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Trigonometric identities and quadratic residues

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Abstract. In this paper, we obtain some novel identities involving trigonometric functions. Let n be any positive odd integer. We mainly show that

$$\sum_{r=0}^{n-1}\frac{1}{1+\sin 2\pi\frac{x+r}{n}+\cos 2\pi\frac{x+r}{n}}=\frac{(-1)^{(n-1)/2}n}{1+(-1)^{(n-1)/2}\sin 2\pi x+\cos 2\pi x}$$

for any complex number with $x+1/2, x+(-1)^{(n-1)/2}/4\notin\mathbb{Z}$, and

$$\sum_{j,k=0}^{n-1}\frac{1}{\sin 2\pi\frac{x+j}{n}+\sin 2\pi\frac{y+k}{n}}=\frac{(-1)^{(n-1)/2}n^2}{\sin 2\pi x+\sin 2\pi y}$$

for all complex numbers x and y with $x+y, x-y-1/2\notin\mathbb{Z}$. We also determine the values of $\prod_{k=1}^{(p-1)/2}(1+\tan\pi\frac{k^2}{p})$ and $\prod_{k=1}^{(p-1)/2}(1+\cot\pi\frac{k^2}{p})$ for any odd prime p. In addition, we pose several conjectures on the values of $\prod_{k=1}^{(p-1)/2}(x-e^{2\pi ik^2/p})$ with p an odd prime and x a root of unity.

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