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The topology of semilocal connectedness and s-continuity of multivalued maps

By JANINA EWERT (Slupsk)

A topological space $(Y,\mathcal{T})$ is said to be semilocally connected if for each open set $U \subset Y$ and each point $x \in U$ there exists an open set V such that $x \in V \subset U$ and $Y \setminus V$ consists of a finite number of components [2]. This notion was introduced by WHYBURN [9. p. 19] under the additional assumption that $(Y,\mathcal{T})$ is a connected T_1-space. Similarly as in [2, 3, 4, 8] we consider the semilocal connectedness of general topological spaces.

In a topological space $(Y,\mathcal{T})$ the family $\{U \in \mathcal{T} : Y \setminus U \text{ is connected}\}$ is a subbase of a topology denoted as $\mathcal{T}^\star$. Obviously, $\mathcal{T}^\star \subset \mathcal{T}$. Further $(Y,\mathcal{T}^\star)$ is semilocally connected [2, Th. 3.1] and $(Y,\mathcal{T})$ is a semilocally connected space iff $\mathcal{T} = \mathcal{T}^\star$ [2, Th. 3.3].

Let X, Y be topological spaces. A function $f : X \to Y$ is called s-continuous at a point $x \in X$ if for each open set $V \subset Y$ containing $f(x)$ and having connected complement there is an open set U satisfying $x \in U \subset f^{-1}(V)$. A function f is called s-continuous if it is s-continuous at each point [2, 3, 4]. One can readily check that:

(A) A function $f : X \to (Y,\mathcal{T})$ is s-continuous if and only if $f : X \to (Y,\mathcal{T}^\star)$ is continuous [8, Prop. 9].

Now, let $F : X \to Y$ be a multivalued map. For a set $B \subset Y$ we denote $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. A multivalued map $F : X \to Y$ is said to be upper (lower) s-continuous at a point $x \in X$ if for each open set $V \subset Y$ with $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) and $Y \setminus V$ connected there exists an open set $U \subset X$ for which $x \in U \subset F^+(V)$, (resp. $x \in U \subset F^-(V)$) holds, [6]. A multivalued map F is called upper (lower) s-continuous if it is upper (lower) s-continuous at each point.

Theorem (A) simply rewritten for multivalued maps is not true in general. For a multivalued map $F : X \to (Y,\mathcal{T})$ the upper (lower) $\mathcal{T}^\star$-semicontinuity implies the upper (lower) s-continuity, but the inverse does not hold.

Example 1. Let $(R, \mathcal{T})$ be the space of real numbers with the natural topology and let $D \subset R$ be a non-Borel set. We define a multivalued map $F : R \to R$ assuming

$$F(x) = \begin{cases} [1,2] \cup [5,7], & \text{if } x \in D. \\ [1,3] \cup [6,7], & \text{if } x \notin D. \end{cases}$$

The map F is lower s-continuous. For $V_1 = (5,6)$, $V_2 = (0,4) \cup (5,8)$ we have $F^-(V_1) = D$, $F^+(V_2) = R \backslash D$, thus F is not lower semicontinuous. Since $\mathcal{T} = \mathcal{T}^\star$ the map F is not lower $\mathcal{T}^\star$-semicontinuous. (Moreover it is not upper nor lower Borel measurable).

Example 2. Let $(R^2, \mathcal{T})$ be the plane with the natural topology, B_n the closed ball with center p_n and radius $\frac{1}{2}$, where $p_n = (0, 2n-1)$ for $n \geq 1$. Let us put

$$A = \bigcup_{n=0}^{\infty} ([-1,1] \times \{2n\}) \cup \{-1,1\} \times R$$

and $A_n = A \cup \{0\} \times [2n-1, \infty)$ for $n \geq 1$. Then we define the multivalued map $F : R \to R^2$ as follows

$$F(x) = \begin{cases} A_n, & \text{if } x = \frac{1}{n},\ n \geq 1 \\ A, & \text{if } x \notin \left\{\frac{1}{n} :\ n \geq 1\right\}. \end{cases}$$

If $V \subset R^2$ is an open set such that $R^2 \backslash V$ is connected, then $F^+(V) = R$ or $F^+(V) = R \backslash \left\{1, \frac{1}{2}, \ldots, \frac{1}{k}\right\}$ for some $k \geq 1$, hence F is upper s-continuous. On the other hand the set $W = R^2 \backslash \bigcup\limits_{n=1}^{\infty} B_n$ is open and $F^+(W) = R \backslash \left\{\frac{1}{n} : n \geq 1\right\}$, so F is not upper semicontinuous. Since we have $\mathcal{T} = \mathcal{T}^\star$, the map F is not upper $\mathcal{T}^\star$-semicontinuous, either.

Theorem 3. *Let $F : X \to (Y, \mathcal{T})$ be a multivalued map with connected values. Then F is lower s-continuous if and only if $F : X \to (Y, \mathcal{T}^\star)$ is lower semicontinuous.*

Proof. Let $\mathcal{B}$ be a base of $\mathcal{T}^\star$ consisting of $\mathcal{T}$-open sets whose complements have a finite number of components in $(Y, \mathcal{T})$. If $W \in \mathcal{B}$, then $Y \backslash W = \bigcup\limits_{j=1}^{n} M_j$ where the M_j are pairwise disjoint connected closed subsets of $(Y, \mathcal{T})$. For a point $x \in F^+(Y \backslash W)$ we have $F(x) \subset M_j$ for some $j \leq n$, thus $F^+(Y \backslash W) = \bigcup\limits_{j=1}^{n} F^+(M_j)$. It follows from the lower s-continuity that $F^+(M_j)$ is a closed set, so $F^-(W)$ is open and the proof is completed.

As Example 2 shows, for upper s-continuity the theorem analogous to the above one is not true. We remind that a multivalued map $F : X \to Y$ is said to be of lower (upper) Baire class 1 if for each open set $W \subset Y$ the set $F^-(W)$ (resp. $F^+(W)$) is an F_σ set, [5].

Now let $S(Y)$ denote the family of all non-empty subsets of Y and let $\tau_\mathcal{T}$ be the Vietoris topology on $S(Y)$ given by a topology $\mathcal{T}$ on Y, [1]. For multivalued maps $F, F_n : X \to Y$, $n \geq 1$, we will write $F = \tau_\mathcal{T} - \lim F_n$ iff for each $x \in X$ the sequence $\{F_n(x) : n \geq 1\}$ converges to $F(x)$ in the space $(S(Y), \tau_\mathcal{T})$. Then we have the following result:

Theorem 4. *Let X be a topological space, $(Y, \mathcal{T})$ a locally connected separable metrizable space, and let $F, F_n : X \to Y$, $n \geq 1$, be multivalued maps with $F = \tau_\mathcal{T}\text{-}\lim F_n$. Then*
(a) *if F_n are upper s-continuous maps, then F is of lower Baire class 1;*
(b) *if $F(x)$ is a compact set for each $x \in X$ and the F_n are lower $\mathcal{T}^\star$-semicontinuous, then F is of upper Baire class 1.*

Proof. Let $W \subset Y$ be a non-empty open set; there exists a sequence $\{G_n : n \geq 1\}$ of connected open sets satisfying $W = \bigcup_{n=1}^{\infty} \bar{G}_n$. Assume $W_1 = G_1$. It follows from our assumptions on Y that we can choose a connected open set U_1 satisfying $\bar{G}_1 \subset U_1 \subset \bar{U}_1 \subset W$, then let us put $W_2 = G_1 \cup U_1$. Thus W_1, W_2 are open sets having finitely many of components and $\bar{W}_1 \subset W_2 \subset \bar{W}_2 \subset W$. Assume that we have constructed open sets $W_1, \dots, W_m$ such that each of them has a finite number of components and $\bar{W}_i \subset W_{i+1} \subset W$ for $i \leq m-1$. Let $S_1, \dots, S_k$ be components of the set W_m. We take open connected sets $V_1, \dots, V_k$ satisfying $\bar{S}_j \subset V_j \subset \bar{V}_j \subset W$ for $j \leq k$ and we denote $W_{m+1} = G_{m+1} \cup \bigcup_{j=1}^{k} V_j$. Thus W_{m+1} is open, it has at most $k+1$ components and $\bar{W}_m \subset W_{m+1} \subset W$. So we have shown:

(1) each open set $W \subset Y$ is of the form $W = \bigcup_{n=1}^{\infty} W_n$, where the W_n are open sets having finitely many components and $\bar{W}_n \subset W_{n+1}$ for $n \geq 1$.

Let W be an open set and $x \in F^-(W)$. Then (1) implies that there exists k_0 such that $F(x) \cap W_k \neq \emptyset$ for $k \geq k_0$. From the $\tau_\mathcal{T}$-convergence of the sequence $\{F_n(x) : n \geq 1\}$ for each $k \geq k_0$ there exists n_k such that $F_n(x) \cap W_k \neq \emptyset$ for $n \geq n_k$. Hence we can choose $k \geq 1$ such that $F_{n+k}(x) \cap \bar{W}_k \neq \emptyset$ for each $n \geq 1$; this leads to the inclusion

$$F^-(W) \subset \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F^-_{n+k}(W_k).$$

If $x \notin F^-(W)$, then $F(x) \subset Y \backslash \bar{W}_k$ for each $k \geq 1$. The $\tau_{\mathcal{T}}$-convergence implies that for each k there exists n_k such that $F_n(X) \subset Y \backslash \bar{W}_k$ for $n \geq n_k$, thus for each $k \geq 1$ there exists $n \geq 1$ for which $F_{n+k}(x) \subset Y \backslash \bar{W}_k$ holds. Hence

$$X \backslash F^-(W) \subset \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_{n+k}^+(Y \backslash \bar{W}_k) = X \backslash \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^-(\bar{W}_k).$$

So we have shown

$$F^-(W) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^-(\bar{W}_k). \tag{2}$$

Let $S_{k,1}, \dots, S_{k,m_k}$ be components of $\bar{W}_k$. If the F_n are upper s-continuous, then

$$F_{n+k}^-(\bar{W}_k) = \bigcup_{j=1}^{m_k} F_{n+k}^-(S_{k,j});$$

this set is closed, so F is of lower Baire class 1 and (a) is proved.

Now let $W \subset Y$ be an open set and let $x \in F^+(W)$. Since $F(x) \subset \bigcup_{k=1}^{\infty} W_k$ where the W_k are open sets such as in (1) and $F(x)$ is compact, there exists $k_0 \geq 1$ such that $F(x) \subset W_k$ for each $k \geq k_0$. Furthermore for $k \geq k_0$ there is n_k such that $F_n(x) \subset W_k$ for $n \geq n_k$; hence we obtain that there is $k \geq 1$ such that for every $n \geq 1$ there holds $F_{n+k}(x) \subset \bar{W}_k$. In the consequence we have

$$F^+(W) \subset \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^+(\bar{W}_k).$$

Finally let $x \notin F^+(W)$. Then $F(x) \cap (Y \backslash \bar{W}_k) \neq \emptyset$ for each $k \geq 1$. Using the $\tau_{\mathcal{T}}$-convergence of the sequence $\{F_n(x) : n \geq 1\}$ for each $k \geq 1$ we can choose $n \geq 1$ for which $F_{n+k}(x) \cap (Y \backslash \bar{W}_k) \neq \emptyset$. This implies

$$X \backslash F^+(W) \subset \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_{n+k}^-(Y \backslash \bar{W}_k)$$

and then

$$F^+(W) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^+(\bar{W}_k). \tag{3}$$

The sets $\bar{W}_k$ are $\mathcal{T}^\star$-closed. If the F_n are lower $\mathcal{T}^\star$-semicontinuous, then according to (3), F is of upper Baire class 1.

Let us observe that in Theorem 4 (b) the lower $\mathcal{T}^\star$-semicontinuity of the maps F_n cannot be replaced by lower s-continuity. For instance, it suffices to take $F_n = F$ for $n \geq 1$, where F is the map from Example 1.

For a topological space $(Y, \mathcal{T})$ we denote by $2_c^{(Y,\mathcal{T})}$ the family of all non-empty connected closed subsets of Y (for simplicity we will write 2_c^Y). For a set $U \subset Y$ we denote

$$U^+ = \{B \in 2_c^Y : B \subset U\}, \qquad U^- = \{B \in 2_c^Y : B \cap U \neq \emptyset\}.$$

Then $\mathcal{T}^+$ and $\mathcal{T}^-$ are topologies on 2_c^Y given by the base $\{U^+ : U \in \mathcal{T}\}$ and the subbase $\{U^- : U \in \mathcal{T}\}$ respectively. Similarly are constructed topologies $(\mathcal{T}^\star)^+$ and $(\mathcal{T}^\star)^-$ on 2_c^Y by taking the sets $U \in \mathcal{T}^\star$.

Corollary 5. *Let $F : X \to (Y, \mathcal{T})$ be a multivalued map with values in 2_c^Y. Then F is lower s-continuous if and only if the function $F : X \to (2_c^Y, (\mathcal{T}^\star)^-)$ is continuous.*

According to Example 2 a similar property for upper s-continuity cannot be formulated.

In the sequel we will show that there exists a topology τ on 2_c^Y such that a multivalued map $F : X \to Y$ with values in 2_c^Y is upper s-continuous if and only if the function $F : X \to (2_c^Y, \tau)$ is continuous. It will be obtained by constructing of some quasi-uniformities on 2_c^Y (for quasi-uniform spaces see [7]).

For subsets U_1, U_2 of a topological space $(Y, \mathcal{T})$ such that $U_1 \subset U_2$ we put:

$$H_{U_1,U_2} = \{(A, B) \in 2_c^Y \times 2_c^Y : A \cap U_1 \neq 0 \Rightarrow B \cap U_2 \neq 0\}$$
$$M_{U_1,U_2} = \{(A, B) \in 2_c^Y \times 2_c^Y : A \subset U_1 \Rightarrow B \subset U_2\}.$$

It can be easily verified that these sets have the following properties:

(P1) $\Delta \subset H_{U_1,U_2}$ and $\Delta \subset M_{U_1,U_2}$;

(P2) if $U_1 \subset U_2 \subset U_3$, then $H_{U_1,U_2} \subset H_{U_1,U_3}$ and $M_{U_1,U_2} \subset M_{U_1,U_3}$; $H_{U_2,U_3} \subset H_{U_1,U_3}$, $M_{U_2,U_3} \subset M_{U_1,U_3}$;

(P3) $H^{-1}_{U_1,U_2} = M_{Y \setminus U_2, Y \setminus U_1}$ and $M^{-1}_{U_1,U_2} = H_{Y \setminus U_2, Y \setminus U_1}$;

(P4) if $U_1 \subset U_2 \subset U_3$ then $H_{U_1,U_2} \cdot H_{U_2,U_3} \subset H_{U_1,U_3}$ and $M_{U_1,U_2} \cdot M_{U_2,U_3} \subset M_{U_1,U_3}$.

Theorem 5. *Let $(Y, \mathcal{T})$ be a locally connected T_4 space and let*

$$\mathcal{G}_1 = \{H_{U_1,U_2} : U_1, U_2 \in \mathcal{T},\ \bar{U}_1 \subset U_2 \text{ and } Y \setminus U_1, Y \setminus U_2 \text{ are connected}\},$$
$$\mathcal{G}_2 = \{M_{U_1,U_2} : U_1, U_2 \in \mathcal{T},\ \bar{U}_1 \subset U_2 \text{ and } Y \setminus U_1, Y \setminus U_2 \text{ are connected}\},$$
$$\mathcal{P}_1 = \{H_{U_1,U_2} : \bar{U}_1 \subset U_2 \text{ and } U_1, U_2 \text{ are connected open sets}\},$$
$$\mathcal{P}_2 = \{M_{U_1,U_2} : \bar{U}_1 \subset U_2 \text{ and } U_1, U_2 \text{ are connected open sets}\}.$$

Then

(a) $\mathcal{G}_1, \mathcal{G}_2, \mathcal{P}_1, \mathcal{P}_2$ *are subbases of quasi-uniformities* $\mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{U}_1$ *and* $\mathfrak{U}_2$ *on* 2^Y_c*;*

(b) $\mathfrak{V}_1^{-1} = \mathfrak{U}_2$ *and* $\mathfrak{V}_2^{-1} = \mathfrak{U}_1$*;*

(c) $\tau_{\mathfrak{V}_1} = (\mathcal{T}^\star)^-$, $\tau_{\mathfrak{U}_1} = \mathcal{T}^-$, $\tau_{\mathfrak{V}_2} \subset (\mathcal{T}^\star)^+$ *and* $\tau_{\mathfrak{U}_2} = \mathcal{T}^+$*;*

(d) Let X *be a topological space and* $F : X \to Y$ *a multivalued map with connected closed values. Then* F *is upper s-continuous if and only if the function* $F : X \to (2^Y_c, \mathfrak{V}_2)$ *is continuous.*

Proof. (a) For any open sets U_1, U_2 such that $\bar{U}_1 \subset U_2$ and $Y\backslash U_1$, $Y\backslash U_2$ are connected there exists an open set U with $Y\backslash U$ connected and $\bar{U}_1 \subset U \subset \bar{U} \subset U_2$. Then $H_{U_1,U}, H_{U,U_2} \in \mathcal{G}_1$, $M_{U_1,U}, M_{U,U_2} \in \mathcal{G}_2$ and $H_{U_1,U} \circ H_{U,U_2} \subset H_{U_1,U_2}$, $M_{U_1,U} \circ M_{U,U_2} \subset M_{U_1,U_2}$. These together with (P.1) give that $\mathcal{G}_1, \mathcal{G}_2$ are subbases of some quasi-uniformities $\mathfrak{V}_1, \mathfrak{V}_2$ on 2^Y_c. Using analogous arguments we conclude that $\mathcal{P}_1$ and $\mathcal{P}_2$ are subbases of some quasi-uniformities $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on 2^Y_c.

(b) Let us take open sets U_1, U_2 such that $\bar{U}_1 \subset U_2$ and $Y\backslash U_1$, $Y\backslash U_2$ are connected. We can choose connected open sets V_1, V_2 satisfying

$$Y\backslash U_2 \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset Y\backslash \bar{U}_1.$$

Then we have $H_{U_1,U_2} \in \mathcal{G}_1$, $M_{U_1,U_2} \in \mathcal{G}_2$ and

$$H^{-1}_{U_1,U_2} = M_{Y\backslash U_2, Y\backslash U_1} \supset M_{V_1,V_2},\ M^{-1}_{U_1,U_2} = H_{Y\backslash U_2, Y\backslash U_1} \supset H_{V_1,V_2}.$$

Since $M_{V_1,V_2} \in \mathcal{P}_2$ and $H_{V_1,V_2} \in \mathcal{P}_1$ we obtain $\mathfrak{V}_1^{-1} \subset \mathfrak{U}_2$ and $\mathfrak{V}_2^{-1} \subset \mathfrak{U}_1$. Now, let U_1, U_2 be connected open sets and let $\bar{U}_1 \subset U_2$. We choose connected open sets V_1, V_2 for which $\bar{U}_1 \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset U_2$ holds. Then we have $H_{U_1,U_2} \in \mathcal{P}_1$, $M_{U_1,U_2} \in \mathcal{P}_2$ and

$$H^{-1}_{U_1,U_2} = M_{Y\backslash U_2, Y\backslash U_1} \supset M_{Y\backslash \bar{V}_2, Y\backslash \bar{V}_1}$$
$$M^{-1}_{U_1,U_2} = H_{Y\backslash U_2, Y\backslash U_1} \supset H_{Y\backslash \bar{V}_2, Y\backslash \bar{V}_1}.$$

Since $M_{Y\backslash \bar{V}_2, Y\backslash \bar{V}_1} \in \mathcal{G}_2$ and $H_{Y\backslash \bar{V}_2, Y\backslash \bar{V}_1} \in \mathcal{G}_1$ we obtain $\mathfrak{U}_1^{-1} \subset \mathfrak{V}_2$ and $\mathfrak{U}_2^{-1} \subset \mathfrak{V}_1$; thus (b) has been shown.

(c) For any set $A \subset Y$ we have

$$H_{U_1,U_2}[A] = \begin{cases} U_2^-, & \text{if } A \cap U_1 \neq 0 \\ 2^Y_c, & \text{if } A \cap U_1 = 0 \end{cases}$$

$$M_{U_1,U_2}[A] = \begin{cases} U_2^+, & \text{if } A \subset U_1 \\ 2^Y_c, & \text{if } A \cap (Y\backslash U_1) \neq 0. \end{cases}$$

Thus

$$\tau_{\mathfrak{V}_1} \subset (\mathcal{T}^\star)^-,\ \tau_{\mathfrak{V}_2} \subset (\mathcal{T}^\star)^+,\ \tau_{\mathfrak{U}_1} \subset \mathcal{T}^- \text{ and } \tau_{\mathfrak{U}_2} \subset \mathcal{T}^+. \tag{1}$$

Let $\mathcal{B}^\star$ be a base of the topology $\mathcal{T}^\star$ consisting of the open sets whose complements have a finite number of components in $(Y, \mathcal{T})$. Then $\{U^- : U \in \mathcal{B}^\star\}$ is a subbase of the topology $(\mathcal{T}^\star)^-$ in 2^Y_c. If $U \in \mathcal{B}^\star$, then $Y \backslash U = \bigcup_{i=1}^{n} D_i$, where the D_i are connected closed pairwise disjoint sets. For any connected set $C \subset \bigcup_{i=1}^{n} D_i$ there exists $i \leq n$ such that $C \subset D_i$, thus $\left(\bigcup_{i=1}^{n} D_i\right)^+ = \bigcup_{i=1}^{n} D_i^+$. From this we obtain $\left(\bigcap_{i=1}^{n} U_i\right)^- = \bigcap_{i=1}^{n} U_i^-$, where $U_i = Y \backslash D_i$ and $U = \bigcap_{i=1}^{n} U_i$. Let $A \in U^-$; then we can choose a point $y \in A \cap \bigcap_{i=1}^{n} U_i$ and open sets V_i with $y \in V_i \subset \bar{V}_i \subset U_i$ and $Y \backslash V_i$ connected for $i \leq n$. So $H_{V_i, U_i} \in \mathcal{G}_1$ and $U^- = \bigcap_{i=1}^{n} U_i^- = \bigcap_{i=1}^{n} H_{V_i, U_i}[A]$, which implies $U^- \in \tau_{\mathfrak{V}_1}$. As a consequence we have $(\mathcal{T}^\star)^- \subset \tau_{\mathfrak{V}_1}$ hence $\tau_{\mathfrak{V}_1} = (\mathcal{T}^\star)^-$.

Now, let $\mathcal{B}$ be a base of the topology $\mathcal{T}$ consisiting of connected open sets. The family $\{U^- : U \in \mathcal{B}\}$ is a subbase of the topology $\mathcal{T}^-$ on 2^Y_c. For any $U \in \mathcal{B}$ and $A \in U^-$ we take a point $y \in A \cap U$ and a set $V \in \mathcal{B}$ such that $y \in V \subset \bar{V} \subset U$. Then $H_{V,U} \in \mathcal{P}_1$ and $U^- = H_{V,U}[A]$; it gives $\mathcal{T}^- \subset \tau_{\mathfrak{U}_1}$ and finally $\tau_{\mathfrak{U}_1} = \mathcal{T}^-$.

Let $U \in \mathcal{T}$ and $A \in U^+$. Since A is a connected closed set, it follows from assumptions on $(Y, \mathcal{T})$ that we can take connected open sets V_1, V_2 for which $A \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset U$ holds. Then $M_{V_1, V_2} \in \mathcal{P}_2$ and $M_{V_1, V_2}[A] = V_2^+ \subset U^+$, so $U^+ \in \tau_{\mathfrak{U}_2}$. Since $\{U^+ : U \in \mathcal{T}\}$ is a base of $\mathcal{T}^+$, in virtue of (1) we have $\tau_{\mathfrak{U}_2} = \mathcal{T}^+$.

(d) Let $F : X \to Y$ be a multivalued map with values in 2^Y_c. Let us assume that F is upper s-continuous, $M_{U_1, U_2} \in \mathcal{G}_2$ and $x_0 \in X$. If $F(x_0) \not\subset U_1$, then $M_{U_1, U_2}[F(x_0)] = 2^Y_c$; hence for each $x \in X$ we have $F(x) \in M_{U_1, U_2}[F(x_0)]$. In the other case, if $F(x_0) \subset U_1$, then according to the upper s-continuity there is a neighbourhood W of x_0 such that $F(x) \subset U_1$ for $x \in W$. Thus $F(x) \in M_{U_1, U_2}[F(x_0)]$ for each $x \in W$ and this implies the continuity of the function $F : X \to (2^Y_c, \mathfrak{V}_2)$.

Conversely, suppose that $F : X \to (2^Y_c, \mathfrak{V}_2)$ is continuous. Let $x_0 \in X$, $U \in \mathcal{T}$, $F(x_0) \subset U$ and let $Y \backslash U$ be connected. We choose an open set V such that $Y \backslash V$ is connected and $F(x_0) \subset V \subset \bar{V} \subset U$. Since $M_{V,U} \in \mathcal{G}_2$ there exists a neighbourhood W of x_0 such that $F(x) \in M_{V,U}[F(x_0)]$ for $x \in W$. But $M_{V,U}[F(x_0)] = U^+$, so $F(x) \subset U$ for $x \in W$ and the proof is completed.

Corollary 6. *Let $(Y, \mathcal{T})$ be a locally connected T_4 space. Then there exist uniformities S_1, S_2 on 2^Y_c such that for any multivalued map $F : X \to Y$ with closed connected values the following conditions are equivalent:*
(a) *F is upper semicontinuous and lower s-continuous (lower semicontinuous and upper s-continuous);*
(b) *the function $F : X \to (2^Y_c, S_1)$, (resp. $F : X \to (2^Y_c, S_2)$) is continuous.*

Proof. Applying Theorem 5 it suffices to take $S_1 = \mathfrak{V}_1 \vee \mathfrak{U}_2$ and $S_2 = \mathfrak{U}_1 \vee \mathfrak{V}_2$.

In theorem 5 (c) the equality $\tau_{\mathfrak{V}_2} = (\mathcal{T}^\star)^+$ does not hold in general.

Example 7. Let $(R^2, \mathcal{T})$ be the plane with the natural topology and let A, W be the subsets of R^2 defined in Example 2. Then A is connected closed, W is open and $A \subset W$. Let $M_{U_{1,j},U_{2,j}} \in \mathcal{G}_2$ for $j \le n$ and

$$\left(\bigcap_{j=1}^n M_{U_{1,j},U_{2,j}}\right)[A] \subset W^+; \tag{1}$$

without loss of generality we can assume $A \subset \bigcap_{j=1}^n U_{1,j}$. Since

$$\left(\bigcap_{j=1}^n M_{U_{1,j},U_{2,j}}\right)[A] = \bigcap_{j=1}^n M_{U_{1,j},U_{2,j}}[A] = \bigcap_{j=1}^n U_{2,j}^+ = \left(\bigcap_{j=1}^n U_{2,j}\right)^+,$$

for each point $x \in \bigcap_{j=1}^n U_{2,j}$ we have $\{x\} \in \left(\bigcap_{j=1}^n U_{2,j}\right)^+$. Hence

$$\bigcap_{j=1}^n U_{2,j} \subset W. \tag{2}$$

The sets $R^2 \setminus U_{2,j}$, $j \le n$, are connected, so $R^2 \setminus \bigcap_{j=1}^n U_{2,j}$ has a finite number of components. Further $A \subset \bigcap_{j=1}^n U_{2,j}$, hence $B_m \not\subset \bigcap_{j=1}^n U_{2,j}$ holds only for a finite number of indices m. Thus $\bigcup_{m=1}^\infty B_m \not\subset R^2 \setminus \bigcap_{j=1}^n U_{2,j}$ and this contradicts (2). Consequently (1) does not hold and $(\mathcal{T}^\star)^+ = \mathcal{T}^+ \not\subset \tau_{\mathfrak{V}_2}$.

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JANINA EWERT
DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
ARCISZEWSKIEGO 22
76–200 SLUPSK
POLAND

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