On the greatest and least prime factors of $n! + 1$, II

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In memory of Béla Brindza

Abstract. Let $\varepsilon$ be a positive real number. We prove that for infinitely many odd integers $n$ the least prime factor of $n! + 1$ is at most $(\sqrt[4]{145} - \varepsilon)n$ and that for infinitely many positive integers $n$ the greatest prime factor of $n! + 1$ exceeds $(\frac{11}{2} - \varepsilon)n$.

1. Introduction

In 1856, Liouville [6] proved that $(p - 1)! + 1$ is not a power of $p$ for any prime $p$ larger than 5. More than a century later Erdős and Graham [2] asked if the equation

$$(p - 1)! + a^{p-1} = p^k$$

has only finitely many solutions in positive integers $a$, $k$, $p$ with $p$ an odd prime. In 1991 Brindza and Erdős [1] resolved the question by proving that all solutions of (1) are smaller than an effectively computable number. A few years later Yu and Liu [10] and then Le [5] determined the complete list of solutions.

Mathematics Subject Classification: 11D75, 11J25.

Key words and phrases: shifted factorials, prime factors.

This research was supported in part by Grant A3528 from the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chairs Program.
In 1976 we investigated with Erdős [3] the arithmetical character of integers of the form \( n! + 1 \) where \( n \) is a positive integer. For any integer \( m \) larger than 1 let \( P(m) \) denote the greatest prime factor of \( m \) and let \( p(m) \) denote the least prime factor of \( m \). By Wilson’s theorem \( p \) divides \( (p-1)! + 1 \) whenever \( p \) is a prime. Since all prime factors of \( n! + 1 \) exceed \( n \) we see that \( p(n! + 1) = n + 1 \) whenever \( n + 1 \) is a prime. We showed with Erdős [3] that if \( n + 1 \) is not a prime then

\[
p(n! + 1) > n + (1 - o(1)) \frac{\log n}{\log \log n}. \tag{2}
\]

Further, for almost all integers \( n \),

\[
p(n! + 1) > n + \varepsilon(n)n^{\frac{1}{2}}, \tag{3}
\]

where \( \varepsilon(n) \) is any positive function that decreases to 0 as \( n \to \infty \).

In [3] we indicated how to prove that for infinitely many integers \( n \) for which \( n + 1 \) is not a prime \( p(n! + 1) \) is less than \( 2n \). We observed, as a direct consequence of Wilson’s theorem, that if \( p \) is a prime then

\[
(p - 1 - i)! \equiv (-1)^{i+1} \pmod{p}, \quad 0 \leq i \leq p - 1. \tag{4}
\]

Thus if \( p \mid i! + 1 \) for some odd integer \( i (> 1) \) then, from (4), \( p \mid (p-i-1)! + 1 \). For any positive real numbers \( \theta \) and \( t \) with \( t > \frac{1}{\theta} \) we have

\[
\max \left( \theta, \frac{1}{t - \theta - 1} \right) \geq \frac{2}{t} \tag{5}
\]

and

\[
\min \left( \theta, \frac{1}{t - \theta - 1} \right) \leq \frac{2}{t}. \tag{6}
\]

Note that \( \frac{p}{p-i-1} = \frac{1}{p-i} \frac{1}{\frac{p}{p-i} - \frac{1}{p}} \) and so on taking \( \theta = \frac{p}{i} \) and \( t = \frac{p-1}{p} \) we see from (5) and (6) that

\[
\max \left( \frac{p}{i}, \frac{p}{p-i-1} \right) \geq \frac{2p}{p-1} = 2 + \frac{2}{p-1},
\]

and

\[
\min \left( \frac{p}{i}, \frac{p}{p-i-1} \right) \leq 2 + \frac{2}{p-1},
\]
for $0 < i < p - 1$. As $i$ tends to infinity so also do $p$ and $p - i$. Thus for each $\varepsilon > 0$,

$$p(n! + 1) < (2 + \varepsilon)n,$$

(7)

for infinitely many composite integers $n + 1$. Further

$$P(n! + 1) > 2n,$$

(8)

for infinitely many positive integers $n$. We indicated in [3] that $2 + \varepsilon$ in (7) could be replaced by $2 - \delta$ for some positive number $\delta$. Our first result will be of this character.

**Theorem 1.** Let $\varepsilon > 0$. There are infinitely many odd integers $n$ for which

$$p(n! + 1) < \left(\frac{\sqrt{145} - 1}{8} + \varepsilon\right)n.$$

(9)

Observe that $\frac{\sqrt{145} - 1}{8} = 1.38019 \ldots$.

With Erdős [3] we proved that (2) holds with $P(n! + 1)$ in place of $p(n! + 1)$ for all positive integers $n$. Of course this only is an improvement on (2) for those integers $n$ for which $n + 1$ is a prime. Recently Luca and Shparlinski [7] sharpened this result by proving that

$$P(n! + 1) > n + \left(\frac{1}{4} + o(1)\right)\log n;$$

(10)

indeed they established (10) with $P(n! + 1)$ replaced by $P(n! + f(n))$ where $f$ is any non-zero polynomial with integer coefficients. In 2002 Murty and Wong [8] showed that if the $abc$ conjecture holds, then

$$P(n! + 1) > (1 + o(1))n\log n.$$

In 1976 we improved on (8) with Erdős [3] by proving that there is a positive number $\delta$ such that

$$P(n! + 1) > (2 + \delta)n,$$

(11)

for infinitely many integers $n$. Luca and Shparlinski [7] established (11) with $(2 + \delta)n$ replaced by $(\frac{5}{2} + o(1))n$ and showed that their result applies with $n! + f(n)$ in place of $n! + 1$ where $f$ is any non-zero polynomial with integer coefficients. Our next result gives a further improvement on (11).
For any set $X$ we denote the cardinality of $X$ by $|X|$. For any set $A$ of positive integers and any positive integer $n$ we put $A(n) = A \cap \{1, \ldots, n\}$. The lower asymptotic density of $A$ is $\lim \inf \frac{|A(n)|}{n}$.

**Theorem 2.** Let $\varepsilon > 0$. The set of positive integers $n$ for which

$$P(n! + 1) > \left(\frac{11}{2} - \varepsilon\right)n$$  \hspace{1cm} (12)

has positive lower asymptotic density.

As we remarked in [3] estimates (2), (3), (7), (8) and (11) hold with $n! + 1$ replaced by $n! - 1$ and the same comment applies to the estimates (9) and (12). Further, the same techniques used to prove Theorems 1 and 2 allow one to prove, for instance, that for each positive real number $\varepsilon$ there exist infinitely many positive integers $n$ for which $P((2n)! + 1) > \left(\frac{17+\sqrt{145}}{8} - \varepsilon\right)2n$ and there exist infinitely many positive integers $n$ for which $P((n! + 1)(n! - 1)) > \left(\frac{11+\sqrt{85}}{2} - \varepsilon\right)n$.

## 2. Preliminary lemmas

Let $p_1, p_2, \ldots$ denote the sequence of prime numbers and put $d_k = p_{k+1} - p_k$ for $k = 1, 2, \ldots$. Our first lemma, due to HEATH-BROWN, gives a bound on the frequency of large differences between consecutive prime numbers.

**Lemma 1.** Let $\varepsilon$ be a positive real number. There is a positive number $c$, which depends on $\varepsilon$, such that

$$\sum_{p_k \leq x} d_k^2 < cx^{23/18 + \varepsilon}.$$  \hspace{1cm} (13)

**Proof.** This is Theorem 1 of [4].

Our next result gives a bound for the size of the greatest common divisor of a collection of terms of the form $k! + 1$.

**Lemma 2.** Let $n$ and $t$ be positive integers with $t \geq 2$ and let $i_1, \ldots, i_t$ be distinct positive integers from a subinterval of $[1, n]$ of length $\ell$. Then

$$\gcd(i_1! + 1, \ldots, i_t! + 1) < n^{\ell/t}. $$  \hspace{1cm} (13)
Further, there exists a positive number $c_1$ such that if $n$ exceeds $c_1$ and $t \geq 3$ then
\[
gcd(i_1! + 1, \ldots, i_t! + 1) < e^{n \frac{2t}{(t-1)^2}}. \tag{14} \]

Furthermore, let $\delta$ and $\varepsilon$ be positive real numbers. There exists a positive number $c_2$, which depends on $\varepsilon$ and $\delta$, such that if $n$ exceeds $c_2$,
\[
3 \leq t < n^{\frac{13}{18} - \delta}, \tag{15} \]
and
\[
\ell > \frac{n}{(\log n)^\frac{2}{3}}, \tag{16} \]
then
\[
gcd(i_1! + 1, \ldots, i_t! + 1) < \exp \left( (1 + \varepsilon)\ell \left( \log t + \log \left( \frac{\ell n}{t} \right) + \frac{2 \log n \max(1, \log \log t)}{(t-1)^2} \right) \right). \tag{17} \]

We remark that it is possible to replace the term $\max(1, \log \log t)$ on the right hand side of inequality (17) by $f(t)$ where $f$ is any real valued function to the real numbers of size at least 1 for which $\lim_{t \to \infty} f(t) = \infty$ provided that $c_2$ is modified to depend on $f$.

**Proof of Lemma 2.** Let $A$ be a positive integer and let $(k_1, k_2)$ and $(k_3, k_4)$ be distinct pairs of integers with $n \geq k_1 > k_2 \geq 1$ and $n \geq k_3 > k_4 \geq 1$ for which
\[
A \mid k_i! + 1, \tag{18} \]
for $i = 1, 2, 3, 4$. Suppose, without loss of generality, that
\[
k_1 - k_2 \geq k_3 - k_4. \tag{19} \]
(Note that $\{k_1, k_2, k_3, k_4\}$ may only contain 3 elements.) Then, by (18),
\[
A \mid k_i! - k_{i+1}!, \quad \text{for } i = 1, 3, \]
and so
\[
A \mid k_{i+1}!(k_i(k_i - 1) \cdots (k_{i+1} + 1) - 1), \quad \text{for } i = 1, 3. \]
But, by (18), \( \gcd(A, k_{i+1}!) = 1 \) for \( i = 1, 3 \) hence
\[
A \mid k_i \cdot (k_{i+1} + 1) - 1, \tag{20}
\]
for \( i = 1, 3 \). Therefore
\[
A \mid k_1 \cdots (k_2 + 1) - k_3 \cdots (k_4 + 1).
\]
Since
\[
k_1 \cdots (k_2 + 1) - k_3 \cdots (k_4 + 1) = (k_3 - k_4)! \left( \frac{(k_1 - k_2)!}{(k_3 - k_4)!} \right) \left( \frac{k_1}{k_2} \right) - \left( \frac{k_3}{k_4} \right),
\]
gcd\( (k_3 - k_4)! \) = 1 and \( k_1 - k_2 \geq k_3 - k_4 \), we find that
\[
A \mid ((k_3 - k_4)!)^{-1} (k_1 \cdots (k_2 + 1) - k_3 \cdots (k_4 + 1)). \tag{21}
\]
Notice that
\[
|k_1 \cdots (k_2 + 1) - k_3 \cdots (k_4 + 1)| < n^{k_1 - k_2}. \tag{22}
\]
Thus, provided that
\[
k_1 \cdots (k_2 + 1) \neq k_3 \cdots (k_4 + 1), \tag{23}
\]
we deduce from (21), (22) and the fact that \( m! \geq \left( \frac{m}{e} \right)^m \), when \( m \) is a positive integer, that
\[
A < n^{(k_1 - k_2) - (k_3 - k_4)} \left( \frac{ne}{k_3 - k_4} \right)^{k_3 - k_4}. \tag{24}
\]
Since \( g(x) = \left( \frac{ne}{x} \right)^x \) attains its maximum value at \( x = n \) we see that, when (23) holds,
\[
A < n^{(k_1 - k_2) - (k_3 - k_4)} e^n. \tag{25}
\]
We now put
\[
A = \gcd(i_1! + 1, \ldots, i_t! + 1).
\]

We shall prove (13) first. Put
\[
\mu = \min \{ i_a - i_b \mid i_a > i_b, \ a, b \in \{ 1, \ldots, t \} \}
\]
and let \( k_1, k_2 \) be elements of \( \{ i_1, \ldots, i_t \} \) with \( k_1 - k_2 = \mu \). Since \( \mu \leq \frac{t}{(t-1)} \)
we see from (20) with \( i = 1 \) that
\[
A < n^{k_1 - k_2} \leq n^{\frac{t}{(t-1)}},
\]
as required.
We shall prove (14) next. There are \( \binom{t}{2} \) pairs of integers \((i, i')\) with \(i > i'\) which can be chosen from \(\{i_1, \ldots, i_t\}\). Associated to each such pair is the difference \(i - i'\) and \(0 < i - i' \leq \ell\). Therefore there are two such pairs, \((k_1, k_2)\) and \((k_3, k_4)\) say, for which

\[
0 \leq (k_1 - k_2) - (k_3 - k_4) \leq \frac{\ell}{\binom{t}{2} - 1}. 
\]

(26)

Thus, provided that (23) holds, by (25) and (26),

\[
A < n^{\binom{t}{2} - 1} e^n,
\]

hence, since \(t \geq 3\) and \(\binom{t}{2} - 1 \geq \frac{(t-1)^2}{2}\),

\[
A < e^n n^{\frac{2^t}{(t-1)^2}}.
\]

It remains only to ensure that (23) holds. We may assume that \(A\) exceeds \(e^n\) since otherwise the result is immediate. Note that if \(k_1 = k_3\) then, since the pairs \((k_1, k_2)\) and \((k_3, k_4)\) are distinct, \(k_2 \neq k_4\) and so (23) holds. Thus we may assume that \(k_1 > k_3\); a similar argument applies if \(k_3 < k_1\). Further, we may assume, after renumbering \(k_2, k_3, k_4\) if necessary, that \(k_2 \geq k_3 > k_4\). Since, as in (20), \(A\) divides \(k_1 \cdots (k_2 + 1) - 1\) we see that \(A \leq n^{k_1 - k_2}\). But \(A\) exceeds \(e^n\) and so

\[
k_1 - k_2 \geq \frac{n}{\log n}.
\]

By a version of the prime number theorem with an explicit error term, for \(n\) sufficiently large there is a prime \(p\) between \(k_1\) and \(k_2\). As a consequence \(p\) divides \(k_1 \cdots (k_2 + 1)\) and not \(k_3 \cdots (k_4 + 1)\) and so (23) holds and (14) follows.

Finally, we shall prove (17). Let \(c_3, c_4, \ldots\) denote positive numbers which are effectively computable in terms of \(\epsilon\) and \(\delta\). Without loss of generality we may suppose that

\[
n \geq i_1 > i_2 > \cdots > i_t \geq 1.
\]

Note that we may also suppose that

\[
t - 1 \geq (\log n)^{\frac{1}{3}},
\]

(27)
since otherwise, by (16),
\[
\frac{2\ell \log n}{(t-1)^2} \geq 2\ell (\log n)^{\frac{3}{2}} \geq 2n (\log n)^{\frac{1}{4}}
\]
and therefore, by (14),
\[
A < \exp \left( (1 + \varepsilon) \frac{2\ell (\log n)}{(t-1)^2} \right),
\]
for \(n\) larger than \(c_3\), whence (17) holds.

We consider the consecutive integers \(i_{j+1} + 1, \ldots, i_j\) for \(j = 1, \ldots, t-1\). Notice that \(A\) divides \(i_j! + 1\) and \(i_{j+1}! + 1\) hence \(A\) divides \(i_j \cdots (i_{j+1}+1) - 1\). Therefore \(A\) is at most \(n^{i_j-i_{j+1}}\). If for some \(j\), with \(1 \leq j \leq t-1\),
\[
i_j - i_{j+1} < \frac{n}{t (\log n)^{\frac{3}{2}}},
\]
then
\[
A \leq \exp \left( \frac{n}{t (\log n)^{\frac{1}{2}}} \right),
\]
and, by (16), (17) holds. Thus we may suppose that
\[
i_j - i_{j+1} \geq \frac{n}{t (\log n)^{\frac{3}{2}}},
\]
for \(j = 1, \ldots, t-1\). Let \(m\) denote the number of the intervals \([i_{j+1} + 1, i_j]\) for \(j = 1, \ldots, t-1\) which do not contain a prime number. Let \(p_1, p_2, \ldots\) denote the sequence of prime numbers and put \(d_k = p_{k+1} - p_k\) for \(k = 1, 2, \ldots\). Then
\[
\sum_{p_k \leq n} d_k^2 \geq m \left( \frac{n}{t (\log n)^{\frac{3}{2}}} \right)^2.
\]
But, by Lemma 1,
\[
\sum_{p_k \leq n} d_k^2 < n^{\frac{23}{18} + \delta},
\]
for \(n > c_4\). In particular
\[
m < \frac{t^2 (\log n)^{3}}{n^{\frac{13}{18} - \frac{\delta}{2}}},
\]
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and by (15), since $t \leq n$,

$$m < tn^{-\frac{\delta}{2}}(\log n)^3 < t^{1-\frac{\delta}{2}},$$  \hspace{1cm} (28)

for $n > c_5$.

Put

$$t_1 = t - 1 - m,$$  \hspace{1cm} (29)

and order the differences $i_j - i_{j+1}$ with $1 \leq j \leq t - 1$ for which there is a prime in the interval $[i_{j+1} + 1, i_j]$ according to size. Let us denote these differences by $\gamma_1, \ldots, \gamma_{t_1}$ so that

$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{t_1}.$$  \hspace{1cm} (30)

Observe that

$$\gamma_1 + \cdots + \gamma_{t_1} \leq i_1 - i_t \leq \ell.$$  \hspace{1cm} (31)

For any real number $x$ let $[x]$ denote the largest integer of size at most $x$. Put

$$t_2 = \left[ t_1 / (\log \log t_1)^{\frac{1}{2}} \right].$$  \hspace{1cm} (32)

Then, by (30) and (31),

$$\gamma_{t_2}(t_1 - t_2) \leq \ell.$$  

Thus, by (27), (28), (29), (30) and (32), for $n > c_6$,

$$\gamma_h < \left( 1 + \frac{\varepsilon}{2} \right)^{\frac{\ell}{\ell}},$$  \hspace{1cm} (33)

for $h = 1, \ldots, t_2$.

Next note that

$$(\gamma_{t_2} - \gamma_{t_2-1}) + (\gamma_{t_2-1} - \gamma_{t_2-2}) + \cdots + (\gamma_1 - 0) = \gamma_{t_2}$$

$$(\gamma_{t_2-1} - \gamma_{t_2-2}) + \cdots + (\gamma_1 - 0) = \gamma_{t_2-1}$$

\hspace{1cm} \cdots \hspace{1cm} \vdots

\hspace{1cm} \cdots \hspace{1cm} \vdots

$$(\gamma_1 - 0) = \gamma_1$$
(\gamma_{t_2} - \gamma_{t_2 - 1}) + 2(\gamma_{t_2 - 1} - \gamma_{t_2 - 2}) + \cdots + t_2\gamma_1 = \gamma t_2 + \cdots + \gamma_1. \quad (34)

Put

\[ \theta = \min(\gamma_{t_2} - \gamma_{t_2 - 1}, \ldots, \gamma_2 - \gamma_1, \gamma_1). \]

Then, by (31) and (34),

\[ \frac{t_2(t_2 + 1)}{2} \theta \leq \ell. \]

Therefore, by (27), (28), (29) and (32), for \( n > c_7 \),

\[ \theta < (1 + \varepsilon)\frac{2\ell \log \log t}{t^2}. \quad (35) \]

We have \( \gamma_1 = i_r - i_{r+1} \) for an integer \( r \) with \( 1 \leq r \leq t - 1 \). Then

\[ A \mid i_r \cdots (i_{r+1} + 1) - 1. \]

If \( \theta = \gamma_1 \), we see that

\[ A < n^\theta, \]

and by (35) our result follows.

Thus we may suppose that \( \theta = \gamma_s - \gamma_{s-1} \) for some integer \( s \) from \( \{2, \ldots, t_2\} \). In particular,

\[ \theta = (i_a - i_{a+1}) - (i_b - i_{b+1}) \]

with \( a \) and \( b \) distinct integers from \( \{1, \ldots, t - 1\} \). Put \( k_1 = i_a, k_2 = i_{a+1}, k_3 = i_b \) and \( k_4 = i_{b+1} \). By construction there is a prime among the integers \( k_2 + 1, \ldots, k_1 \) and another prime among the integers \( k_4 + 1, \ldots, k_3 \). Thus the larger of the two primes divides one of \( k_1 \cdots (k_2 + 1) \) and \( k_3 \cdots (k_4 + 1) \) and not the other whence (23) holds. Note also that \( \left(\frac{\ln x}{x}\right)^2 \) is an increasing function of \( x \) for \( x \) positive and less than \( n \). Therefore, by (24), (27), (33) and (35), we find that

\[ A < n^{(1 + \varepsilon)\frac{2\ell \log \log t}{t^2}} \left( \frac{net}{(1 + \varepsilon)^4} \right)^{(1 + \varepsilon)\frac{4}{t}}, \]

hence that

\[ A < \exp \left( (1 + \varepsilon)\ell \left( \frac{2\log n \log \log t}{t^2} + \frac{\log t}{t} + \frac{\log \left(\frac{ne}{(1 + \varepsilon)^4}\right)}{t} \right) \right) \quad (36) \]

for \( n > c_8 \), as required. \( \square \)
For any prime $p$ let $t(p)$ denote the number of positive integers $k$ for which $p \mid k! + 1$. In [9, Theorem 7.5] we noted that

$$t(p) < \frac{(m + 1)(m + 2)}{2} \quad \text{where} \quad m = (3p)^{\frac{3}{4}}. \quad (37)$$

To see this observe that if $n$ and $s$ are positive integers and $p$ divides both $n! + 1$ and $(n + s)! + 1$ then $p$ divides $(n + s)! - n!$ hence $(n + s) \cdots (n + 1) \equiv 1 \pmod{p}$. In particular, $n$ is a solution of the polynomial congruence $(x + s) \cdots (x + 1) \equiv 1 \pmod{p}$, and by Lagrange’s theorem the number of such solutions is at most $s$. Let $I$ be an interval of length $\ell \geq 1$ and let $n_1 < n_2 < \cdots < n_k$ denote all the solutions of $x! + 1 \equiv 0 \pmod{p}$ in $I$. Plainly

$$\sum_{i=1}^{k-1} (n_{i+1} - n_i) \leq \ell, \quad (38)$$

and by our earlier observation at most $s$ of the terms in brackets in the above sum are equal to $s$. Therefore

$$\sum_{i=1}^{k-1} (n_{i+1} - n_i) \geq \sum_{s=1}^{u} s^2, \quad (39)$$

where $u$ is defined by the inequalities

$$\sum_{s=1}^{u} s \leq k - 1 < \sum_{s=1}^{u+1} s.$$

Thus $k$ is at most $\frac{(u+1)(u+2)}{2}$ and by (38) and (39)

$$\ell \geq \frac{u(u + 1)(2u + 1)}{6} > \frac{u^3}{3}. \quad (40)$$

Since all integers $n$ for which $p \mid n! + 1$ lie in the interval $[1, p - 1]$, (37) follows from (40) with $\ell = p$. Further, from (40) we obtain Lemma 3 below, a result which is the special case of Lemma 2 of LUCA and SHPARLINSKI [7] with $f(x)$ equal to 1.

**Lemma 3.** There exists a positive number $c$ such that if $p$ is a prime number and $I$ is an interval of the positive real numbers of length $\ell$ with $\ell \geq 1$ then the number of integers $k$ in $I$ for which $p$ divides $k! + 1$ is at most $c\ell^\frac{2}{3}$. 


For any non-zero integer \( m \) and any prime \( p \) we denote by \( \text{ord}_p m \) the exponent of the largest power of \( p \) which divides \( m \). As usual \( |m|_p \) is the \( p \)-adic absolute value of \( m \) normalized so that
\[
|m|_p = p^{-\text{ord}_p m}.
\]

**Lemma 4.** There exists a positive number \( c_1 \) such that if \( p \) is a prime number, \( n \) a positive integer and \( I \) a subinterval of \([1, n]\) of length \( \ell \geq 2 \) then
\[
(\log p) \, \text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) < \frac{2}{3} \ell \log \ell \log n + c_1 \ell \log n + n \log n. \tag{41}
\]
Further, for each pair of positive real numbers \( \varepsilon \) and \( \varepsilon_1 \) there exist positive numbers \( c_2 \) and \( c_3 \) such that if \( \ell \) exceeds \( \varepsilon_1 n \) and \( n \) exceeds \( c_3 \) then
\[
(\log p) \, \text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) < (1 + \varepsilon)^2 \left( \frac{2}{9} \ell \log \ell \right)^2 + c_2 n \log n. \tag{42}
\]

**Proof.** Let \( i_1, \ldots, i_u \) be the integers \( i \) in \( I \) for which \( p \) divides \( i! + 1 \). By Lemma 3 there is a positive number \( c \) such that
\[
u \leq c \ell^{3/2}. \tag{43}
\]
Put \( h_t = \text{ord}_p (i_t! + 1) \) for \( t = 1, \ldots, u \). We may suppose that
\[h_1 \geq \cdots \geq h_u.
\]
Then, by (13) of Lemma 2,
\[
p^{h_t} < n^{\frac{\ell}{\ell - 1}}, \tag{44}
\]
for \( t = 2, \ldots, u \). In particular by (43) and (44),
\[
(\log p)(h_2 + \cdots + h_u) < \ell \log n \left( 1 + \int_2^{c_3^{3/2}} \frac{1}{t - 1} \, dt \right)
< \frac{2}{3} \ell \log \ell \log n + c_4 \ell \log n, \tag{45}
\]
where \( c_4 \) is a positive number. Since
\[
h_1 \log p \leq n \log n \tag{46}
\]
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and

$$\text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) = h_1 + \cdots + h_u, \quad (47)$$

(41) follows from (45) and (46).

Let $c_5, c_6, \ldots$ denote positive numbers which depend on $\varepsilon$ and $\varepsilon_1$ and suppose that $\ell$ exceeds $\varepsilon_1 n$. Then by (43),

$$u \leq c n^{\frac{2}{3}} < n^{\frac{13}{18} - \frac{1}{36}},$$

for $n > c_5$. Thus for $n > c_6$, (15) of Lemma 2 holds with $\delta = \frac{1}{36}$ and, since $\ell > \varepsilon_1 n$, (16) of Lemma 2 also holds. Therefore by (17) of Lemma 2, for $n > c_7$,

$$(\log p) h_t < (1 + \varepsilon) \ell \left( \frac{\log t}{t} + \frac{\log (t^{-1})}{t} + \frac{2 \log n \cdot \max(1, \log \log t)}{(t - 1)^2} \right), \quad (48)$$

for $t = 3, \ldots, u$. Since the expression on the right hand side of (48) is a decreasing function of $t$ for $t > \varepsilon$, we see that

$$(\log p)(h_4 + \cdots + h_u) < (1 + \varepsilon) \ell \int_{3}^{c \ell^{\frac{2}{3}}} \log t \cdot \frac{\log t}{t} + \frac{\log (t^{-1})}{t} + \frac{2 \log n \cdot \max(1, \log \log t)}{(t - 1)^2} dt$$

and so

$$(\log p)(h_4 + \cdots + h_u) < (1 + \varepsilon) \frac{2}{9} \ell (\log \ell)^2 + c_8 n \log n. \quad (49)$$

Since

$$(h_1 + h_2 + h_3) \log p < 3 n \log n, \quad (50)$$

(42) now follows from (47), (49) and (50). □

**Lemma 5.** Let $\varepsilon$ be a positive real number. There exists a positive number $c$, which depends on $\varepsilon$, such that if $p$ is a prime number, $n$ an integer with $n \geq 2$ and $I$ a subinterval of $[1, n]$ of length $\ell$ then

$$(\log p) \text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) < \frac{2}{9} \ell (\log n)^2 + \varepsilon n (\log n)^2 + cn \log n. \quad (51)$$
Proof. Let \( c_1, c_2, \ldots \) denote positive numbers which depend on \( \varepsilon \). By (42) of Lemma 4, if \( \ell \) exceeds \( \varepsilon n \) and \( n \) exceeds \( c_1 \),

\[
\left( \log p \right) \text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) < \frac{2}{9} \ell (\log n)^2 + \frac{2}{9} \varepsilon n (\log n)^2 + c_2 n \log n. \tag{52}
\]

On the other hand if \( 2 \leq \ell \leq \varepsilon n \) then by (41) of Lemma 4,

\[
\left( \log p \right) \text{ord}_p \left( \prod_{i \in I} (i! + 1) \right) < \frac{2}{3} \varepsilon n (\log n)^2 + c_3 n \log n; \tag{53}
\]

plainly (53) holds if \( \ell \leq 2 \). Therefore from (52) and (53), we obtain (51) with \( c_4 \) in place of \( c \) for \( n > c_1 \), hence (51) holds for \( n \geq 2 \) and our result follows. \( \square \)

3. Proof of Theorem 1

Let \( \delta \) be a positive real number with \( \delta < \frac{1}{100} \). Put \( \delta' = 10\delta \),

\[
\lambda = \frac{\sqrt{145} - 1}{8} + \delta', \tag{54}
\]

and \( \lambda_1 = \lambda + \delta' \). Note that \( \lambda < \frac{3}{2} \). Let \( c_1, c_2, \ldots \) denote positive numbers which depend on \( \delta \). We may suppose that there exist only finitely many odd positive integers \( n \) for which

\[
p(n! + 1) \leq \lambda_1 n,
\]

since otherwise the theorem holds. Thus there exists a positive integer \( c_1 \) such that for each odd integer \( n \) with \( n > c_1 \),

\[
p(n! + 1) > \lambda_1 n. \tag{55}
\]

We shall show that this leads to a contradiction and the theorem then follows.

Since (55) holds, we also have

\[
P(n! + 1) < \frac{\lambda}{\lambda - 1} n, \tag{56}
\]
for all odd integers $n$ with $n > c_2$. To see this, observe that if $q = P(n!+1)$ with $n > 1$ and
\[ q \geq \frac{\lambda}{\lambda-1} n, \tag{57} \]
then $q$ is odd and, by (4),
\[ q \mid (q - n - 1)! + 1. \]
But then
\[ p((q - n - 1)! + 1) \leq q = \frac{1}{1 - \frac{n+1}{q}} (q - n - 1). \]
We have $q > \lambda$ and, by (57), $\frac{n}{q} \leq \frac{\lambda - 1}{\lambda}$ hence
\[ \frac{1}{1 - \frac{n+1}{q}} \leq \frac{1}{1 - \frac{1}{q} - \left(\frac{\lambda - 1}{\lambda}\right)} = \frac{q\lambda}{q - \lambda}. \]
But $\frac{q\lambda}{q - \lambda} < \lambda_1$ for $n > c_3$ since $q > n$. Thus
\[ p((q - n - 1)! + 1) \leq \lambda_1 (q - n - 1). \]
Furthermore, $q - n - 1 > c_1$ for $n > c_4$ by (57) and this contradicts (55). Therefore (56) holds.

The proof now proceeds by a comparison of estimates for
\[ Z = \prod_{n \text{ odd}, n > c_2}^N (n! + 1). \]
Put $R = \{ n \in \mathbb{Z} \mid n \text{ odd, } c_2 < n \leq N \}$. Observe that if $p \mid n! + 1$ with $n$ in $R$ then, by (55), $p > \lambda_1 n$ and, by (56), $p < \frac{\lambda}{\lambda - 1} n$. In particular,
\[ \frac{\lambda - 1}{\lambda} p < n < \frac{1}{\lambda_1} p. \]
Put
\[ I_p = \left( \frac{\lambda - 1}{\lambda} p, \min\left( N, \frac{1}{\lambda_1} p \right) \right). \]
Since $n! \geq \left( \frac{n}{e} \right)^n$, \[ Z > \exp\left( \sum_{n \in R} (n \log n - n) \right) \]

so

\[ \log Z > (1 - \delta) \frac{N^2}{4} \log N, \] (58)

provided that \( N \) exceeds \( c_5 \).

On the other hand

\[ Z = \prod_p |Z|_p^{-1} \leq \prod_p \left| \prod_{n \in I_p \cap R} (n! + 1) \right|_p^{-1}. \]

Put

\[ A(p) = (\log p) \text{ord}_p \left( \prod_{n \in I_p \cap R} (n! + 1) \right). \]

Then

\[ Z \leq \exp \left( \sum_{p < \frac{\lambda}{\lambda - 1} N} A(p) \right). \]

Thus by (51) of Lemma 5, with \( \varepsilon = \delta \),

\[ \log Z \leq \frac{2}{9} (\log N)^2 \sum_{p < \frac{\lambda}{\lambda - 1} N} \ell(p) \]

\[ + (\delta N (\log N)^2 + c_6 N \log N) \pi \left( \frac{\lambda}{\lambda - 1} N \right), \] (59)

where \( \ell(p) \), the length of \( I_p \), is given by

\[ \ell(p) = \left( \frac{1}{\lambda_1} - \frac{\lambda - 1}{\lambda} \right) p \quad \text{when } p \leq \lambda_1 N \]

and by

\[ \ell(p) = N - \left( \frac{\lambda - 1}{\lambda} \right) p \quad \text{when } p \geq \lambda_1 N. \]

By (54), (59) and the prime number theorem,

\[ \log Z \leq \frac{2}{9} (\log N)^2 \sum_{p < \frac{\lambda}{\lambda - 1} N} \ell(p) + 4\delta N^2 \log N + c_7 N^2. \] (60)
Further
\[ \sum_{p < \frac{\lambda_1}{\lambda - 1} N} \ell(p) = \sum_{p \leq \lambda_1 N} \left( \frac{1}{\lambda_1} - \frac{\lambda - 1}{\lambda} \right) p + \sum_{\lambda_1 N < p < \frac{\lambda_1}{\lambda - 1} N} \left( N - \frac{\lambda - 1}{\lambda} p \right) \]
\[ = \frac{1}{\lambda_1} \sum_{p \leq \lambda_1 N} p + N \left( \sum_{\lambda_1 N < p < \frac{\lambda_1}{\lambda - 1} N} 1 \right) - \frac{\lambda - 1}{\lambda} \sum_{p < \frac{\lambda_1}{\lambda - 1} N} p. \]

Thus by the prime number theorem and Abel summation, for \( N > c_8 \),
\[ \sum_{p < \frac{\lambda_1}{\lambda - 1} N} \ell(p) < (1 + \delta) \left( \frac{\lambda_1}{2} + \left( \frac{\lambda}{\lambda - 1} - \lambda_1 \right) - \frac{\lambda - 1}{2(\lambda - 1)} \right) \frac{N^2}{\log N} \]
\[ < (1 + \delta) \left( \frac{\lambda}{2(\lambda - 1)} - \frac{\lambda_1}{2} \right) \frac{N^2}{\log N}, \]
and so by (60), and the fact that \( \lambda_1 \) exceeds \( \lambda \),
\[ \log Z < \frac{(1 + \delta)}{9} \left( \frac{\lambda}{\lambda - 1} - \lambda \right) N^2 \log N + 4\delta N^2 \log N + c_7 N^2. \]
(61)

Comparing (58) and (61) we find that for \( N > c_9 \),
\[ \frac{1 - \delta}{4} < \frac{(1 + \delta)}{9} \frac{\lambda(2 - \lambda)}{(\lambda - 1)} + 4\delta + \frac{c_8}{\log N}. \]

By (54) we obtain a contradiction for \( N \) sufficiently large and the result now follows.

### 4. Proof of Theorem 2

We may suppose that \( 0 < \varepsilon < \frac{1}{4} \). Put \( \gamma = \frac{11}{2} - 18\varepsilon \) and let \( B(\gamma) \) be the set of positive integers for which
\[ P(n! + 1) \geq \gamma n. \]
(62)

We shall show that for \( n \) sufficiently large the set \( B(\gamma) \cap \{1, \ldots, n\} \) has cardinality at least \( \frac{\varepsilon}{3} n \) and hence the result follows. Accordingly suppose that \( N \) is a positive integer for which
\[ |B(\gamma) \cap \{1, \ldots, N\}| \leq \frac{\varepsilon}{3} N. \]
(63)
Let $c_1, c_2, \ldots$ denote positive numbers which depend on $\varepsilon$. Our proof proceeds by a comparison of estimates for

$$Z = \prod_{n=1 \atop n \notin B(\gamma)}^{N} (n! + 1).$$

Since $n! \geq \left(\frac{n}{e}\right)^n$,

$$Z > \exp \left( \sum_{n=2}^{N} (n \log n - n) - |B(\gamma) \cap \{1, \ldots, N\}| N \log N \right)$$

whence, by (63),

$$\log Z > (1 - \varepsilon) \frac{N^2 \log N}{2},$$

provided that $N > c_1$.

Notice that if $p \mid n! + 1$ and $n \notin B(\gamma)$ then $n < p < \gamma n$ hence $\frac{1}{\gamma} p < n < p$. Put

$$I_p = \left( \frac{1}{\gamma} p, \min\{N, p\} \right),$$

and

$$A(p) = (\log p) \text{ ord}_p \left( \prod_{n \in I_p} (n! + 1) \right).$$

Then

$$Z \leq \exp \left( \sum_{p<\gamma N} A(p) \right).$$

Thus, by (51) of Lemma 5 with $\frac{\varepsilon}{6}$ in place of $\varepsilon$,

$$\log Z < \frac{2}{9} (\log N)^2 \sum_{p<\gamma N} \ell(p) + \left( \frac{\varepsilon}{6} N (\log N)^2 + c_2 N \log N \right) \pi(\gamma N),$$

(65)

where $\ell(p)$, the length of $I_p$, is given by

$$\ell(p) = \left( 1 - \frac{1}{\gamma} \right) p \quad \text{for } p \leq N$$

and

$$\ell(p) = N - \frac{1}{\gamma} p \quad \text{for } p > N.$$
Since \( \gamma < \frac{11}{2} \) it follows from (65) and the prime number theorem that
\[
\log Z < \frac{2}{9} (\log N)^2 \sum_{p < \gamma N} \ell(p) + \varepsilon N^2 \log N + c_3 N^2.
\] (66)

Further
\[
\sum_{p < \gamma N} \ell(p) = \sum_{p \leq N} \left( 1 - \frac{1}{\gamma} \right) p + \sum_{N < p < \gamma N} \left( N - \frac{1}{\gamma} p \right)
\]
\[
= \sum_{p \leq N} p + \sum_{N < p < \gamma N} N - \frac{1}{\gamma} \sum_{p < \gamma N} p.
\]
Thus, by the prime number theorem and Abel summation, for \( N > c_4 \),
\[
\sum_{p < \gamma N} \ell(p) < (1 + \varepsilon) \left( \frac{N^2}{2 \log N} + \frac{(\gamma - 1)N^2}{\log N} - \frac{\gamma N^2}{2 \log N} \right)
\]
and so by (66),
\[
\log Z < \frac{(1 + \varepsilon)(\gamma - 1)N^2 \log N + \varepsilon N^2 \log N + c_3 N^2}{9}.
\] (67)
Comparing (64) and (67) we find that for \( N > c_5 \),
\[
\frac{1 - \varepsilon}{2} < (1 + \varepsilon) \left( \frac{\gamma - 1}{9} \right) + \varepsilon + \frac{c_3}{\log N}.
\]
But \( \gamma < \frac{11}{2} \) and so for \( N \) sufficiently large we obtain a contradiction. Thus (63) does not hold for \( N \) sufficiently large and the result now follows.

References


