) and \( \kappa'(t) \) be given by

\[
\kappa(2) = 8, \quad \kappa(3) = 9, \quad \kappa(4) = 11, \quad \kappa(5) = 15, \quad \kappa(6) = 16, \quad \kappa(7) = 24
\]  

(12)

and

\[
\kappa'(2) = 11, \quad \kappa'(3) = 25, \quad \kappa'(4) = 28, \\
\kappa'(5) = 30, \quad \kappa'(6) = 46, \quad \kappa'(7) = 50.
\]  

(13)
We prove

**Theorem 3.** Let $2 \leq t \leq 7$ and $k \geq 2 + t$. Let $d_1, d_2, \ldots, d_{k-t}$ be distinct integers in $[0, k-1]$. Assume that

$$(n + d_1)(n + d_2)\cdots(n + d_{k-t}) = z^2$$

where $z > 0$ is an integer. If $n > k^2$, then the solution of (14) are given by

$$240.243.245 = 3780^2, \quad 242.245.250 = 3850^2,$$

$$240.242.243.250 = 59400^2.$$  \hspace{1cm} (15)

Further

$$k \leq \kappa(t) \quad \text{if } k < n \leq k^2$$

and

$$k \leq \kappa'(t) \quad \text{if } n \leq k$$  \hspace{1cm} (17)

The proof of Theorem 3 depends on Theorem 2 and inequalities (1), (3), (5). The values $\kappa(t)$ and $\kappa'(t)$ given in (12) and (13) are optimal. For $3 \leq t \leq 7$, we can compute all squares which are products of $k-t$ distinct terms out of $k$ consecutive positive integers such that $t$ is minimal. But the number of these squares turn out to be much larger than given by (11) in the case $t = 2$. We shall follow the notation introduced in Section 1 throughout the paper. We shall use MATHEMATICA for computations in this paper.

We thank Shanta Laishram and the referee for their comments on an earlier version of this paper.

2. Lemmas

This section consists of lemmas for the proof of Theorem 2. We shall assume that $n > k^2$ throughout this section so that $a_i$ with $0 \leq i \leq k-1$ and $i \notin G$ are distinct. We begin with the following result which will be applied inductively on $g$ to assume without loss of generality that $k$ is prime if $k \geq 7$, $g = 2; k \geq 11, g = 3, 4, 5; k \geq 13, g = 6$ and $k \geq 17, g = 7$. This decreases the computational load considerably.
Lemma 1. Let \( n > 0 \) and \( t \geq 1 \) be integers. Assume that \( k_1 < k_2 \) be consecutive primes. Suppose that
\[
g(n, k) \neq t \quad \text{for } k \geq k_1
\] (18)
and
\[
g(n, k) \neq t + 1 \quad \text{for } k = k_1.
\] (19)

Then
\[
g(n, k) \neq t + 1 \quad \text{for } k_1 < k < k_2.
\]

Proof. Let \( k \) with \( k_1 < k < k_2 \) be given and assume that \( g(n, k) = t + 1 \). We put \( i_{r+1} = k - r \) and \( k = k_1 + j \) with \( j, r \) positive integers. If \( r \geq j + 1 \), we observe that \( t + 1 = g(n, k) = g(n, k_1) \neq t + 1 \) by (19). Thus \( r \leq j \). Then \( k_1 \leq k - r \) and \( g(n, k - r) = t \). This contradicts (18). \( \square \)

Let \( m \geq 1 \) be an integer. We denote by \( f(k, m, G) \) the number of \( a_i \)'s with \( 0 \leq i \leq k - 1 \) and \( i \not\in G \) composed of the first \( m \) primes \( 2 = p_1 < \cdots < p_m \). Then
\[
f(k, m, G) \geq f_0(k, m, g) := k - g - \sum_{j \geq m+1} \left( \left\lfloor \frac{k}{p_j} \right\rfloor + \epsilon_j \right)
\]
where \( \epsilon_j = 0 \) if either \( p_j > k \) or if \( p_j \mid k \) and \( \epsilon_j = 1 \) otherwise. Since \( a_i \)'s are square free, we see that \( f(k, m, G) \leq 2^m \) and hence
\[
f_0(k, m, g) \leq 2^m. \tag{20}
\]

This function with \( G = \Phi \) was introduced by Erdős and Selfridge [4]. We check the values of this function given in the next two lemmas.

Lemma 2. We have
\[
\begin{align*}
f_0(k, 3, 2) & \geq 9 & \text{for } 29 \leq k \leq 73 & \text{and} \\
f_0(k, 4, 2) & \geq 17 & \text{for } 74 \leq k \leq 216, & \\
f_0(k, 4, 3) & \geq 17 & \text{for } 53 \leq k \leq 263 & \text{and} \\
f_0(k, 5, 3) & \geq 33 & \text{for } 264 \leq k \leq 276, & \\
f_0(k, 4, 4) & \geq 17 & \text{for } 59 \leq k \leq 233 & \text{and}
\end{align*}
\]
We have

\begin{align*}
f_0(k, 5, 4) & \geq 33 \quad \text{for } 234 \leq k \leq 338, \\
f_0(k, 4, 5) & \geq 17 \quad \text{for } 67 \leq k \leq 229, \quad \text{and} \\
f_0(k, 5, 5) & \geq 33 \quad \text{for } 230 \leq k \leq 401, \\
f_0(k, 4, 6) & \geq 17 \quad \text{for } 83 \leq k \leq 211, \\
f_0(k, 5, 6) & \geq 33 \quad \text{for } 212 \leq k \leq 433, \quad \text{and} \\
f_0(k, 6, 6) & \geq 65 \quad \text{for } 434 \leq k \leq 466, \\
f_0(k, 4, 7) & \geq 17 \quad \text{for } 97 \leq k \leq 197, \\
f_0(k, 5, 7) & \geq 33 \quad \text{for } 198 \leq k \leq 433, \quad \text{and} \\
f_0(k, 6, 7) & \geq 65 \quad \text{for } 434 \leq k \leq 533.
\end{align*}

**Lemma 3.** We have

(i) \(f_0(5, 3, 2) = 3, \ f_0(6, 3, 2) = f_0(7, 3, 2) = 4, \ f_0(11, 3, 2) = f_0(13, 3, 2) = 6, \ f_0(17, 3, 2) = f_0(19, 3, 2) = f_0(23, 3, 2) = 7.\)

(ii) \(f_0(6, 3, 3) = 3, \ f_0(7, 3, 3) = 3, \ f_0(11, 3, 3) = f_0(13, 3, 3) = 5, \ f_0(17, 3, 2) = f_0(19, 3, 3) = f_0(23, 3, 3) = 6, \ f_0(29, 3, 3) = f_0(31, 3, 3) = 8.\)

(iii) \(f_0(7, 4, 4) = 3, \ f_0(11, 4, 4) = f_0(13, 4, 4) = 6, \ f_0(17, 4, 4) = f_0(19, 4, 4) = 8, \ f_0(23, 4, 4) = 9, \ f_0(29, 4, 4) = f_0(31, 4, 4) = 12, \ f_0(37, 4, 4) = f_0(41, 4, 4) = f_0(43, 4, 4) = f_0(47, 4, 4) = 14.\)

(iv) \(f_0(8, 4, 5) = 3, \ f_0(9, 4, 5) = 4, \ f_0(10, 4, 5) = 5, \ f_0(11, 4, 5) = f_0(13, 4, 5) = 5, \ f_0(17, 4, 5) = f_0(19, 4, 5) = 7, \ f_0(23, 4, 5) = 8, \ f_0(29, 4, 5) = f_0(31, 4, 5) = 11, \ f_0(37, 4, 5) = f_0(41, 4, 5) = f_0(43, 4, 5) = f_0(47, 4, 5) = 13, \ f_0(53, 4, 5) = 15.\)

(v) \(f_0(9, 4, 6) = 3, \ f_0(10, 4, 6) = 4, \ f_0(11, 4, 6) = f_0(13, 4, 6) = 4, \ f_0(17, 4, 6) = f_0(19, 4, 6) = 6, \ f_0(23, 4, 6) = 7, \ f_0(29, 4, 6) = f_0(31, 4, 6) = 10, \ f_0(37, 4, 6) = f_0(41, 4, 6) = f_0(43, 4, 6) = f_0(47, 4, 6) = 12, \ f_0(53, 4, 6) = 14, \ f_0(59, 4, 6) = f_0(61, 4, 6) = 15.\)

(vi) \(f_0(10, 4, 7) = f_0(11, 4, 7) = f_0(13, 4, 7) = 3, \ f_0(17, 4, 7) = f_0(19, 4, 7) = 5, \ f_0(23, 4, 7) = 6, \ f_0(29, 4, 7) = f_0(31, 4, 7) = 9,\)
\[ f_0(37, 4, 7) = f_0(41, 4, 7) = f_0(43, 4, 7) = f_0(47, 4, 7) = 11, \]
\[ f_0(53, 4, 7) = 13, \quad f_0(59, 4, 7) = f_0(61, 4, 7) = 14, \]
\[ f_0(67, 4, 7) = f_0(71, 4, 7) = f_0(73, 4, 7) = f_0(79, 4, 7) = 15. \]

The following result is due to Rosser and Schoenfeld [8, p. 69, 71].

**Lemma 4.** We have

(i) \[ \pi(2x) - \pi(x) > \frac{3x}{5 \log x} \text{ for } x \geq 20.5 \]

(ii) \[ \prod_{p \leq x} p < (2.78)^x. \]

We apply Lemmas 2 and 4 in the next result.

**Lemma 5.** Let \( 2 \leq g \leq 7, \ n > k^2 \) and \( k \) prime. Then \( k \leq k_0(g) \) where \( k_0(2) = 23, \ k_0(3) = 31, \ k_0(4) = 47, \ k_0(5) = 53, \ k_0(6) = 61 \) and \( k_0(7) = 79. \)

**Proof.** Suppose that the assumptions of Lemma 5 are satisfied. We recall that (7) holds and \( a_i \)'s are square free and they are distinct since \( n > k^2 \). Let \( R \) be the set of integers in \([0, k-1]\) which do not belong to \( G \). We give an upper bound and a lower bound for \( \prod_{i \in R} a_i \). For a prime \( p_0 \leq k \), we write

\[ \gamma_{p_0} = \text{ord}_{p_0} \left( \prod_{i \in \mathbb{R}} a_i \right). \]

Then

\[ \gamma_{p_0} \leq \left\lfloor \frac{k - 1}{p_0} \right\rfloor + 1. \]

Since

\[ \prod_{i \in \mathbb{R}} a_i = \prod_{p_0 \leq k} p_0^{\gamma_{p_0}}, \]

it follows that

\[ \prod_{i \in \mathbb{R}} a_i \bigg| \prod_{p_0 \leq k} p_0^{\left\lfloor \frac{k - 1}{p_0} \right\rfloor + 1}. \]

Thus

\[ \prod_{i \in \mathbb{R}} a_i \bigg| (k - 1)! \prod_{p_0 \leq k} p_0. \]
Let
\[\gamma'_p = \text{ord}_p \left( (k-1)! \prod_{p_0 \leq k} p_0 \right).\]

Let \(p_0^h \leq k - 1 < p_0^{h+1}\). Then
\[\gamma'_p = \left\lfloor \frac{k-1}{p_0} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p_0^h} \right\rfloor + 1.\]

We observe that \(\gamma_{p_0}\) is equal to the number of terms \(n + i\) with \(i \in \mathbb{R}\) divisible by \(p_0\) to an odd power. Let \(n + J\) for \(J \in \mathbb{R}\) be a term divisible by the maximal power of \(p_0\). We consider the set \(S = \{n + i : i \in \mathbb{R}, i \neq J\}\) and let \(\mu\) be a positive integer. Then the number of elements of \(S\) divisible by \(p_0^\mu\) is at most \([\left\lfloor (k-1)/p_0^\mu \right\rfloor]\) and at least \([\left\lfloor (k-1)/p_0^\mu \right\rfloor] - g - 1\). Thus
\[\gamma_{p_0} - \gamma'_p \leq (g + 1) \frac{(h + \epsilon - 1)}{2}
- 2 \left( \left\lfloor \frac{k-1}{p_0^h} \right\rfloor + \left\lfloor \frac{k-1}{p_0^{h+1}} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p_0^{h+\epsilon-1}} \right\rfloor \right)
\leq (g + 1) \frac{(h + \epsilon - 1)}{2} - 2 \left( \frac{k-1}{p_0^h} + \cdots + \frac{k-1}{p_0^{h+\epsilon-1}} - \frac{h + \epsilon - 1}{2} \right)
\leq (g + 3) \frac{(h + \epsilon - 1)}{2} - 2 \left( \frac{k-1}{p_0^h} - 1 \right) \left( 1 - \frac{1}{p_0^{h+\epsilon-1}} \right).
\]

Since \(p_0^{h+1} > k - 1\) and \(h < \log k / \log p_0\), we get
\[\gamma_{p_0} - \gamma'_p < \frac{\log k}{2 \log p_0} (g + 3) - \frac{2k}{p_0^2 - 1} + \delta_{p_0}\]
where
\[\delta_{p_0} = \frac{2 + 2p_0^2}{p_0^2 - 1}.\]
Square free part of products of consecutive integers

We observe that
\[ \prod_{i \in \mathbb{R}} a_i \bigg| (k - 1)! \prod_{p_0 \leq k} p_0 \prod_{p_0 \leq 7} \gamma_{p_0} - \gamma'_{p_0}. \]

We compute that
\[ \prod_{p_0 \leq 7} p_0^{\gamma_{p_0} - \gamma'_{p_0}} \leq 296001 k^{2g+6} (2.5907)^{-k}. \]

Thus
\[ \prod_{i \in \mathbb{R}} a_i \leq 296001 (k - 1)! k^{2g+6} (1.07307)^k \quad (21) \]

by Lemma 4. On the other hand, we see that
\[ \prod_{i \in \mathbb{R}} a_i \geq \prod_{i=1}^{k-g} s_i \quad (22) \]

where \( s_i \) denotes the \( i \)-th square free integer. Further
\[ \prod_{i=1}^{k-g} s_i \geq (k - g)! (1.5)^{k-g} \quad \text{for } k \geq 79. \quad (23) \]

We check (23) for \( k = 79 \). Then (23) follows immediately by induction on \( k \) from an inequality of ERDŐS [2] that \( s_i \geq (1.5)i \) for \( i \geq 39 \).

By combining (21), (22) and (23), we get
\[ (1.3978)^k \leq 296001 (1.5)^g k^{3g+5} \]

which implies that \( k \leq 216 \) if \( g = 2 \); \( k \leq 276 \) if \( g = 3 \); \( k \leq 338 \) if \( g = 4 \); \( k \leq 401 \) if \( g = 5 \); \( k \leq 466 \) if \( g = 6 \) and \( k \leq 533 \) if \( g = 7 \).

Now we apply Lemma 2 and (20). We conclude that \( k \leq 23 \) if \( g = 2 \); \( k \leq 47 \) if \( g = 3 \); \( k \leq 53 \) if \( g = 4 \); \( k \leq 61 \) if \( g = 5 \); \( k \leq 79 \) if \( g = 6 \) and \( k \leq 89 \) if \( g = 7 \). Thus it remains to exclude the cases \( k = 37, 41, 43, 47 \) if \( g = 3 \); \( k = 53 \) if \( g = 4 \); \( k = 59, 61 \) if \( g = 5 \); \( k = 67, 71, 73, 79 \) if \( g = 6 \) and \( k = 83, 89 \) if \( g = 7 \).

Let \( g = 3 \). We observe that \( f_0(37, 3, 3) = f_0(41, 3, 3) = 9 \) which imply that \( k \neq 37, 41 \) by (20). Let \( k = 43 \). Then the primes 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7 divide 1, 2, 2, 2, 2, 2, 3, 3, 4, 4, 7 distinct \( a_i \)’s,
or a which is not possible. When the latter possibility holds, we observe that divides a. This is not possible. Let \( k = 47 \). Then exactly 1, 2, 2, 2, 2, 3, 3, 3, 4, 5, 7 distinct \( a_i \)'s are divisible by 47, 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7, respectively, and none of these \( a_i \)'s is divisible by more than one of these primes. Hence 23 divides \( a_0, a_{23}, a_{46} \). Then 11 divides either \( a_1, a_{12}, a_{23}, a_{34}, a_{45} \) or \( a_2, a_{13}, a_{24}, a_{35}, a_{46} \) leading to a contradiction in either of the cases.

The proofs for the other cases are similar and we suppress some details. Let \( g = 4 \). We have \( f_0(53, 3, 4) = 8 \). Hence 13 divides \( a_0, a_{13}, a_{26}, a_{39}, a_{52} \) and 17 divides \( a_1, a_{18}, a_{35}, a_{52} \), a contradiction.

Let \( g = 5 \). Then \( f_0(61, 4, 5) = f_0(59, 4, 5) = 16 \). If \( k = 61 \), then 59 divides \( a_0, a_{59} \) or \( a_1, a_{60} \). If 59 divides \( a_0, a_{59} \), then 29 divides \( a_2, a_{31}, a_{60} \), 19 divides \( a_1, a_{20}, a_{39}, a_{58} \) and 11 divides \( a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58} \) which is not possible. If 59 divides \( a_1, a_{60} \), then 29 divides \( a_0, a_{29}, a_{58} \), 19 divides \( a_2, a_{21}, a_{40}, a_{50} \) and 11 divides \( a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58} \) which is impossible. When \( k = 59 \), then 29 divides \( a_0, a_{29}, a_{58} \) and 19 divides \( a_1, a_{20}, a_{39}, a_{58} \). This is a contradiction.

Let \( g = 6 \). We see that \( f_0(79, 4, 6) = f_0(73, 4, 6) = f_0(71, 4, 6) = f_0(67, 4, 6) = 16 \). Let \( k = 79 \). Then 13 divides \( a_0, a_{13}, a_{26}, a_{39}, a_{52}, a_{65}, a_{73} \) and 11 divide \( a_1, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{67}, a_{78} \). This is impossible. Let \( k = 73 \). Then 71 divides \( a_0, a_{71} \) or \( a_1, a_{72} \). Let 71 divide \( a_0, a_{71} \). Then 23 divides \( a_1, a_{24}, a_{47}, a_{70} \) or \( a_3, a_{26}, a_{49}, a_{72} \). Suppose that the first possibility holds. Then 17 divides \( a_4, a_{21}, a_{38}, a_{55}, a_{72} \) and 13 divides \( a_2, a_{15}, a_{28}, a_{41}, a_{54}, a_{67} \). Therefore 11 divides either \( a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58}, a_{69} \) or \( a_4, a_{15}, a_{26}, a_{37}, a_{48}, a_{59}, a_{70} \) but neither is possible. Let 23 divide \( a_3, a_{26}, a_{49}, a_{72} \). Then 17 divides either \( a_1, a_{18}, a_{35}, a_{52}, a_{69} \) or \( a_2, a_{19}, a_{36}, a_{53}, a_{70} \). In case of the former possibility, we see that 13 divides \( a_2, a_{15}, a_{28}, a_{41}, a_{54}, a_{67} \), 11 divides \( a_1, a_{15}, a_{26}, a_{37}, a_{48}, a_{59}, a_{70} \) which is not possible. When the latter possibility holds, we observe that 13 divides \( a_4, a_{17}, a_{30}, a_{43}, a_{56}, a_{69} \), 11 divides \( a_1, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{67} \), a contradiction. The case 71 dividing \( a_1, a_{72} \) is excluded similarly by considering divisibility of \( a_i \)'s by primes 23, 17, 13, 11 and 67. Let \( k = 71 \). We observe that 23 divides \( a_0, a_{23}, a_{46}, a_{69} \) or \( a_1, a_{24}, a_{47}, a_{70} \). In the first case, 17 divides \( a_2, a_{19}, a_{36}, a_{53}, a_{70} \). 67 divides \( a_1, a_{68} \) and hence 11 can not divide 7 distinct \( a_i \)'s. This is also the case whenever the latter
possibility holds. This is a contradiction. Let $k = 67$. We observe that 11 divides $a_0, a_{11}, a_{22}, a_{33}, a_{44}, a_{55}, a_{66}$ and 13 divides $a_1, a_{14}, a_{27}, a_{40}, a_{53}, a_{66}$ which is not possible.

Let $g = 7$. Then we have $f_0(83, 4, 7) = f_0(89, 4, 7) = 16$. Let $k = 89$. Then 11 divides $a_0, a_{11}, a_{22}, a_{33}, a_{44}, a_{55}, a_{66}, a_{77}, a_{88}$ and 29 divides $a_1, a_{30}, a_{59}, a_{88}$, a contradiction. Let $k = 83$. Then 41 divides $a_0, a_{41}, a_{82}$.

If 79 divides $a_1, a_{80}$, then 13 divides $a_3, a_{16}, a_{29}, a_{42}, a_{55}, a_{68}, a_{79}$ and 11 divides $a_2, a_{13}, a_{24}, a_{35}, a_{46}, a_{57}, a_{68}, a_{79}$ which is impossible. The case 79 dividing $a_2, a_{81}$ is excluded similarly. □

3. Proof of Theorem 2

Let $2 \leq g_1 \leq 7$, $k \geq 3 + g_1$ and $n > k^2$. Assume that $g = g_1$. We recall that (7) is valid. We first give a proof of Theorem 2 under the assumption that $k$ is prime if $k \geq 7$, $g = 2$; $k \geq 11$, $g = 3, 4, 5$; $k \geq 13$, $g = 6$ and $k \geq 17$, $g = 7$. We conclude from Lemma 5 that $k \leq k_0(g)$.

Let $g = 2$. We first consider $k = 23$. We have at least 7 $a_i$’s composed only of 2, 3, 5 by Lemma 3 (i). Hence there are at least 3 $a_i$’s such that the corresponding $i$’s belong to exactly one of the intervals $[0, 7], [8, 15], [16, 22]$. Therefore we see from (8) that

$$(n + i_0)(n + i_0 + p)(n + i_0 + q) = by^2, \quad 1 \leq p < q \leq 7, \quad P(b) \leq 5.$$  

We shall always denote by $i_0$ a non-negative integer and $X = b(n + i_0)$ in the proof of Theorem 2. Putting $b^2y = Y$, we get the following set of elliptic curves

$$X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 7, \quad P(b) \leq 5. \quad (24)$$

For $k = 19, 17, 13, 11, 7, 6, 5$, we divide $0 \leq i \leq k - 1$ into 3, 3, 2, 2, 1, 1, 1 parts, respectively, and apply Lemma 3(i) as above to obtain elliptic curves (24). Thus we need to solve (24) in integers.

We apply Lemma 3(ii), (iii), (iv), (v), (vi) as above according as $g = 3, 4, 5, 6, 7$, respectively. Then we obtain the following set of elliptic equations

$$X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 11, \quad P(b) \leq 5. \quad (25)$$
\[X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 7, \quad P(b) \leq 7, \quad (26)\]

\[X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 9, \quad P(b) \leq 7 \quad (27)\]

and

\[X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 12, \quad P(b) \leq 7 \quad (28)\]

according as \(g = 3, g = 4, 5, g = 6\) and \(g = 7\), respectively. For the preceding assertion, we need to make few additional observations in the cases \(k = 9, g = 5\) and \(k = 11, 13, g = 6\). Let \(k = 9, g = 5\). Then \(f_0(9, 4, 5) = 4\) and there are at least 3 \(a_i\)'s with \(1 \leq i \leq 8\) composed only of 2, 3, 5, 7. Let \(k = 11, g = 6\). Then \(f_0(11, 4, 6) = 4\) and there are at least 3 \(a_i\)'s with \(1 \leq i \leq 10\) composed only of 2, 3, 5, 7. Finally let \(k = 13, g = 6\). Then \(f_0(13, 4, 6) = 4\). We may assume that 11 divides \(a_0, a_11\) or \(a_1, a_12\). Thus we find at least three \(3 a_i\)'s with \(1 \leq i \leq 10\) or \(2 \leq i \leq 11\) composed only of 2, 3, 5, 7.

Now we use simath to solve the equations (24), (25), (26), (27) and (28). This was used for the first time in a similar context by Filakovszky and Hajdu [5]. Further we describe how to obtain Table 1 from the above solutions. Let \(2 \leq g \leq 7\) be given and we restrict to (24), (25), (26), (27), (28) according as \(g = 2, g = 3, g = 4, 5, g = 6\) and \(g = 7\), respectively. We observe that \(n + i_0 = X/b\) is an integer. Further \(k \leq \lfloor \sqrt{X/b} \rfloor\) since \(n > k^2\). Let \(K = \min\{\lfloor \sqrt{X/b} \rfloor, k_0(g)\}\). Thus

\[k \leq K. \quad (29)\]

Now \(i_0 + q \leq k - 1 \leq K - 1\) implying that \(0 \leq i_0 \leq K - q - 1\). Therefore

\[n \in [X/b - K + q + 1, X/b]. \quad (30)\]

For \(n, k\) satisfying (29) and (30), we include \((g, n, k)\) in Table 1 if and only if the number of \(i\) with \(0 \leq i \leq k - 1\) such that \(n + i\) is divisible by a prime > \(k\) to odd power is exactly equal to \(g\). We explain the above argument in the case \(g = 2\). Thus we need to solve (24). For example, we consider (24) with \(p = 3, q = 4\) and \(b = 15\). We have \(X = 15(n + i_0) > 15k^2\) and \(k \geq 5\). Now we conclude by simath that \(X = 675\). Then \(K = 6\) and \(k = 5, 6\) by (29). Further \(n = 44, 45\) by (30). Thus we need to consider only the pairs \((n, k) = (44, 5), (44, 6), (45, 5), (45, 6)\). The first two pairs
are excluded since \( g = 3 \) for both. On the other hand, we find that \( g = 2 \) for the last two pairs. Hence the values of \( n \) and \( k \) corresponding to these pairs are included in Table 1 against \( g = 2 \).

For a composite \( k \), it remains to show that \( g(n, k) \neq 2, 3, 4, 5, 6, 7 \) according as \( k \) exceeds 7, 11, 11, 13, 17, respectively. Let \( k' > 7 \) be composite. Let \( k_1 \geq 7 \) and \( k_2 \) be consecutive primes such that \( k_1 < k' < k_2 \). As shown above, \( g(n, k_1) \neq 2 \) and \( g(n, k) \neq 1 \) for every \( k \geq k_1 \) by (6). Therefore the assumptions of Lemma 1 with \( t = 1 \) are satisfied. Hence we derive from Lemma 1 with \( t = 1 \) that \( g(n, k') \neq 2 \). Thus \( g(n, k) \neq 2 \) for every \( k \geq 7 \). As proved above, \( g(n, k) \neq 3 \) whenever \( k \geq 11 \) is prime. Now we conclude from Lemma 1 with \( t = 2 \) that \( g(n, k) \neq 3 \) for every \( k \geq 11 \). Further we apply Lemma 1 inductively with \( t = 3, 4, 5, 6 \) to complete the proof of Theorem 2.

\[ \Box \]

4. Proof of Theorem 3

Let \( 2 \leq t \leq 7 \), \( k \geq 2 + t \) and we assume (14). Let \( k = 2 + t \). We may assume that all the solutions of (14) are given by \( 1 \cdot 4 = 2^2 \), \( 1 \cdot 9 = 3^2 \), \( 2 \cdot 8 = 4^2 \), \( 4 \cdot 9 = 6^2 \), \( 9 \cdot 16 = 12^2 \). The last one is covered by (17) and the remaining ones by (16). Thus we may assume that \( k > 2 + t \). Further we observe from (7) that \( g \leq t \).

Let \( n > k^2 \). Then \( g \geq 2 \) by (6) and the assertion of Theorem 2 holds. If \( t = 7 \), then \( k \geq 10 \) and \( g \geq 5 \) by Table 1. Similarly \( g \geq 4 \) if \( t = 5, 6 \) and \( g \geq 3 \) if \( t = 4 \). Further we check whether every possible product of \( k - t \) distinct integers out of \( k - g \) integers \( n + i \) with \( 0 \leq i \leq k - 1 \) and \( i \notin G \) is a square. We find that all the solutions of (14) are given by (15). Thus we may assume that \( n \leq k^2 \).

Let \( n > k \). Then \( g' = g \leq t \). Now we apply (1) to derive \( k \leq \kappa_1(t) \) where

\[ \kappa_1(2) = 8, \kappa_1(3) = 12, \kappa_1(4) = 22, \]

\[ \kappa_1(5) = 36, \kappa_1(6) = 46, \kappa_1(7) = 60. \]

Further we apply (3) to sharpen \( k \leq \kappa_1(t) \) to \( k \leq \kappa_2(t) \) with \( t \geq 4 \) where

\[ \kappa_2(2) = 8, \kappa_2(3) = 12, \kappa_2(4) = 18, \]

\[ \kappa_2(5) = 28, \kappa_2(6) = 30, \kappa_2(7) = 36. \]
While applying (3), we check that the exceptions given in (4) with $\kappa_1(t) < k \leq \kappa_2(t)$, $t \geq 5$ are excluded by noting that $\pi(2k) - \pi(k) - 1 > t$ and (5) holds. Further the exceptions given in (4) with $\kappa_1(4) < k \leq \kappa_2(4)$ are excluded by direct computations. Finally we conclude (16) from $k \leq \kappa_2(t)$ by computations as above in the case $n > k^2$.

Therefore we may suppose that $n \leq k$. Then $n \leq (n+k)/2 < n+k-1$ and we see from (14) that

$$\pi(n+k-1) - \pi\left(\frac{n+k}{2}\right) \leq t.$$ 

This implies that $n+k \leq 122$ by Lemma 4. This is improved to $n+k \leq 66$ by using exact values of $\pi$ function. Thus $k \leq 65$ which we sharpen to (17) by checking whether a product of $k-t$ distinct integers $n+i$ with $0 \leq i \leq k-1$ such that $n+i$ is composite whenever $i > (k-n)/2$, is a square.

5. Proof of Corollary 3

Assume (14) with $t = 2$. Then we conclude from Theorem 3 that $n \leq k^2$. Further $k \leq 8$ if $n > k$, $k \leq 11$ if $n \leq k$ and the assertion of Corollary 3 follows by computations as in the proof of Theorem 3. 

References

Square free part of products of consecutive integers


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