Transverse totally geodesic submanifolds
of the tangent bundle

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Abstract. It is well-known that if \( \xi \) is a smooth vector field on a given Riemannian manifold \( M^n \) then \( \xi \) naturally defines a submanifold \( \xi(M^n) \) transverse to the fibers of the tangent bundle \( TM^n \) with Sasaki metric. In this paper, we are interested in transverse totally geodesic submanifolds of the tangent bundle. We show that a transverse submanifold \( N^l \) of \( TM^n \) \((1 \leq l \leq n)\) can be realized locally as the image of a submanifold \( F^l \) of \( M^n \) under some vector field \( \xi \) defined along \( F^l \). For such images \( \xi(F^l) \), the conditions to be totally geodesic are presented. We show that these conditions are not so rigid as in the case of \( l = n \), and we treat several special cases (\( \xi \) of constant length, \( \xi \) normal to \( F^l \), \( M^n \) of constant curvature, \( M^n \) a Lie group and \( \xi \) a left invariant vector field).

Introduction

Let \( (M^n,g) \) be a Riemannian manifold and \( (TM^n,g_s) \) its tangent bundle equipped with the Sasaki metric [12]. Let \( \xi \) be a given smooth vector field on \( M^n \). Then \( \xi \) naturally defines a mapping \( \xi : M^n \rightarrow TM^n \) such that the submanifold \( \xi(M^n) \subset TM^n \) is transverse to the fibers. This fact allows to ascribe to the vector field \( \xi \) some geometrical characteristics from the geometry of submanifolds. We say that the vector field \( \xi \) is minimal,
totally umbilic or totally geodesic if $\xi(M^n)$ possesses the same property. In a similar way we can say about the sectional, Ricci or scalar curvature of a vector field. For the case of a unit vector field this approach has been proposed by H. Gluck and W. Ziller [6]. They proved that the Hopf vector field $h$ on three-sphere $S^3$ is one with globally minimal volume, i.e. $h(S^3)$ is a globally minimal submanifold in the unit tangent bundle $T_1S^3$. Corresponding local consideration leads to the notion of the mean curvature of a unit vector field and a number of examples of locally minimal unit vector fields were found based on a preprint version of [5] (see [1], [2], [7] and references). In a different way, the second author found examples of unit vector fields of constant mean curvature [18] and completely described the totally geodesic unit vector fields on 2-dimensional manifolds of constant curvature [19]. The energy of a mapping $\xi : M^n \to T_1M^n$ can also be ascribed to the vector field $\xi$ and we can say about the energy of a unit vector field (see [17], [4], [15] and references).

In contrast to unit vector fields, there are few results (both of local or global aspects) on the geometry of general vector fields treated as submanifolds in the tangent bundle. It is known [10] that if $\xi$ is the zero vector field, then $\xi(M^n)$ is totally geodesic in $TM^n$. P. Walczak [14] treated the case when $\xi$ is a non-zero vector field on $M^n$ and proved that if $\xi$ is a parallel vector field on $M^n$, then $\xi(M^n)$ is totally geodesic in $TM^n$. Moreover, if $\xi$ is of constant length, then $\xi(M^n)$ is totally geodesic in $TM^n$ if and only if $\xi$ is a parallel vector field on $M^n$. The latter condition is rather burdensome. The basic manifold $M^n$ should be a metrical product $M^{n-k} \times E^k (k \geq 1)$, where $E^k$ is a Euclidean (flat) factor.

Remark that $\xi(M^n)$ has maximal dimension among submanifolds in the tangent bundle, transverse to the fibers. In this paper, we study submanifolds $N^l$ of $TM^n$ with $l \leq n$ which are transverse to the fibers. We show in Section 2 that any transverse submanifold $N^l$ of $TM^n$ can be realized locally as the image of a submanifold $F^l$ of $M^n$ under some vector field $\xi$ defined along $F^l$. We also investigate some cases when the image can be globally realized. Mainly, we are interested in submanifolds among this class which are totally geodesic. In this way, we get a chain of inclusions:

$$\xi(F^l) \subset \xi(M^n) \subset TM^n.$$ 

In comparison with the case when $\xi$ is defined over the whole $M^n$ or, at
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least, over a domain $D^n \subset M^n$ as in [14], the picture becomes different, because $\xi(F^l)$ can be totally geodesic in $TM^n$ while $\xi(M^n)$ is not. Our considerations include also the case when the vector field is defined only on $F^l$, so that $\xi$ defines a “direct” embedding $\xi : F^l \to TM^n$.

For $l = 1$ we get nothing else but a vector field along a curve in $M^n$ which generates a geodesic in $TM^n$. S. Sasaki [12] described geodesic lines in $TM^n$ in terms of vector fields along curves in $M^n$ and found the differential equations on the curve and the corresponding vector field. Moreover, in the case when $M^n$ is of constant curvature, K. Sato [13] explicitly described the curves and the vector fields.

Evidently, our approach takes an intermediate position between the above mentioned considerations for $l = 1$ and $l = n$. Necessary and sufficient conditions on $\xi(F^l)$ to be totally geodesic, that we make explicit in Section 3 (Proposition 3.1), have a clearer geometrical meaning if we suppose that $\xi$ is of constant length along $F^l$ (Theorem 3.2) or is a normal vector field along $F^l$ (Theorem 3.3). Indeed, an application of Theorem 3.3 to the specific case of foliated Riemannian manifolds allows us to clarify the geometrical structure of $\xi(M^n)$ (Corollary 3.5).

The case of a base space $M^n$ of constant curvature is discussed in detail in Section 4. An application to the case of a Riemannian manifold of constant curvature enlightens us as to the non rigidity of the totally geodesic property of $\xi(F^l), l < n$, contrary to the case $l = n$.

Finally, an application of our results to Lie groups endowed with bi-invariant metrics gives a clear geometrical picture of our problem.

**Remark.** Throughout the paper

- $M^n$ is a given Riemannian manifold with metric $\bar{g}$, $F^l$ is a submanifold of $M^n$ with the induced metric $g$, $TM^n$ is the tangent bundle of $M^n$ equipped with the Sasaki metric $g_s$;
- $\bar{\nabla}, \nabla, \tilde{\nabla}$ are the Levi–Civita connections with respect to $\bar{g}, g, g_s$ respectively;
- the indices range is fixed as $a, b, c = 1 \ldots n; i, j, k = 1 \ldots l$;
- all the vector fields are supposed sufficiently smooth, say of class $C^\infty$. 


1. Local geometry of $\xi(F)$

1.1. Tangent bundle of $\xi(F)$. Let $(M^n, \bar{g})$ be an $n$-dimensional Riemannian manifold with metric $\bar{g}$. Denote by $\bar{g}(\cdot, \cdot)$ the scalar product with respect to $\bar{g}$. The Sasaki metric $g_s$ on $TM^n$ is defined by the following scalar product: if $\tilde{X}, \tilde{Y}$ are tangent vector fields on $TM^n$, then

$$g_s(\tilde{X}, \tilde{Y}) = \bar{g}(\pi_* \tilde{X}, \pi_* \tilde{Y}) + \bar{g}(K \tilde{X}, K \tilde{Y})$$

(1)

where $\pi_* : TTM^n \to TM^n$ is the differential of the projection $\pi : TM^n \to M^n$ and $K : TTM^n \to TM^n$ is the connection map [3]. The local representations for $\pi_*$ and $K$ are the following ones. Let $(x^1, \ldots, x^n)$ be a local coordinate system on $M^n$. Denote by $\partial/\partial x^a$ the natural tangent coordinate frame. Then, at each point $x \in M^n$, any tangent vector $\xi$ can be decomposed as $\xi = \xi_a \partial/\partial x^a(x)$. The set of parameters $\{x^1, \ldots, x^n; \xi^1, \ldots, \xi^n\}$ forms the natural induced coordinate system in $TM^n$, i.e. for a point $z = (x, \xi) \in TM^n$, with $x \in M^n$, $\xi \in T_x M^n$, we have $x = (x^1, \ldots, x^n)$, $\xi = \xi_a \partial/\partial x^a(x)$. The natural frame in $T_z TM^n$ is formed by $\{\partial/\partial x^a(z), \partial/\partial \xi^a(z)\}$ and for any $\tilde{X} \in T_z TM^n$ we have the decomposition $\tilde{X} = \tilde{X}^a \partial/\partial x^a(z) + \tilde{X}^{n+a} \partial/\partial \xi^a(z)$. Now locally, the horizontal and vertical projections of $\tilde{X}$ are given by

$$\pi_* \tilde{X} = \tilde{X}^a \partial/\partial x^a(\pi(z)),$$

$$K \tilde{X} = (\tilde{X}^{n+a} + \tilde{\Gamma}_{bc}^a(\pi(z))\xi^b \tilde{X}^c) \partial/\partial x^a(\pi(z)),$$

(2)

where $\tilde{\Gamma}_{bc}^a$ are the Christoffel symbols of the metric $\bar{g}$. The inverse operations are called lifts. If $\tilde{X} = \tilde{X}^a \partial/\partial x^a$ is a vector field on $M^n$ then the vector fields on $TM$ given by

$$\tilde{X}^h = \tilde{X}^a \partial/\partial x^a - \tilde{\Gamma}_{bc}^a \xi^b \tilde{X}^c \partial/\partial \xi^a,$$

$$\tilde{X}^v = \tilde{X}^a \partial/\partial \xi^a$$

are called the horizontal and vertical lifts of $X$ respectively. Remark that for any vector field $\tilde{X}$ on $M^n$ it holds

$$\pi_* \tilde{X}^h = \tilde{X}, \quad K \tilde{X}^h = 0,$$

$$\pi_* \tilde{X}^v = 0, \quad K \tilde{X}^v = \tilde{X}.$$  

(3)
Let \( F^l \) be an \( l \)-dimensional submanifold in \( M^n \) with a local representation given by
\[
x^a = x^a(u^1, \ldots, u^l).
\]

Let \( \xi \) be a vector field on \( M^n \) defined in some neighborhood of (or only on) the submanifold \( F^l \). Then the restriction of \( \xi \) to the submanifold \( F^l \), called a vector field on \( M^n \) along \( F^l \), generates a submanifold \( \xi(F^l) \subset TM^n \) with a local representation of the form
\[
\xi(F^l) : \begin{cases} 
x^a = x^a(u^1, \ldots, u^l), \\
\xi^a = \xi^a(x^1(u^1, \ldots, u^l), \ldots, x^n(u^1, \ldots, u^l)).
\end{cases}
\]

In what follows we will refer to the submanifold (4) as to one generated by a vector field on \( M^n \) along \( F^l \).

The following proposition describes the tangent space of \( \xi(F^l) \).

**Proposition 1.1.** A vector field \( \tilde{X} \) on \( TM^n \) is tangent to \( \xi(F^l) \) along \( \xi(F^l) \) if and only if its horizontal-vertical decomposition is of the form
\[
\tilde{X} = X^h + (\tilde{\nabla}_X \xi)^v,
\]
where \( X \) is a tangent vector field on \( F^l \), \( \tilde{\nabla}_X \xi \) is the covariant derivative of \( \xi \) in the direction of \( X \) with respect to the Levi–Civita connection of \( M^n \) and the lifts are considered as those on \( TM^n \).

**Proof.** Let us denote by \( \tilde{e}_i \) the vectors of the coordinate frame of \( \xi(F^l) \). Then, evidently,
\[
\tilde{e}_i = \left\{ \frac{\partial x^1}{\partial u^i}, \ldots, \frac{\partial x^n}{\partial u^i} ; \frac{\partial \xi^1}{\partial u^i}, \ldots, \frac{\partial \xi^n}{\partial u^i} \right\}.
\]

Applying (2), we have
\[
\pi_* \tilde{e}_i = \frac{\partial x^a}{\partial u^i} \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u^i},
\]
\[
K \tilde{e}_i = \left( \frac{\partial \xi^a}{\partial u^i} + \tilde{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i} \right) \frac{\partial}{\partial x^a} = \left( \frac{\partial \xi^a}{\partial x^c} \frac{\partial x^c}{\partial u^i} + \tilde{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i} \right) \frac{\partial}{\partial x^a}
\]
\[
= \frac{\partial x^c}{\partial u^i} \left( \frac{\partial \xi^a}{\partial x^c} + \tilde{\Gamma}^a_{bc} \xi^b \right) \frac{\partial}{\partial x^a} = \tilde{\nabla}_i \xi,
\]
where $\bar{\Gamma}^{a}_{bc}$ are the Christoffel symbols of the metric $\bar{g}$ taken along $F^l$ and $\bar{\nabla}_i$ means the covariant derivative of a vector field on $M^n$ with respect to the Levi–Civita connection of $\bar{g}$ along the $i$-th coordinate curve of the submanifold $F^l \subset M^n$. Summing up, we have

$$\tilde{e}_i = \left( \frac{\partial}{\partial u^i} \right)^h + (\bar{\nabla}_i \xi)^v.$$  \hspace{1cm} (5)

Let $\tilde{X}$ be a vector field on $TM^n$ tangent to $\xi(F^l)$ along $\xi(F^l)$. Then the following decomposition holds $\tilde{X} = \tilde{X}_i \tilde{e}_i$. Set $X = \tilde{X}_i \partial/\partial u^i$. The vector field $X$ is tangent to $F^l$ and, taking into account (5), the decomposition of $\tilde{X}$ can be represented as $\tilde{X} = X^h + (\bar{\nabla}_X \xi)^v$, which completes the proof. □

**Corollary 1.1.** Let $(F^l,g)$ be a submanifold of a Riemannian manifold $(M^n,\bar{g})$ with the induced metric. Let $\xi$ be a vector field on $M^n$ along $F^l$. Then the metric on $\xi(F^l)$, induced by the Sasaki metric of $TM^n$, is defined by the following scalar product

$$g_s(\tilde{X},\tilde{Y}) = g(X,Y) + \bar{g}(\bar{\nabla}_X \xi, \bar{\nabla}_Y \xi),$$

for all vector fields $\tilde{X} = X^h + (\bar{\nabla}_X \xi)^v$ and $\tilde{Y} = Y^h + (\bar{\nabla}_Y \xi)^v$ on $\xi(F^l)$, where $X, Y$ are vector fields on $F^l$.

**1.2. Normal bundle of $\xi(F^l)$.** To describe the normal bundle of $\xi(F^l)$, we need one auxiliary notion. Let $\xi$ be a given vector field on a submanifold $F^l \subset M^n$. Then $\bar{\nabla}$ enables us to define a point-wise linear mapping $\bar{\nabla}_\xi : T_xF^l \to T_xM^n$, $X \mapsto \bar{\nabla}_X \xi$, for all $x \in M^n$. Its dual mapping, with respect to the corresponding scalar products induced by $g$ and $\bar{g}$, gives rise to the linear mapping $(\nabla_\xi)^* : T_xM^n \to T_xF^l$ defined by the formula

$$g((\nabla_\xi)^* W, X) = \bar{g}(\bar{\nabla}_X \xi, W)$$

for all $W \in T_xM^n$ and $X \in T_xF^l$. \hspace{1cm} (6)

We call the mapping $(\nabla_\xi)^* : T_xM^n \to T_xF^l$ the conjugate derivative mapping, or simply conjugate derivative. Remark, that if $W$ is a vector field on $M^n$, then the application of $(\nabla_\xi)^*$ gives rise to a vector field $(\nabla_\xi)^* W$ on $F^l$ by $[(\nabla_\xi)^* W]_x = (\nabla_\xi)^* W_x \in T_xF^l$ for all $x \in F^l$.

Now we can prove
Proposition 1.2. Let η and Z be normal and tangent vector fields on $F^l$ respectively. Then the lifts

\[ \eta^h, \eta^v - ((\bar{\nabla}\xi)^*\eta)^h, Z^v - ((\bar{\nabla}\xi)^*Z)^h \]

to the points of $\xi(F^l)$ span the normal bundle of $\xi(F^l)$ in $TM^n$.

Proof. Let $\tilde{X} = X^h + (\bar{\nabla}_X\xi)^v$ be a vector field on $\xi(F^l)$. Let η and Z be vector fields on $F^l$ which are normal and tangent to $F^l$ respectively. Taking into account (1), (3) and (6), we have

\[ g_s(\tilde{X}, \eta^h) = \bar{g}(X, \eta) = 0 \]
\[ g_s(\tilde{X}, \eta^v - [(\bar{\nabla}_X\xi)^*\eta]^h) = -\bar{g}(X, (\bar{\nabla}_X\xi)^*\eta) + \bar{g}(\bar{\nabla}_X\xi, \eta) = 0 \]
\[ g_s(\tilde{X}, Z^v - [(\bar{\nabla}_X\xi)^*Z]^h) = -\bar{g}(X, (\bar{\nabla}_X\xi)^*Z) + \bar{g}(\bar{\nabla}_X\xi, Z) = 0 \]

Let $\eta_1, \ldots, \eta_p$ ($p = 1, \ldots, n-l$) be a normal frame of $F^l$ while $f_1, \ldots, f_i$ span $T_x F^l$ at each point $x \in F^l$. Consider the vector fields

\[ N_\alpha = \eta_\alpha^h, \quad P_\alpha = \eta_\alpha^v - ((\bar{\nabla}_X\xi)^*\eta_\alpha)^h, \quad F_i = f_i^v - ((\bar{\nabla}_X\xi)^*e_i)^h, \]

where $\alpha = 1, \ldots, n-l; i = 1, \ldots, n$. Let us show that these are linearly independent. Indeed, suppose that

\[ \lambda^\alpha N_\alpha + \mu^\alpha P_\alpha + \nu^i F_i \]
\[ = \{\lambda^\alpha \eta_\alpha - \mu^\alpha (\bar{\nabla}_X\xi)^*\eta_\alpha - \nu^i (\bar{\nabla}_X\xi)^*e_i\}^h + \{\mu^\alpha \eta_\alpha + \nu^i f_i\}^v = 0. \]

Because of the fact that the horizontal and vertical components are linearly independent, we see that $\mu^\alpha \eta_\alpha + \nu^i f_i = 0$ which is possible iff $\mu^\alpha = 0$, $\nu^i = 0$. Then, from the horizontal part of the decomposition above we see that $\lambda^\alpha = 0$. So, $N_\alpha, P_\alpha$ and $F_i$ are linearly independent, which completes the proof. \(\square\)

Remark. In the case when $\xi$ is a normal vector field, the images $(\bar{\nabla}_X\xi)^*\eta$ and $(\bar{\nabla}_X\xi)^*Z$ have a simple and natural meaning, namely

\[ (\bar{\nabla}_X\xi)^*\eta = g^{ik} g(\bar{\nabla}^+_k \xi, \eta) \frac{\partial}{\partial u^l}, \quad (\bar{\nabla}_X\xi)^*Z = -A_\xi Z, \]
where $\nabla_{\perp}$ is the normal bundle connection of $F^l$ and $A_\xi$ is the shape operator of $F^l$ with respect to the normal vector field $\xi$. In fact, $(\nabla_\xi)^* \eta$ is the vector field on $F^l$ dual to the $1$-form $g(\nabla_{\perp}^* \xi, \eta) du^k$.

2. Characterization of submanifolds of $TM^n$ transverse to fibers

It is clear that all totally geodesic vector fields along submanifolds of $M^n$ generate submanifolds in $TM^n$ which are transverse to the fibers of $TM^n$. We study in this section the converse question. We start with the local case.

**Proposition 2.1.** Let $N^l$ be an embedded submanifold in the tangent bundle of a Riemannian manifold $M^n$, which is transverse to the fiber at a point $z \in N^l$, then there is a submanifold $F^l$ of $M^n$ containing $x = \pi(z)$, a neighborhood $U$ of $x$ in $M^n$, a neighborhood $V$ of $z$ in $TM^n$ and a vector field $\xi$ on $M^n$ along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$.

**Proof.** Since $T_x N^l$ is transverse to the vertical subspace $V_x TM^n$ of $TTM^n$ at $z$, $\pi_* | T_x N^l : T_x N^l \to T_x M^n$ is injective, and so there is an open neighborhood $W$ of $z$ in $TM^n$ such that $\pi_* | T_{z'} N^l : T_{z'} N^l \to T_{\pi(z')} M^n$ is injective for all $z' \in W \cap N^l$. Hence $\pi | W \cap N^l : W \cap N^l \to M^n$ is an immersion, and thus there exist a cubic centered coordinate system $(U, \varphi)$ about $x = \pi(z)$ and a neighborhood $V$ of $z$ in $W$ such that $\pi | V \cap N^l$ is $1:1$ and $\pi(V \cap N^l)$ is a part of a slice $F^l$ of $(U, \varphi)$ [16, p. 28]. The slice $F^l$ is a submanifold of $M^n$ and we have $\pi | V \cap N^l : V \cap N^l \to U \cap F^l$ is an imbedding onto, and so there is a $C^\infty$-mapping $\xi : F^l \cap U \to N^l \cap V$ such that $\pi \circ \xi = Id_{F^l \cap U}$. In other words, $\xi$ is a vector field on $M^n$ along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$. \qed

The global version of the last result requires further conditions.

**Theorem 2.1.** Let $N^n$ be a connected compact $n$-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold $M^n$, which is everywhere transverse to the fibers of $TM^n$. Then $M^n$ is also compact, and there is a vector field $\xi$ on $M^n$ such that $\xi(M^n) = N^n$. 


Proof. The fact that $N^n$ is everywhere transverse to the fibers of $TM^n$ implies that $\pi \upharpoonright N^n : N^n \to M^n$ is an immersion. Since $M^n$ and $N^n$ are connected of the same dimension and $N^n$ is compact, then $M^n$ is compact and $\pi \upharpoonright N^n$ is a covering projection (cf. [8, Vol. 1, p. 178]). Now, $M^n$ is simply connected and so $\pi \upharpoonright N^n$ is a diffeomorphism. Let $\xi : M^n \to N^n$ be the inverse of $\pi \upharpoonright N^n$. Then $\xi$ is a vector field on $M^n$ and $\xi(M^n) = N^n$. □

In a similar way, we can show the following:

Theorem 2.2. Let $N^l$ be a connected compact submanifold of the tangent bundle of a connected simply connected manifold $M^n$, which is transverse to the fibers it meets and projects onto a simply connected submanifold $F^l$ of $M^n$. Then $F^l$ is compact and there is a vector field $\xi$ on $M^n$ along $F^l$ such that $\xi(F^l) = N^l$.

In the particular case of horizontal totally geodesic submanifolds of $TM^n$, i.e. whose tangent space at any point is horizontal, we can state the following:

Theorem 2.3. Let $N^l$ be a connected complete totally geodesic horizontal submanifold of the tangent bundle of a connected Riemannian manifold $M^n$ which projects into a simply connected Riemannian submanifold $F^l$ of $M^n$. Then $F^l$ is also complete and totally geodesic in $M^n$ and there is a parallel vector field $\xi$ on $M^n$ along $F^l$ such that $\xi(F^l) = N^l$.

Proof. By hypothesis, for all $z \in N^l$, $T_z N^l$ is a horizontal subspace of $T_z TM^n$ with respect to the Levi–Civita connection of $\bar{g}$. Hence $\pi \upharpoonright N^l : N^l \to F^l$ is an isometric submersion of $N^l$ into $F^l$, with $N^l$ and $F^l$ connected and of the same dimension. Since $N^l$ is complete, also $F^l$ is complete and $N^l$ is a covering space of $F^l$ (cf. [8, Vol. 1, p. 176]). The fact that $F^l$ is simply connected implies that $\pi \upharpoonright N^l : N^l \to F^l$ is an isometry, and there is an isometry $\xi : F^l \to N^l$ such that $\pi \upharpoonright N^l \circ \xi = Id_{F^l}$, i.e. $\xi$ is a vector field on $M^n$ along $F^l$.

Now, $F^l$ is totally geodesic. Indeed, let $X$ and $Y$ be vector fields on $F^l$, and denote by the same letters some of their extensions to $M^n$. If we denote by $X^h$ and $Y^h$ their horizontal lifts to $TM^n$, then $X^h \upharpoonright N^l$ and $Y^h \upharpoonright N^l$ are vector fields on $TM^n$ along $N^l$. For all $z \in N^l$, $T_z N^l$ being horizontal, $\pi_* \upharpoonright T_z N^l : T_z N^l \to T_z M^n$ is bijective. Since $\pi_*(X^h(z)) =
$X(\pi(z))$ and $\pi_*(Y^h(z)) = Y(\pi(z))$, we have that $X^h(z)$ and $Y^h(z)$ are tangent to $N^l$. Thus $(\nabla_X Y^h) \mid N^l$ is tangent to $N^l$ and hence horizontal. Consequently $(\nabla_X Y^h) \mid N^l = (\nabla_X Y)^h \mid N^l$ and is tangent to $N^l$. Hence $\nabla_X Y = \pi_*(\nabla_X Y)^h$ is tangent to $F^l$ and so $F^l$ is totally geodesic. It remains to prove that $\xi$ is parallel along $F^l$. In fact, for all $x \in F^l$ and $X \in T_x F^l$, the vector $X^h + (\bar{\nabla}_X \xi)^v$ is tangent to $\xi(F^l) = N^l$ at $\xi(x)$ and is mapped onto $X$. Since $T_{\xi(x)} N^l$ is a horizontal space, $\bar{\nabla}_X \xi = 0$. Therefore, $\xi$ is parallel along $F^l$. \hfill \Box

**Corollary 2.1.** Let $N^n$ be a connected complete totally geodesic horizontal $n$-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold $M^n$. Then $M^n$ is also complete and there is a parallel vector field $\xi$ on $M^n$ such that $\xi(M^n) = N^n$.

### 3. The conditions on $\xi(F^l)$ to be totally geodesic

Evidently, geometrical properties of the submanifold $\xi(F^l)$ depend on the submanifold $F^l$ and the vector field $\xi$. If one does not pose any restrictions on them, the geometry of $\xi(F^l)$ becomes rather intricate. Nevertheless, it is possible to formulate the conditions on $\xi(F^l)$ to be totally geodesic in more or less geometrical terms.

To do this, we introduce the notion of a $\xi$-connection on the Riemannian manifold $M^n$.

**Definition 3.1.** Let $M^n$ be a Riemannian manifold with Riemannian connection $\bar{\nabla}$ and curvature tensor $\bar{R}$. Let $\xi$ be a fixed smooth vector field on $M^n$. Denote by $\mathfrak{X}(M^n)$ the set of all smooth vector fields on $M^n$. The mapping $\bar{\nabla}: \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ defined by

$$
\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2} \left[ \bar{R}(\xi, \nabla_X \xi) Y + \bar{R}(\xi, \nabla_Y \xi) X \right]
$$

(7)

is a torsion-free affine connection on $M^n$. It is called the $\xi$-connection.

Remark that if $\xi$ is a parallel vector field or the manifold $M^n$ is flat, then the $\xi$-connection is the same as the Levi–Civita connection of $M^n$.

It is easy to check that (7) indeed defines a torsion-free affine connection. Now we can state the main technical tool for the further considerations.
Proposition 3.1. Let $F^l$ be a submanifold in a Riemannian manifold $M^n$. Let $\xi$ be a vector field on $M^n$ along $F^l$. Then $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if

(a) $F^l$ is totally geodesic with respect to the $\xi$-connection (7);

(b) for any vector fields $X, Y$ on $F^l$,

$$\nabla_X \nabla_Y \xi = \nabla_{\nabla_X Y} \xi + \frac{1}{2} \bar{R}(X, Y) \xi.$$  

Proof. By definition, the submanifold $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if $g_\ast(\nabla_X \nabla_Y \xi, \nabla_z \xi) = 0$ for any vector fields $\tilde{X}, \tilde{Y}$ tangent to $\xi(F^l)$ along $\xi(F^l)$ and $\tilde{N}$ normal to $\xi(F^l)$. To calculate $\nabla_{\nabla_X \nabla_Y}$, we use the Kowalski formulas [9].

For any vector fields $\tilde{X}, \tilde{Y}$ on $M^n$, the covariant derivatives of various combinations of lifts to the point $(x, \xi) \in TM^n$ can be found as follows

$$\begin{align*}
\nabla_{\nabla_X \nabla_Y} \xi &= \nabla_{\nabla_X Y} \xi + \frac{1}{2} \bar{R}(X, Y) \xi, \\
\nabla_{\nabla_X \nabla_Y} \xi &= \frac{1}{2} \bar{R}(\xi, \nabla_X \xi) Y + \frac{1}{2} \bar{R}(\xi, \nabla_Y \xi) X.
\end{align*}$$  

(8)

where $\nabla$ and $\bar{R}$ are the Levi–Civita connection and the curvature tensor of $M^n$ respectively.

Let $\tilde{X} = X^h + (\nabla_X \xi)^v$ and $\tilde{Y} = Y^h + (\nabla_Y \xi)^v$ be vector fields tangent to $\xi(F^l)$. Then, applying (8), we easily find

$$\begin{align*}
\nabla_{\nabla_X \tilde{Y}} &= \left( \nabla_X Y + \frac{1}{2} \bar{R}(\xi, \nabla_X \xi) Y + \frac{1}{2} \bar{R}(\xi, \nabla_X \xi) X \right)^h \\
&\quad + \left( \nabla_X \nabla_Y \xi - \frac{1}{2} \bar{R}(X, Y) \xi \right)^v
\end{align*}$$

or

$$\begin{align*}
\nabla_{\nabla_X \tilde{Y}} &= (\nabla_X Y)^h + \left( \nabla_X \nabla_Y \xi - \frac{1}{2} \bar{R}(X, Y) \xi \right)^v.
\end{align*}$$

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Using Proposition 1.2, we see that the totally geodesic property of \( \xi(F^l) \) is equivalent to

\[
\begin{aligned}
\bar{g}(\nabla_X Y, \eta) &= 0, \\
\bar{g}(\nabla_X Y, (\nabla \xi)^* \eta) &= \bar{g}(\tilde{\nabla}_X \tilde{\nabla}_Y \xi - \frac{1}{2} \tilde{R}(X, Y) \xi, \eta), \\
\bar{g}(\nabla_X Y, (\nabla \xi)^* Z) &= \bar{g}(\tilde{\nabla}_X \tilde{\nabla}_Y \xi - \frac{1}{2} \tilde{R}(X, Y) \xi, Z),
\end{aligned}
\tag{9}
\]

for any vector fields \( X, Y, Z \) tangent to \( F^l \) and any vector field \( \eta \) orthogonal to \( F^l \).

From (9), we see that \( F^l \) must be autoparallel with respect to \( \ast \nabla \) and hence totally geodesic [8]. Thus, \( \ast \nabla X \) is tangent to \( F^l \) and it is possible to apply (6). Therefore, we can rewrite the equations (9) as

\[
\begin{aligned}
\bar{g}(\tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_X \tilde{\nabla}_Y \xi + \frac{1}{2} \tilde{R}(X, Y) \xi, \eta) &= 0, \\
\bar{g}(\tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_X \tilde{\nabla}_Y \xi + \frac{1}{2} \tilde{R}(X, Y) \xi, Z) &= 0
\end{aligned}
\]

for any vector fields \( \eta \) normal and \( Z \) tangent to \( F^l \) along \( F^l \). Thus, we conclude

\[
\tilde{\nabla}_X \tilde{\nabla}_Y \xi = \tilde{\nabla}_X \tilde{\nabla}_Y \xi + \frac{1}{2} \tilde{R}(X, Y) \xi,
\]

which completes the proof.

For the cases when \( l = 1 \) and \( l = n \), we get the known conditions for the totally geodesic property of \( \xi(F^l) \).

**Corollary 3.1.** If \( l = 1 \) and \( \xi(F^l) \) is a curve \( \Gamma \) in \( TM^n \) then this curve is a geodesic if and only if

\[
\begin{aligned}
x'' + \tilde{R}(\xi, \xi') x' &= 0, \\
\xi'' &= 0,
\end{aligned}
\]

where \( (\cdot') \) means the covariant derivative with respect to the natural parameter of \( \Gamma \) and \( x(\sigma) = (\pi \circ \Gamma)(\sigma) \) (cf. [12]);

**Proof.** Indeed, in this case \( \tilde{X} = \tilde{Y} = \Gamma' = (x')^h + (\xi')^v \), \( \tilde{X} = \tilde{Y} = x' \) and \( \tilde{\nabla}_X \tilde{Y} = x'' + \tilde{R}(\xi, \xi') x' \). Thus, \( x(\sigma) \) is geodesic with respect to the \( \xi \)-connection iff \( x'' + \tilde{R}(\xi, \xi') x' = 0 \) and the rest of the proof is evident. \( \square \)
Corollary 3.2. If \( l = n \) and \( F^l = M^n \), then \( \xi(M^n) \) is totally geodesic in \( TM^n \) if and only if for any vector fields \( \bar{X}, \bar{Y} \) on \( M^n \) (cf. [14])

\[
\nabla_{\bar{X}} \nabla_{\bar{Y}} \xi = \nabla_{\nabla_{\bar{X}} \bar{Y}} \xi + \frac{1}{2} \bar{R} (\bar{X}, \bar{Y}) \xi.
\]

Proof. In this case, only (b) of Proposition 3.1 should be checked, which completes the proof. \( \square \)

The result of Corollary 3.2 can be expressed in more geometrical terms. To do this, introduce a symmetric bilinear mapping \( h_\xi : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \rightarrow \mathfrak{X}(M^n) \) by

\[
h_\xi (\bar{X}, \bar{Y}) = \frac{1}{2} \left[ \bar{R}(\xi, \nabla_{\bar{X}} \xi) \bar{Y} + \bar{R}(\xi, \nabla_{\bar{Y}} \xi) \bar{X} \right],
\]

(10)

for all \( \bar{X}, \bar{Y} \in \mathfrak{X}(M^n) \). Then the definition of the \( \xi \)-connection takes as similar form as the Gauss decomposition

\[
\nabla^* \bar{X} \bar{Y} = \nabla_{\bar{X}} \bar{Y} + h_\xi (\bar{X}, \bar{Y}).
\]

(11)

Define a “shape operator” \( A_\xi \) for the field \( \xi \) by

\[
A_\xi \bar{Y} = -\nabla_{\bar{Y}} \xi, \quad \text{for all } \bar{Y} \in \mathfrak{X}(M^n).
\]

(12)

Then the covariant derivative of the \((1,1)\)-tensor field \( A_\xi \) is given by

\[
(nabla_{\bar{X}} A_\xi) \bar{Y} = -\nabla_{\bar{X}} \nabla_{\bar{Y}} \xi + \nabla_{\nabla_{\bar{X}} \bar{Y}} \xi.
\]

Hence we see that the Codazzi-type equation \( \bar{R}(\bar{X}, \bar{Y}) \xi = (\nabla_{\bar{Y}} A_\xi) \bar{X} - (\nabla_{\bar{X}} A_\xi) \bar{Y} \) holds. In these notations

\[
\nabla_{\bar{X}} \bar{Y} \xi + \frac{1}{2} \bar{R}(\bar{X}, \bar{Y}) \xi - \nabla_{\bar{X}} \nabla_{\bar{Y}} \xi = \nabla_{h_\xi (\bar{X}, \bar{Y})} \xi + \frac{1}{2} \left[ (\nabla_{\bar{X}} A_\xi) \bar{Y} + (\nabla_{\bar{Y}} A_\xi) \bar{X} \right].
\]

If we introduce a symmetric bilinear mapping \( \Omega_\xi : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \rightarrow \mathfrak{X}(M^n) \) defined by

\[
\Omega_\xi (\bar{X}, \bar{Y}) = \nabla_{h_\xi (\bar{X}, \bar{Y})} \xi + \frac{1}{2} \left[ (\nabla_{\bar{X}} A_\xi) \bar{Y} + (\nabla_{\bar{Y}} A_\xi) \bar{X} \right],
\]

then Corollary 3.2 can be reformulated as
Corollary 3.3. If $\xi$ is a smooth vector field on a Riemannian manifold $M^n$ then $\xi(M^n)$ is totally geodesic in $TM^n$ if and only if for any vector fields $\bar{X}, \bar{Y}$ on $M^n$

$$\Omega_\xi(\bar{X}, \bar{Y}) = \nabla_{h_\xi(\bar{X}, \bar{Y})}\xi + \frac{1}{2} \left[ (\nabla_{\bar{X}} A_\xi)\bar{Y} + (\nabla_{\bar{Y}} A_\xi)\bar{X} \right] \equiv 0,$$  \hspace{1cm} (13)

where $h_\xi$ and $A_\xi$ are defined by (10) and (12) respectively.

Remark. The statement of Proposition 3.1 can also be reformulated in these terms, namely, let $F^l$ be a submanifold in a Riemannian manifold $M^n$ and $\xi$ be a vector field on $M^n$ along $F^l$. Then $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if $F^l$ is totally geodesic with respect to the $\xi$-connection (7) and $\Omega_\xi$ vanishes on the tangent bundle of $F^l$.

Now, combining Theorem 2.1 with Proposition 3.1, we obtain

Corollary 3.4. On a connected simply connected compact $n$-dimensional Riemannian manifold, vector fields satisfying (b) of Proposition 3.1 generate the only connected compact totally geodesic $n$-dimensional submanifolds of the tangent bundle which are transverse to fibers.

As has been shown in [20], for the case of the unit tangent bundle, the Hopf vector fields on odd dimensional spheres generate totally geodesic submanifolds in $T_1S^n$. For the tangent bundle the situation is different.

Theorem 3.1. A non-zero Killing vector field on a space of non-zero constant curvature $(M^n, c)$ never generates a totally geodesic submanifold in $TM^n$. Moreover, a manifold with positive sectional curvature does not admit a non-zero Killing vector field with totally geodesic property.

Proof. Let $\xi$ be a Killing vector field on a space $M^n$ of constant curvature $c$. Then $A_\xi$ is a skew-symmetric linear operator, i.e.

$$\bar{g}(A_\xi \bar{X}, \bar{Y}) + \bar{g}(\bar{X}, A_\xi \bar{Y}) = 0,$$  \hspace{1cm} (14)

and moreover,

$$(\nabla_{\bar{X}} A_\xi)\bar{Y} = \bar{R}(\xi, \bar{X})\bar{Y}$$  \hspace{1cm} (15)

for all vector fields $\bar{X}, \bar{Y}$ on $M^n$ (cf. [8]). Since $M^n$ is of non-zero constant curvature, the equation (13) can be simplified in the following way:

$$(\nabla_{\bar{X}} A_\xi)\bar{Y} + (\nabla_{\bar{Y}} A_\xi)\bar{X} = \bar{R}(\xi, \bar{X})\bar{Y} + \bar{R}(\xi, \bar{Y})\bar{X}$$
\[
\bar{R}(\xi, \nabla_X \xi) \bar{Y} + \bar{R}(\xi, \nabla_Y \xi) \bar{X} = c \left[ \bar{g}(\nabla_X \xi, \bar{Y}) + \bar{g}(\bar{X}, \nabla_Y \xi) \bar{X} \right] \xi \\
- c \left[ (\bar{g}(\xi, \bar{X}) \nabla_Y \xi + \bar{g}(\xi, \bar{Y}) \nabla_X \xi) \right] \\
= c \left[ \bar{g}(\xi, \bar{X}) A_\xi \bar{Y} + \bar{g}(\xi, \bar{Y}) A_\xi \bar{X} \right].
\]

So, \( \xi \) is totally geodesic if
\[
\bar{g}(\xi, \bar{X}) \bar{Y} + \bar{g}(\xi, \bar{Y}) \bar{X} - \nabla_{\bar{g}(\xi, \bar{X}) A_\xi \bar{Y} + \bar{g}(\xi, \bar{Y}) A_\xi \bar{X}} \xi = 2\bar{g}(\bar{X}, \bar{Y}) \xi,
\]
or
\[
\bar{g}(\xi, \bar{X}) \left[ \bar{Y} + A_\xi (A_\xi \bar{Y}) \right] + \bar{g}(\xi, \bar{Y}) \left[ \bar{X} + A_\xi (A_\xi \bar{X}) \right] = 2\bar{g}(\bar{X}, \bar{Y}) \xi,
\]
for all vector fields \( \bar{X}, \bar{Y} \) on \( M^n \). Choosing \( \bar{X}, \bar{Y} \) such that \( \bar{X}_x \neq 0 \) and \( \bar{X}_x = \bar{Y}_x \perp \xi_x \), we get \( 2|\bar{X}_x|^2 \xi = 0 \). Therefore, \( \xi = 0 \) for all \( x \in M^n \).

Let \( \xi \) be a non-zero Killing vector field on a manifold with \textit{positive} (non-constant) sectional curvature. From (14) it follows that \( A_\xi \xi \perp \xi \).

If \( A_\xi \xi = 0 \), then, after setting \( Y = \xi \) in (14), we conclude that \( \xi \) has a constant length and therefore can be totally geodesic if it is a parallel vector field [14]. In this case, \( M^n = M^{n-1} \times E^1 \) and we come to a contradiction. Suppose that \( A_\xi \xi \neq 0 \). Then \( \xi \wedge A_\xi \xi \) is a non-zero bivector field. Setting \( \bar{Y} = \bar{X} \) in (13) and using (15), we have
\[
A_\xi \left[ \bar{R}(\xi, A_\xi \bar{X}) \bar{X} \right] + \bar{R}(\xi, \bar{X}) \bar{X} = 0.
\]

Taking a scalar product in both sides with \( \xi \) and applying (14), we get
\[
-\bar{g}(\bar{R}(\xi, A_\xi \bar{X}) \bar{X}, A_\xi \xi) + K_{\xi \wedge \bar{X}} |\xi \wedge \bar{X}|^2 = 0.
\]

Finally, setting \( \bar{X} = A_\xi \xi \), we have \( K_{\xi \wedge \bar{X}} = 0 \) and come to a contradiction.

The next theorem is analogous to the one proved by P. Walczak [14], but does not have similar rigid consequences for the structure of \( M^n \).

**Theorem 3.2.** Let \( \xi \) be a vector field of constant length along a submanifold \( F^l \subset M^n \). Then \( \xi(F^l) \) is a totally geodesic submanifold in \( TM^n \) if and only if \( F^l \) is totally geodesic in \( M^n \) and \( \xi \) is a parallel vector field on \( M^n \) along \( F^l \).
Proof. The condition $|\xi| = \text{const}$ implies $\bar{g}(\bar{\nabla}_X \xi, \xi) = 0$ for any vector field $X$ tangent to $F^l$. As $\xi(F^l)$ is supposed to be totally geodesic, it follows from the second condition of Proposition 3.1 that $\bar{g}(\bar{\nabla}_X \bar{\nabla}_Y \xi, \xi) = 0$. Hence $\bar{g}(\bar{\nabla}_X \xi, \bar{\nabla}_Y \xi) = 0$ for any $X, Y \in T_x F^l$, $x \in F^l$. Supposing $X = Y$, we see that $\bar{\nabla}_X \xi = 0$, i.e. $\xi$ is parallel along $F^l$ in the ambient space and the second condition of Proposition 3.1 is fulfilled. Moreover, the condition $\bar{\nabla}_X \xi = 0$ means that the $\xi$-connection (7) coincides with the Levi–Civita connection of $M^n$, so that by Proposition 3.1 $F^l$ is totally geodesic in $M^n$.

On the other hand, if $F^l$ is totally geodesic in $M^n$ and $\bar{\nabla}_X \xi = 0$ for any tangent vector field $X$ on $F^l$, then both conditions from Proposition 3.1 are satisfied evidently. □

Giving more restrictions on the vector field, we can a more geometrical result.

Theorem 3.3. Let $\xi$ be a normal vector field on a submanifold $F^l \subset M^n$, which is parallel in the normal bundle. Then $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if $F^l$ is totally geodesic in $M^n$.

Proof. If $\xi$ is a normal vector field to $F^l$ and parallel in the normal bundle, then $\bar{\nabla}_X \xi = -A_\xi X$ for each vector field $X$ on $F^l$, where $A_\xi$ is the shape operator of $F^l$ with respect to $\xi$, and hence $\bar{g}(\bar{\nabla}_X \xi, \xi) = 0$. This means that $|\xi| = \text{const}$ along $F^l$.

Let $\xi(F^l)$ be totally geodesic in $TM^n$. Then from (b) of Proposition 3.1 we see that $\bar{g}(\bar{\nabla}_X \bar{\nabla}_Y \xi, \xi) = 0$, which implies $|\bar{\nabla}_X \xi| = 0$ for each $X$ tangent to $F^l$. In this case, along $F^l$ the $\xi$-connection (7) coincides with the Levi–Civita connection of $M^n$ and (a) of Proposition 3.1 implies the totally geodesic property of $F^l$.

Conversely, if $\xi$ is a normal vector field which is parallel in the normal bundle of $F^l$ and $F^l$ is totally geodesic, then $\bar{\nabla}_X \xi = 0$ for any vector field $X$ tangent to $F^l$. Evidently, both conditions of Proposition 3.1 are fulfilled. □

The application of Theorem 3.3 to the specific case of a foliated Riemannian manifold allows to clarify the geometrical structure of $\xi(M^n)$. The manifold $M^n$ is said to be $\nu$-foliated if it admits a family $\mathcal{F}$ of connected $\nu$-dimensional submanifolds $\{\mathcal{F}_\alpha; \alpha \in A\}$ called leaves such that (i) $M^n = \bigcup_{\alpha \in A} \mathcal{F}_\alpha$; (ii) $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for $\alpha \neq \beta$; (iii) there exists a coordinate
covering $U$ of $M^n$ such that in each local chart $U \in \mathcal{U}$ the leaves can be expressed locally as level submanifolds, i.e. $u^{n+1} = c_{n+1}, \ldots, u^n = c_n$.

The family $\mathcal{F}$ is called a $\nu$-foliation and hyperfoliation for $\nu = n - 1$. The hyperfoliation is said to be transversally orientable if $M^n$ admits a vector field $\xi$ transversal to the leaves. Moreover, with respect to the Riemannian metric on $M^n$, this vector field can be chosen as a field of unit normals for each leaf.

A submanifold $F^{k+\nu} \subset M^n$ is called $\nu$-ruled if $F^{k+\nu}$ admits a $\nu$-foliation $\{\mathcal{F}_\alpha; \alpha \in A\}$ such that each leaf $\mathcal{F}_\alpha$ is totally geodesic in $M^n$. The leaves $\mathcal{F}_\alpha$ are called elements or generators [11].

**Corollary 3.5.** Let $M^n$ be a Riemannian manifold admitting a totally geodesic transversally orientable hyperfoliation $\mathcal{F}$. Let $\xi$ be a field of normals of the foliation having constant length. Then $\xi(M^n)$ is an $(n - 1)$-ruled submanifold in $TM^n$ with the elements $\xi(\mathcal{F}_\alpha)$.

**Proof.** Indeed, let $\mathcal{F}_\alpha$ be a leaf of the hyperfoliation and $\xi$ be a vector field of constant length on $M^n$ which is a field of normals along each leaf. Applying Theorem 3.3, we get that $\xi(\mathcal{F}_\alpha)$ is totally geodesic in $TM^n$ for each $\alpha$. Since $\xi : M^n \to \xi(M^n)$ is a homeomorphism, $\xi(\mathcal{F}_\alpha) \cap \xi(\mathcal{F}_\beta) = \emptyset$ for $\alpha \neq \beta$ and $\xi(M^n) = \bigcup_{\alpha \in A} \xi(\mathcal{F}_\alpha)$. Finally, if $\mathcal{F}_\alpha$ is given by $u^n = c_n$ within a local chart $U$ then from (4) we see that $\xi(\mathcal{F}_\alpha)$ is given by the same equalities within the local chart $\xi(U)$. So, $\xi(\mathcal{F}) = \{\xi(\mathcal{F}_\alpha); \alpha \in A\}$ form a hyperfoliation on $\xi(M^n)$ with totally geodesic leaves in $TM^n$. □

4. The case of a base space of constant curvature

If the ambient space is of constant curvature $c \neq 0$ and $\xi$ is a normal vector field on a submanifold $F^l \subset M^n$, then the necessary and sufficient condition on $\xi$ to generate a totally geodesic submanifold in $TM^n$ takes a rather simple form.

**Theorem 4.1.** Let $F^l$ be a submanifold of a space $M^n(c)$ of constant curvature $c \neq 0$. Let $\xi$ be a normal vector field on $F^l$. Then $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if $F^l$ is totally geodesic in $M^n(c)$ and $\xi$ is parallel in the normal bundle.
Proof. The curvature tensor of $M^n(c)$ is of the form \[ \bar{R}(\bar{X}, \bar{Y}) \bar{Z} = c(\bar{g}(\bar{Y}, \bar{Z}) \bar{X} - \bar{g}(\bar{X}, \bar{Z}) \bar{Y}). \] (16)

If $\xi$ is a normal vector field on $F^l$ then $\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi$. As $A_\xi X$ is tangent and $\nabla_X^\perp \xi$ is normal to $F^l$, from (16) we find

\[ \bar{R}(\xi, \nabla_X \xi) \bar{Y} = -c g(A_\xi X, Y) \xi \]

for any vector fields $X, Y$ on $F^l$. Thus, the conditions from Proposition 3.1 mean that

\[ \begin{cases} 
\nabla_X Y - c g(A_\xi X, Y) \xi & \text{is tangent to } F^l, \\
\nabla_{\nabla_X Y} - c g(A_\xi X, Y) \xi & = \nabla_X \nabla_Y \xi. 
\end{cases} \tag{17} \]

Multiplying (17)$_1$ by $\xi$ and by normal vector field $\eta$ orthogonal to $\xi$, we have

\[ \begin{cases} 
g(A_\xi X, Y)(1 - c|\xi|^2) = 0, \\
g(A_\eta X, Y) = 0. 
\end{cases} \]

If $\xi$ is of constant length $|\xi|^2 = \frac{1}{c} (c > 0)$ then by Theorem 3.2, $F^l$ is totally geodesic in $M^n$, otherwise $F^l$ is totally geodesic immediately.

So, $F^l$ is totally geodesic and therefore $\nabla_X \xi = \nabla_X^\perp \xi$, $\nabla_X Y = \nabla_X Y$. The condition (17)$_2$ now takes the form

\[ \nabla_{\nabla_X \nabla_Y} \xi = \nabla_{\nabla_X \nabla_Y} \xi. \tag{18} \]

Set $Y = \nabla_V Z$, where $V$ and $Z$ are arbitrary vector fields tangent to $F^l$. Then from (18), we get

\[ \nabla_{\nabla_X \nabla_Y} \xi = \nabla_{\nabla_X \nabla_Y} \xi. \]

Applying (18) to $\nabla_{\nabla_Y Z} \xi$ in the right-hand side of the above equation, we see that $\nabla_{\nabla_Y Z} \xi = \nabla_{\nabla_Y Z} \xi$ and therefore,

\[ \nabla_{\nabla_X \nabla_Y} \xi = \nabla_{\nabla_X \nabla_Y} \xi. \tag{19} \]

Interchanging the roles of $X$ and $V$, we get

\[ \nabla_{\nabla_Y \nabla_X} \xi = \nabla_{\nabla_Y \nabla_X} \xi. \tag{20} \]
Finally, applying again (18) to the bracket $[X, V]$ and $Z$, we get
\[\nabla_{\nabla_{[X,V]}Z}^\perp \xi = \nabla_{[X,V]}^\perp \nabla_{Z}^\perp \xi.\] (21)

Combining (19), (20) and (21), we obtain
\[\nabla_{R(X,V)}^\perp Z \xi = R^\perp(X,V)\nabla_{Z}^\perp \xi\]
where $R$ is the curvature tensor of $F^i$ and $R^\perp$ is the normal curvature tensor. Since $F^i$ is totally geodesic and $M^n(c)$ is of constant curvature, $R^\perp(X,Y)\eta \equiv 0$ for any normal vector field $\eta$ and, moreover,
\[R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y).\]

So, we have
\[c\nabla_{g(Y,Z)X - g(X,Z)Y}^\perp \xi = 0.\]

Setting $X$ orthogonal to $Y$ and $Y = Z$ we get $\nabla_{X}^\perp \xi = 0$ for any vector field $X$ on $F^i$, which completes the necessary part of the proof. The sufficient part is trivial. \hfill \square

The application of Theorem 4.1 to the case of a space of constant curvature shows the difference between our considerations and Walczak’s [14]. Let $S^n$ be the unit sphere and $S^{n-1}$ be the unit totally geodesic great sphere in $S^n$. Denote by $D^n$ an open equatorial zone around $S^{n-1}$ where the unit geodesic vector field orthogonal to $S^{n-1}$ is regularly defined. Then $D^n$ is a Riemannian manifold of constant positive curvature and $S^{n-1}$ is a totally geodesic submanifold in $D^n$.

Let $\xi$ be a unit (or of constant length) geodesic vector field on $D^n \subset S^n$ which is normal to the totally geodesic great sphere $S^{n-1}$. Then $\xi(D^n)$ is not totally geodesic in $TD^n$ while the restriction of $\xi$ to $S^{n-1}$ generates the totally geodesic submanifold $\xi(S^{n-1})$ in $TD^n$.

Indeed, $\xi$ is of constant length and by Walczak’s result, $\xi(D^n)$ can be totally geodesic in $TD^n$ only if $\xi$ is a parallel vector field on $D^n$ [14], which is impossible due to positive curvature of $D^n$. On the other hand, $\xi$ is parallel in the normal bundle of $S^{n-1} \subset D^n$ and we can apply Theorem 4.1 to see that $\xi(S^{n-1})$ is totally geodesic in $TD^n$.

As concerns flat Riemannian manifolds, Walczak has shown that every totally geodesic vector field on a flat Riemannian manifold is harmonic.
(cf. [14]) and that, consequently, on a compact flat Riemannian manifold, a vector field is totally geodesic if and only if it is parallel. We shall give a similar result for vector fields along submanifolds.

**Theorem 4.2.** Let $F^l$ be a compact oriented submanifold in a flat Riemannian manifold $M^n$. Let $\xi$ be a vector field on $F^l$. Then $\xi(F^l)$ is totally geodesic in $TM^n$ if and only if $F^l$ is totally geodesic in $M^n$ and $\xi$ is parallel along $F^l$.

**Proof.** Since $M^n$ is flat, the $\xi$-connection is the same as the Levi–Civita connection on $M^n$. So, by Proposition 3.1, $\xi(F^l)$ is totally geodesic if and only if $F^l$ is totally geodesic and

$$\bar{\nabla}_X \bar{\nabla}_Y \xi = \bar{\nabla}_{\bar{\nabla}_X Y} \xi$$

(22)

for all vector fields $X$ and $Y$ on $F^l$.

Suppose now that $\xi(F^l)$ is totally geodesic. Then $F^l$ is totally geodesic and is thus flat. Hence locally we can choose vector fields $X_1, X_2, \ldots, X_l$ tangent to $F^l$ such that $\nabla_{X_i} X_j = \nabla_{X_j} X_i = 0$, and $\bar{g}(X_i, X_j) = \delta_{ij}$, for all $i, j = 1, \ldots, l$. Putting $X = Y = X_i$ in the identity (22), we obtain $\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi = 0$. Hence, $\sum_{i=1}^l \bar{g}(\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi, \xi) = 0$, i.e.

$$\sum_{i=1}^l X_i.\bar{g}(\bar{\nabla}_{X_i} \xi, \xi) = \sum_{i=1}^l |\bar{\nabla}_{X_i} \xi|^2.$$ 

(23)

If we consider the function $f$ defined by $f(x) = \frac{1}{2}\bar{g}_x(\xi, \xi)$, for all $x \in F^l$, then we can define a global vector field $X_f$ on $F^l$ by the local formula $X_f = \bar{g}(\bar{\nabla}_{X_i} \xi, \xi)X_i$. Formula (23) can thus be written locally as $\text{div} X_f = \sum_{i=1}^l |\bar{\nabla}_{X_i} \xi|^2$.

Integrating both sides of the last equality and applying Green’s theorem, we obtain $\sum_{i=1}^l \int_{F^l} |\bar{\nabla}_{X_i} \xi|^2 \, dv = 0$, and hence $\bar{\nabla}_{X_i} \xi = 0$, for all $i = 1, \ldots, l$. Therefore $\xi$ is parallel along $F^l$.

The sufficient part of the theorem is trivial. \qed

**Remarks.** 1. If in Theorem 4.2 the field $\xi$ is a normal vector field along $F^l$, then $\bar{\nabla}_{X_i} \xi$ is also normal for each vector field $X$ on $F^l$. Indeed, for the $X_i$’s constructed in the proof of the theorem, we have $\bar{g}(\bar{\nabla}_{X_i} \xi, X_j) = X_i.\bar{g}(\xi, X_j) = 0$, and so $\bar{\nabla}_{X_i} \xi$ is normal to $F^l$. Hence the identity (22) can
be written as
\[ \bar{\nabla}_X \bar{\nabla}_Y \xi = \nabla_{\nabla_X Y} \xi. \] (24)

Also, \( \xi \) is parallel if and only if it is parallel in the normal bundle. Hence \( \xi(F^l) \) is totally geodesic if and only if \( F^l \) is totally geodesic and \( \xi \) is parallel in the normal bundle.

2. The condition of compactness is necessary. Indeed, if we consider \( \mathbb{R}^n \) with its canonical coordinates \((x_1, x_2, \ldots, x_n)\) and its canonical Euclidean metric, and the hypersurface \( \mathbb{R}^{n-1} \) which is identified with the subspace given by: \( x_n = 0 \), then \( \mathbb{R}^{n-1} \) is an oriented totally geodesic submanifold of \( \mathbb{R}^n \). We have \( \bar{\nabla}_{\partial/\partial x_i} \bar{\nabla}_{\partial/\partial x_j} = 0 \) for all \( i, j = 1, \ldots, n \).

Let the vector field \( \xi \) on \( \mathbb{R}^n \) along \( \mathbb{R}^{n-1} \) defined by \( \xi(x) = x_1 \partial/\partial x_n \), where \( x_1 \) is the first component of \( x \). Now, to show that \( \xi(\mathbb{R}^{n-1}) \) is totally geodesic, it suffices to check that (22) is verified. In fact, \( \bar{\nabla}_{\partial/\partial x_i} \bar{\nabla}_{\partial/\partial x_j} \xi = \bar{\nabla}_{\partial/\partial x_i} \delta_{ij} \partial/\partial x_n = 0. \) But \( \bar{\nabla}_{\partial/\partial x_1} \xi = \partial/\partial x_n \), and so \( \xi \) is not parallel.

5. The case of Lie groups with bi-invariant metrics

Let us consider a connected Lie group \( G^n \) equipped with a bi-invariant metric \( \bar{g} \), i.e. invariant by both left and right translations. We shall generalize the results of P. Walczak [14] on totally geodesic left invariant vector fields on \( G^n \) to left invariant vector fields along Lie subgroups.

Let \( H^l \) be a Lie subgroup of \( G^n \). The metric \( g \) induced from \( \bar{g} \) on \( H^l \) is a bi-invariant metric. If we denote by \( \bar{\nabla} \) and \( \nabla \) the Levi–Civita connections on \( G^n \) and \( H^l \) respectively, then we have \( \bar{\nabla}_X Y = \frac{1}{2}[X, Y] \), for all \( X, Y \) of \( \mathfrak{g} \), the Lie algebra of \( G^n \), and \( \nabla_X Y = \frac{1}{2}[X, Y] \), for all \( X, Y \) of \( \mathfrak{h} \), the Lie algebra of \( H^l \).

**Lemma 5.1.** A connected complete submanifold \( F^l \) of \( G^n \) containing the identity element \( e \) of \( G^n \), such that \( T_e F^l \) is a subalgebra of \( \mathfrak{g} \), is totally geodesic if and only if \( F^l \) is a Lie subgroup \( H^l \) of \( G^n \).

**Proof.** If we denote by exp the exponential mapping \( \exp : \mathfrak{g} \to G^n \) of the Lie group \( G^n \), and by \( \exp_x : T_x G^n \to G^n \) the exponential map at a point \( x \) of \( G^n \) with respect to the Levi–Civita connection of the metric \( g \), then for all \( x \in G^n \), \( \exp_x = \exp \circ (L_{x-1})_* \), where \( L_x \) is the left translation.
of $G^n$ by $x$. Indeed, we show firstly that $\exp_e = \exp$. Let $X \in \mathfrak{g} \equiv T_e G^n$ and $\gamma(t) = \exp tX$. It suffices to check that $\gamma$ is a geodesic. We have $\dot{\gamma}(t) = (L_{\gamma(t)})_* \dot{\gamma}(0) = (L_{\gamma(t)})_* (X)$, and thus $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \nabla_{X(\gamma(t))} X(\gamma(t))$, where $X$ denotes also the left invariant vector field on $G^n$ corresponding to $X$. Hence $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \frac{1}{2} [X, X] (\gamma(t)) = 0$, and so $\exp_e = \exp$. Now, our assertion follows from the fact that left translations are isometries.

We consider a Lie subgroup $H^l$ of $G^n$ and $\mathfrak{h} = T_e H^l$ its Lie algebra. If $X \in \mathfrak{h}$, then $\exp_e tX = \exp tX \in H^l$, for all $t$ in a neighborhood of 0, i.e. $H^l$ contains the geodesic starting from $e$ and with initial condition $X$, and by the left translations, $H^l$ contains all geodesics starting from points of $H^l$ with initial vectors tangent to $H^l$ at these points. Thus $H^l$ is totally geodesic.

Conversely, suppose that $F^l$ is a connected complete submanifold of $G^n$ such that $e \in F^l$ and $T_e F^l =: \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let $H^l$ be the connected subgroup of $G^n$ with Lie algebra $\mathfrak{h}$. $H^l$ is then a connected totally geodesic submanifold of $G^n$ with $T_e H^l = T_e F^l$. Therefore $H^l = F^l$.

**Proposition 5.1.** A left invariant vector field on $G^n$ along a submanifold $F^l$ generates a totally geodesic submanifold of $TG^n$ if and only if it is parallel along $F^l$ and $F^l$ is totally geodesic.

**Proof.** A left invariant vector field on $G^n$ is necessarily of constant length, and we apply Theorem 3.2.

**Corollary 5.1.** A left invariant vector field $\xi$ on $G^n$ along a Lie subgroup $H^l$ is totally geodesic if and only if it is an element of the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

**Proof.** By Lemma 5.1, $H^l$ is a totally geodesic submanifold in $G^n$. Thus, by virtue of Proposition 5.1, $\xi$ is totally geodesic if and only if $\xi$ is parallel along $H^l$.

Suppose that $\xi$ is totally geodesic. Then $\nabla_X \xi = 0$, for all $X \in \mathfrak{h}$; i.e. $\xi$ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Conversely, if $\xi$ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, then $\nabla_X \xi = 0$, for all $X \in \mathfrak{h}$. Let $x \in H^l$ and $z \in T_x H^l$. It suffices to prove that $\nabla_x \xi = 0$. But $X := (L_{x^{-1}})_*(z) \in T_e H^l \equiv \mathfrak{h}$, and consequently $\nabla_x \xi = (\nabla_X \xi)(x) = 0$. □
Corollary 5.2. (a) There are no non-zero left invariant totally geodesic vector fields on a semi-simple Lie subgroup of a Lie group with a bi-invariant Riemannian metric.

(b) Every left invariant vector field along a subgroup of an abelian Lie group with a bi-invariant Riemannian metric generates a totally geodesic submanifold of the tangent bundle.

Theorem 5.1. Let $N^l$ be a connected complete totally geodesic embedded submanifold of the tangent bundle of a connected Lie group $G^n$ equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a Lie subgroup of $G^n$. Suppose that $N^l$ is horizontal at a point $z$ of $T_eG^n$.

(a) If $z \in T_eH^l$, then $N^l$ is the image of $H^l$ by a left invariant vector field on $H^l$ which belongs to the center of $\mathfrak{h}$. In particular, if $H^l$ is semi-simple, then $H^l$ is the only connected totally geodesic embedded submanifold of $TG^n$ which is tangent to $H^l$ at $e$ and orthogonal to the fiber at a point of $T_eG^n$.

(b) If $H^l$ is simple, then $N^l$ is the image of $H^l$ by a left invariant vector field on $G^n$ along $H^l$ which belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Proof. Using Proposition 2.1, there is a neighborhood $U$ of $e$ in $G^n$, a neighborhood $V$ of $z$ in $TG^n$, and a vector field $Y$ on $M^n$ along $H^l \cap U$ such that $N^l \cap V = Y(H^l \cap U), Y(e) = z$. We have $T_zN^l = T_z(N^l \cap V) = T_zY(H^l \cap U)$. Then each vector of $T_zN^l$ can be written as $X^h + \langle \nabla_X Y \rangle^n$, for some $X \in \mathfrak{h}$. But $T_zN^l$ is a subset of the horizontal subspace of $TTG^n$ at $z$, so at $e$ we have $\nabla_X Y = 0$ for all $X \in \mathfrak{h}$. On the other hand, since $N^l \cap V = Y(H^l \cap U)$ is totally geodesic, the second assertion of Proposition 3.1 reduces at $e$ to the identity

$$\nabla_{X_1} \nabla_{X_2} Y = \frac{1}{2} \tilde{R}(X_1, X_2) Y,$$

for all vector fields $X_1, X_2$ on $H^l$.

Then for all $W \in \mathfrak{g} = T_eG^n$, we have

$$\tilde{g}(\nabla_{X_1(e)} \nabla_{X_2} Y, W) = \frac{1}{2} \tilde{g}(\tilde{R}(X_1(e), X_2(e)) Y(e), W).$$

If we extend $W$ to a vector field $X_3$ along $H^l$, which is orthogonal to $\nabla_{X_2} Y$ in a neighborhood of $e$ in $H^l$, then we can write

$$\tilde{g}(\nabla_{X_1(e)} \nabla_{X_2} Y, W) = -\tilde{g}(\nabla_{X_2(e)} Y, \nabla_{X_1(e)} X_3) = 0,$$
and consequently, $\bar{g}(\bar{R}(X_1(e), X_2(e))Y(e), W) = 0$, for all $X_1(e), X_2(e) \in \mathfrak{h} = T_eH^l$ and $W \in \mathfrak{g} = T_eG^n$. Therefore we have

$$R(\cdot, \cdot)Y(e) = 0,$$

when applied to vectors in $T_eH^l$.

Let us denote by $\xi$ the left invariant vector field on $G^n$ along $H^l$ such that $Y(e) = \xi(e)$. Then $\bar{R}(\cdot, \cdot)\xi(e) = 0$ when applied to vectors in $T_eH^l$, and hence

$$\bar{R}(\cdot, \cdot)\xi = 0,$$

when applied to elements of $\mathfrak{h}$. (25)

Consider now two cases.

(a) If $\xi(e) = z \in T_eH^l$, then $\xi \in \mathfrak{h}$, and we have, by virtue of (25), $\bar{R}(X, \xi)\xi = 0$, for all $X \in \mathfrak{h}$. Thus $||[\xi, X]|^2 = 4\bar{g}(\bar{R}(\xi, X)X, \xi) = 0$ for all $X \in \mathfrak{h}$. It follows that $\xi$ belongs to the center of $\mathfrak{h}$.

(b) If $H^l$ is simple, then $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. But $\bar{\nabla}_{[X_1, X_2]}\xi = \frac{1}{2}[X_1, X_2]\xi = -2R(X_1, X_2)\xi = 0$, for all $X_1, X_2 \in \mathfrak{h}$, by virtue of (25). Since $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, we deduce easily that $\bar{\nabla}_X\xi = 0$, for all $X \in \mathfrak{h}$, or equivalently $[X, \xi] = 0$, for all $X \in \mathfrak{h}$. It follows that $\xi$ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

In both cases, $\xi$ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Hence, by Lemma 5.1, $H^l$ is complete and totally geodesic in $G^n$, and Proposition 5.1 implies then that $\zeta(H^l)$ is a complete totally geodesic submanifold of $TG^n$. Therefore $\zeta(H^l) = N^l$, because $\zeta_*(T_eH^l) = T_eN^l$ and $N^l$ and $H^l$ are connected. □

**Corollary 5.3.** Let $N^l$ be a connected complete horizontal totally geodesic submanifold of the tangent bundle of a connected Lie group $G^n$ equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a simply connected submanifold of $G^n$ containing the identity element. Suppose that $\mathfrak{h} := \pi_*(T_zN^l)$ is a Lie subalgebra of $\mathfrak{g}$ for a point $z$ of $T_eG^n \cap N^l$. If $Z \in T_zH^l$ (resp. $\mathfrak{h}$ is simple), then $H^l$ is a Lie subgroup of $G^n$ and $N^l$ is the image of $H^l$ by a left invariant vector field on $H^l$ (resp. on $G^n$ along $H^l$) which belongs to the center of $\mathfrak{h}$ (resp. centralizer of $\mathfrak{h}$ in $\mathfrak{g}$).

**Proof.** By Theorem 2.3, $H^l$ is complete and totally geodesic. It follows from Lemma 5.1 that $H^l$ is a Lie subgroup of $G^n$. Now, our corollary follows from Theorem 5.1. □
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References


