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Number of solutions to $a^x + b^y = c^z$ with $\gcd(a, b) > 1$

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Abstract. We show that there are at most two solutions in positive integers (x, y, z) to the equation $a^x + b^y = c^z$ for positive integers a, b, and c all greater than one, with just one exceptional case when $\gcd(a, b) = 1$, and just one exceptional infinite family of cases when $\gcd(a, b) > 1$ (two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) are considered the same solution if $\{a^{x_1}, b^{y_1}\} = \{a^{x_2}, b^{y_2}\}$). The case in which $\gcd(a, b) = 1$ has been handled in a series of successive results by Scott and Styer, Hu and Le, and Miyazaki and Pink, who showed that there are at most two solutions, excepting $(\{a, b\}, c) = (\{3, 5\}, 2)$, which gives three solutions. So, here we treat the case $\gcd(a, b) > 1$, showing that in this case there are at most two solutions, excepting $(a, b, c) = (2^u, 2^v, 2^w)$ with $\gcd(uv, w) = 1$, which gives an infinite number of solutions. This generalizes the work of Bennett, who proved, for both $\gcd(a, b) = 1$ and $\gcd(a, b) > 1$, there are at most two solutions (y, z) to the equation $a + b^y = c^z$, and conjectured there are exactly eleven (a, b, c) giving two solutions to this equation (assuming b and c are not perfect powers).

For both $\gcd(a, b) = 1$ and $\gcd(a, b) > 1$, there are an infinite number of (a, b, c) giving two solutions (x, y, z) to the title equation, which are described in detail in this and a cited previous paper.

In a further result, in which we no longer say that two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) are considered the same solution if $\{a^{x_1}, b^{y_1}\} = \{a^{x_2}, b^{y_2}\}$, we list all cases with more than two solutions.

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