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Limiting behavior of weighted sums of heavy-tailed random vectors

By CHEN PINGYAN (Guangzhou) and HANS-PETER SCHEFFLER (Dortmund)

Abstract. We present an integral test to determine the limiting behavior of weighted sums of i.i.d. $\mathbb{R}^d$-valued random vectors belonging to the (generalized) domain of operator semistable attraction of some nonnormal law, and deduce a version of Chover's law of the iterated logarithm for them.

1. Introduction and Main Results

Let $X, X_1, X_2, \dots$ be i.i.d. $\mathbb{R}^d$-valued random vectors. We assume that X belongs to the strict generalized domain of semistable attraction of a full operator semistable Y having nonnormal component (see [9] for details). Then, by definition, there exists a constant $c > 1$ and a sequence (k_n) of natural numbers tending to infinity with $k_{n+1}/k_n \to c$ as $n \to \infty$ and linear operators $A_n \in GL(\mathbb{R}^d)$ such that for $S_n = \sum_{i=1}^n X_i$ we have

$$A_n S_{k_n} \Rightarrow Y \quad \text{as} \quad n \to \infty. \tag{1.1}$$

Here $\Rightarrow$ denotes convergence in distribution. The distribution ν of the limit Y is then strictly (c^E, c)-operator semistable (E an invertible $d \times d$ matrix), that is

$$\nu^c = (c^E \nu) \tag{1.2}$$

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where ν^c denotes the c-fold convolution power and $(c^E\nu)(A) = \nu(c^{-E}A)$ is the image measure. Note that if ν is strictly operator stable with exponent E, then (1.2) holds for any $c > 1$, but the class of operator semistable laws is much larger than that of operator stable laws.

Then it is shown in [11], that there exists a sequence $(B_n) \subset GL(\mathbb{R}^d)$ regularly varying with exponent $-E$, that is $B_{[\lambda n]}B_n^{-1} \to \lambda^{-E}$ as $n \to \infty$, such that

$$B_{k_n}S_{k_n} \Rightarrow Y \quad \text{as} \quad n \to \infty. \tag{1.3}$$

Moreover the whole sequence $(B_nS_n)_n$ is stochastically compact with distributions in $\{\lambda^{-E}\nu^{\lambda} : \lambda \in [1,c]\}$. Given any unit vector $\theta \in \mathbb{R}^d$, we can project the random walk (S_n) onto the direction θ, that is we consider the one-dimensional random walk

$$\langle S_n, \theta\rangle = \sum_{i=1}^{n} \langle X_i, \theta\rangle.$$

Then it is shown in [11] that for any $\|\theta\| = 1$ there exists a sequence $r_n = r_n(\theta) > 0$ such that $(r_n\langle S_n, \theta\rangle)_n$ is stochastically compact. The norming sequence (r_n) behaves roughly like $n^{-1/\alpha(\theta)}$, where the tail index $0 < \alpha(\theta) < 2$ depends on the exponent E in (1.2). See [9] for details.

The tail behavior of $\langle X, \theta\rangle$ is well understood. In fact, if we let $V_0(t, \theta) = P\{|\langle X, \theta\rangle| > t\}$ is follows from Theorem 6.4.15 of [9] that for any $\delta > 0$ there exist constants $C_1, C_2 > 0$ and a $t_0 > 0$ such that

$$C_1\lambda^{-\alpha(\theta)-\delta} \leq \frac{V_0(\lambda t, \theta)}{V_0(t, \theta)} \leq C_2\lambda^{-\alpha(\theta)+\delta} \tag{1.4}$$

for any $t \geq t_0$ and any $\lambda \geq 1$. Especially, for some $B_1, B_2 > 0$ and some $t_0 > 0$ we have

$$B_1\lambda^{-\alpha(\theta)-\delta} \leq V_0(t, \theta) \leq B_2\lambda^{-\alpha(\theta)+\delta} \tag{1.5}$$

for all $t \geq t_0$. Here the tail-index $\alpha(\theta)$ is as above.

(1.4) and (1.5) are weaker than the well known tail behavior $P\{|Z| > t\} \sim Ct^{-\alpha}$ of an α-stable variable Z, but sufficiently sharp enough for our purpose.

In the following, let $B[0,1]$ and $BV[0,1]$ denote, respectively, the set of all bounded measurable functions and all functions of bounded variation on $[0,1]$.

The law of the iterated logarithm for sums of α-stable random variables was first discovered in [7] and then generalized in various ways. See e.g. [1]–[5], [10]. In this paper we generalize the results in [1] and [10] in the following way:

Let X belong to the strict generalized domain of semistable attraction of some full (c^E, c) operator semistable Y having no normal component. Then we have

Theorem 1.1. *Let $f : [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. Then:*

(a) *If there exists a $\varepsilon_0 > 0$ such that $\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty$, then for any $h \in BV[0,1]$, any $\|\theta\| = 1$ we have for $r_n = r_n(\theta)$ and $\alpha(\theta)$ as above, that*

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^n h(i/n)\langle X_i, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \quad a.s. \tag{1.6}$$

and especially for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^n h(i/n)\langle X_i, \theta\rangle|}{(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \quad a.s. \tag{1.7}$$

(b) *If there exists a $\varepsilon_0 > 0$ such that $\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty$, then for any function h satisfying that there exists a $x_0 \in (0,1]$ with $h(x_0) \neq 0$ and h is continuous in x_0 and any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^n h(i/n)\langle X_i, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = \infty \quad a.s. \tag{1.8}$$

and especially for any $0 < \delta < 1$

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^n h(i/n)\langle X_i, \theta\rangle|}{(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \quad a.s. \tag{1.9}$$

As a corollary the following law of the iterated logarithm (LIL) holds true:

Corollary 1.2. *Let h as in* (a) *and* (b) *of Theorem 1.1. Then for any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \left| r_n \sum_{i=1}^n h(i/n)\langle X_i, \theta\rangle \right|^{1/\log\log n} = e^{1/\alpha(\theta)} \quad a.s. \tag{1.10}$$

where r_n is as above.

If we restrict θ to the set $\{\theta : \|\theta\| = 1,\ 0 < \alpha(\theta) < 1\}$, we can weaken the condition on h.

Theorem 1.3. *Let $f : [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. If there exists a $\varepsilon_0 > 0$ such that $\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty$, then for any $h \in B[0, 1]$, any $\theta \in \{\theta : \|\theta\| = 1,\ 0 < \alpha(\theta) < 1\}$, (1.6) holds, and especially for any $\delta > 0$, (1.7) holds.*

Corollary 1.4. *Let $h \in B[0, 1]$ satisfy* (b) *of Theorem 1.1, then for any $\theta \in \{\theta : \|\theta\| = 1,\ 0 < \alpha(\theta) < 1\}$, (1.10) holds true.*

Additionally to the results above on weighted sums of $\langle X_k, \theta\rangle$ we also derive the limiting behavior of weighted sums of $\langle S_k, \theta\rangle$ as in [1].

Theorem 1.5. *Let $h \in B[0, 1]$ such that for some $0 < t_0 < 1$, $\int_{t_0}^1 h(x)dx \neq 0$. Moreover let $f : [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. Then*

(a) *If there exists a $\varepsilon_0 > 0$ such that $\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty$, then for any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \frac{|r_n \sum_{k=1}^n h(k/n)\langle S_k, \theta\rangle|}{nf(n)^{1/\alpha(\theta)}} = 0 \quad \text{a.s.} \tag{1.11}$$

(b) *If there exists a $\varepsilon_0 > 0$ such that $\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty$, then for any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \frac{|r_n \sum_{k=1}^n h(k/n)\langle S_k, \theta\rangle|}{nf(n)^{1/\alpha(\theta)}} = \infty \quad \text{a.s.} \tag{1.12}$$

Corollary 1.6. *Let $h(x)$ be as in Theorem 1.5. Then for any $0 < \delta < 1$ and any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \frac{|r_n \sum_{k=1}^n h(k/n)\langle S_k, \theta\rangle|}{n(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \quad \text{a.s.} \tag{1.13}$$

and

$$\limsup_{n\to\infty} \frac{|r_n \sum_{k=1}^n h(k/n)\langle S_k, \theta\rangle|}{n(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \quad \text{a.s.} \tag{1.14}$$

and especially

$$\limsup_{n\to\infty}\left|\frac{r_n\sum_{k=1}^n h(k/n)\langle S_k,\theta\rangle}{n}\right|^{1/\log\log n}=e^{1/\alpha(\theta)}\quad a.s. \tag{1.15}$$

Complementary to our results on the limiting behavior of weighted sums of $\langle X_i,\theta\rangle$ given above, we also consider the behavior of the norm of the partial sum S_n of the X_i's. Our result extends Theorem 2.6 in [10] by dropping the symmetry assumption and by allowing one to be an eigenvalue of the exponent E. Recall that the distribution ν of Y is a full (c^E,c) operator semistable law without normal component and let $\mathbb{R}^d=V_1\oplus\cdots\oplus V_p$ denote the spectral decomposition of $\mathbb{R}^d$ with respect to E. Recall that $E=E^{(1)}\oplus\cdots\oplus E^{(p)}$ and that every eigenvalue of $E^{(i)}$ has real part $1/\alpha_i$ for $1\le i\le p$. Then Theorem 1 in [6] implies that $0<\alpha_p<\cdots<\alpha_1<2$.

In the following let X belong to the strict generalized domain of semistable attracttion of a (c^E,c) semistabel law ν such that (1.3) holds. In view of Theorem 8.3.7 of [9] we can assume without loss generality that the distribution of X is spectrally compatible with ν. Then the spaces V_i are B_n invariant for all n and all $1\le i\le p$, so that $B_n=B_n^{(1)}\oplus\cdots\oplus B_n^{(p)}$. We write $X=X^{(1)}+\cdots+X^{(p)}$ with respect to the spectral decomposition of $\mathbb{R}^d$ obtained above and for $1\le i\le p$ set $X^{(1,\dots,i)}=X^{(1)}+\cdots+X^{(i)}$ and $B_n^{(1,\dots,i)}=B_n^{(1)}\oplus\cdots\oplus B_n^{(i)}$.

Theorem 1.7. *Suppose that X is in the strict generalized domain of semistable attraction of some full (c^E,c) operator semistable law without normal component, where $c>1$. Moreover let $f:[1,\infty)\to(0,\infty)$ be nondecreasing with $\lim_{x\to\infty}f(x)=\infty$. Then*

(a) *If there exists a $\varepsilon_0>0$ such that $\int_2^\infty\frac{dx}{xf(x)^{1-\varepsilon_0}}<\infty$, then for any $1\le i\le p$ we have that*

$$\limsup_{n\to\infty}\frac{\left\|B_n^{(1,\dots,i)}S_n^{(1,\dots,i)}\right\|}{f(n)^{1/\alpha_i}}=0\quad a.s. \tag{1.16}$$

and especially for any $\delta>0$

$$\limsup_{n\to\infty}\frac{\left\|B_n^{(1,\dots,i)}S_n^{(1,\dots,i)}\right\|}{(\log n)^{(1+\delta)/\alpha_i}}=0\quad a.s. \tag{1.17}$$

(b) *If there exists a* $\varepsilon_0 > 0$ *such that* $\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty$, *then for any* $1 \leq i \leq p$ *we have*

$$\limsup_{n\to\infty} \frac{\left\|B_n^{(1,\ldots,i)} S_n^{(1,\ldots,i)}\right\|}{f(n)^{1/\alpha_i}} = \infty \quad \text{a.s.} \tag{1.18}$$

and especially for any $0 < \delta < 1$

$$\limsup_{n\to\infty} \frac{\left\|B_n^{(1,\ldots,i)} S_n^{(1,\ldots,i)}\right\|}{(\log n)^{(1-\delta)/\alpha_i}} = \infty \quad \text{a.s.} \tag{1.19}$$

Corollary 1.8. *Under the assumptions of Theorem 1.7 we have*

$$\limsup_{n\to\infty} \left\|B_n^{(1,\ldots,i)} S_n^{(1,\ldots,i)}\right\|^{1/\log\log n} = e^{1/\alpha_i} \quad \text{a.s.} \tag{1.20}$$

Results similar to Theorem 1.1–1.5 and the respective corollaries also hold true for $X^{(1,\ldots,i)}$. We leave the formulation and proofs to the interested reader.

2. Proofs

Before we give the proofs of our results in Section 1, we recall for easier reference some notations and results of [9].

For any unit vector $\theta \in \mathbb{R}^d$ and $b \geq 0$ let

$$U_b(t,\theta) = E\big(|\langle X,\theta\rangle|^b I(|\langle X,\theta\rangle| \leq t)\big).$$

Then it is shown in Corollary 6.4.16 of [9] that for any $\|\theta\| = 1$ and $b > \alpha(\theta)$ there exists a $t_0 > 0$ and constants $m, M > 0$ such that

$$m \leq \frac{t^b V_0(t,\theta)}{U_b(t,\theta)} \leq M \quad \text{for all } t \geq t_0. \tag{2.1}$$

where the tail-function $V_0(t,\theta) = P\{|\langle X,\theta\rangle| > t\}$.

We also need the following large deviation result presented in [10] (see also [9], Theorem 9.1.3):

For any sequence $x_n \to \infty$ we have

$$
\begin{aligned}
0 &< \liminf_{n\to\infty} \frac{P\{|\langle S_n, \theta\rangle| > r_n^{-1} x_n\}}{nP\{|\langle X, \theta\rangle| > r_n^{-1} x_n\}} \\
&\le \limsup_{n\to\infty} \frac{P\{|\langle S_n, \theta\rangle| > r_n^{-1} x_n\}}{nP\{|\langle X, \theta\rangle| > r_n^{-1} x_n\}} < \infty
\end{aligned}
\tag{2.2}
$$

where r_n is the norming sequence for $\langle S_n, \theta\rangle$ as above. This large deviation result replaces the stability property of stable random variables.

Finally, some technical estimates on $nP\{|\langle X, \theta\rangle| > r_n^{-1}\}$ as in (9.21) and (9.22) of [9] together with some asymptotic results on r_n as in Lemma 4.1 of [10] are needed. In fact, for any $\|\theta\| = 1$ we have

$$
0 < \inf_{n\ge 1} nP\bigl\{|\langle X, \theta\rangle| > r_n^{-1}\bigr\} \le \sup_{n\ge 1} nP\bigl\{|\langle X, \theta\rangle| > r_n^{-1}\bigr\} < \infty, \tag{2.3}
$$

where $r_n = r_n(\theta)$ is as above.

The following lemma generalizes Lemma 2.1 in [1] and (4.8) of [10].

Lemma 2.1. *Let f be as in* (a) *of Theorem 1.1. Then for any $\|\theta\| = 1$ we have*

$$
\limsup_{n\to\infty} \frac{\max_{1\le k\le n} |r_n \sum_{i=1}^{k} \langle X_i, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \quad a.s.
$$

Proof. Given any $\lambda > 0$ let

$$
A_n = \Bigl\{\max_{1\le k\le n} |\langle S_k, \theta\rangle| > \lambda r_n^{-1} f(n)^{1/\alpha(\theta)}\Bigr\}
$$

and for $n_k = 2^k$ let

$$
B_m = \Bigl\{\max_{n_m \le n < n_{m+1}} |\langle S_n, \theta\rangle| > \lambda r_{n_m}^{-1} f(n_m)^{1/\alpha(\theta)}\Bigr\}.
$$

In view of Remark 4.5 of [11] it follows that r_n^{-1} is eventually increasing to infinity and hence $\limsup A_n \subset \limsup B_m$. Now argue as in the proof of (4.8) in [10] and Lemma 2.1 in [1], using (1.4), (2.2) and (2.3) to complete the proof. □

We need the next lemma to prove the divergent part of the main theorem.

Lemma 2.2. *Let f be as in* (b) *of Theorem 1.1. Then there exists a nondecreasing function $g : [1, \infty) \to (0, \infty)$ such that $\lim_{n\to\infty} g(x) = \infty$ and $\int_2^\infty \frac{dx}{x(f(x)g(x))^{1+\varepsilon_0}} = \infty$.*

Proof. The assertion follows from Lemma 2.2 of [1]. □

Proof of Theorem 1.1. (a) By the Abel's partial summation method, we get

$$\begin{aligned}
&\left|\sum_{k=1}^{n} h(k/n)\langle X_k, \theta\rangle\right| \\
&\quad= \left|\sum_{k=1}^{n-1}(h(k/n) - h((k+1)/n))\sum_{i=1}^{k}\langle X_i, \theta\rangle + h(1)\sum_{i=1}^{n}\langle X_i, \theta\rangle\right| \\
&\quad\le \max_{1\le k\le n}\left|\sum_{i=1}^{k}\langle X_i, \theta\rangle\right|\left(\sum_{k=1}^{n-1}|h(k/n) - h((k+1)/n)| + |h(1)|\right)
\end{aligned}$$

By the assumption there exists a constant $0 < C_0 < \infty$, such that $\sum_{k=1}^{n-1} |h(k/n) - h((k+1)/n)| + |h(1)| < C_0$ for all $n \ge 1$. So, using Lemma 2.1 (1.16) follows.

(b) Suppose that

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^{n} h(i/n)\langle X_i, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = \infty \quad \text{a.s.}$$

does not hold. Then by Kolmogorov 0-1 law, there exists a $d_0 \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{|r_n \sum_{i=1}^{n} h(i/n)\langle X_i, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = d_0 \quad \text{a.s.}$$

Hence, for the function g obtained in Lemma 2.2 we have

$$\lim_{n\to\infty} \frac{|r_n \sum_{i=1}^{n} h(i/n)\langle X_i, \theta\rangle|}{(f(n)g(n))^{1/\alpha(\theta)}} = 0 \quad \text{a.s.} \tag{2.4}$$

Let $\{X', X'_n, n \ge 1\}$ be independent copies of $\{X, X_n, n \ge 1\}$. Since equation (2.4) above also holds for (X'_i) instead of (X_i), it follows

$$\lim_{n\to\infty} \frac{|r_n \sum_{i=1}^{n} h(i/n)\langle X_i - X'_i, \theta\rangle|}{(f(n)g(n))^{1/\alpha(\theta)}} = 0 \quad \text{a.s.}$$

Let $n(m) = \inf\{n : [x_0 n] = m\}$ for all $m \geq 1$. By the same argument as on page 371 of [1], we get

$$\lim_{m\to\infty} \frac{r_{n(m)}|\langle X_m - X'_m, \theta\rangle|}{\big(f(n(m))g(n(m))\big)^{1/\alpha(\theta)}} = 0 \quad \text{a.s.}$$

Then the Borel–Cantelli lemma implies

$$\sum_{m=1}^{\infty} P\Big\{ r_{n(m)}|\langle X - X', \theta\rangle| \geq \frac{1}{2}\big(f(n(m))g(n(m))\big)^{1/\alpha(\theta)}\big)\Big\} < \infty.$$

For all m large enough, using (6.1) in [8], it follows that

$$\begin{aligned} &P\Big\{ r_{n(m)}|\langle X, \theta\rangle| \geq \big(f(n(m))g(n(m))\big)^{1/\alpha(\theta)}\big)\Big\} \\ &\qquad\qquad \leq 2P\Big\{ r_{n(m)}|\langle X - X', \theta\rangle| \geq \frac{1}{2}\big(f(n(m))g(n(m))\big)^{1/\alpha(\theta)}\big)\Big\} \end{aligned}$$

and hence

$$\sum_{m=1}^{\infty} P\Big\{ r_{n(m)}|\langle X, \theta\rangle| \geq \big(f(n(m))g(n(m))\big)^{1/\alpha(\theta)}\big)\Big\} < \infty.$$

In view of (1.4) together with (2.3), the formula above implies that

$$\sum_{m=1}^{\infty} \frac{1}{n(m)(f(n(m))g(n(m)))^{1+\varepsilon_0}} < \infty. \tag{2.5}$$

Since $\sup_{m\geq 1}(n(m+1) - n(m)) \leq [1/x_0] + 1$, we have

$$\sum_{k=n(m)}^{n(m+1)-1} \frac{1}{k(f(k)g(k))^{1+\varepsilon_0}} \leq \frac{[1/x_0]+1}{n(m)(f(n(m))g(n(m)))^{1+\varepsilon_0}}$$

and then

$$\int_2^{\infty} \frac{dx}{x(f(x)g(x))^{1+\varepsilon_0}} \leq 2 + \sum_{n=2}^{\infty} \frac{1}{n(f(n)g(n))^{1+\varepsilon_0}} < \infty.$$

This leads to a contradiction and completes the proof. $\square$

Proof Theorem 1.3. Since

$$f(n)^{-1/\alpha(\theta)} r_n \left| \sum_{i=1}^{n} h(i/n)\langle X_i, \theta\rangle \right| \leq \sup_{0\leq x\leq 1} |h(x)| f(n)^{-1/\alpha(\theta)} r_n \sum_{i=1}^{n} |\langle X_i, \theta\rangle|$$

and

$$\max_{2^k\leq n<2^{k+1}} f(n)^{-1/\alpha(\theta)} r_n \sum_{i=1}^{n} |\langle X_i, \theta\rangle| \leq C_0 r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} \sum_{i=1}^{2^{k+1}} |\langle X_i, \theta\rangle|$$

where $C_0 = \sup_{k\geq 1} \sup_{2^k\leq n\leq 2^{k+1}} r_n r_{2^{k+1}}^{-1} < \infty$ by Lemma 4.1 in [10], it suffices to prove that

$$r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} \sum_{i=1}^{2^{k+1}} |\langle X_i, \theta\rangle| \to 0 \quad \text{a.s.}$$

By the Borel–Cantelli lemma, this follows if we can show that

$$\sum_{k=1}^{\infty} P\left\{ r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} \sum_{i=1}^{2^{k+1}} |\langle X_i, \theta\rangle| > \varepsilon \right\} < \infty \quad \text{for all } \varepsilon > 0. \tag{2.6}$$

Observe that

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{ r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} \sum_{i=1}^{2^{k+1}} |\langle X_i, \theta\rangle| > \varepsilon \right\} \\
&\leq \sum_{k=1}^{\infty} 2^{k+1} P\left\{ |\langle X, \theta\rangle| > r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)} \right\} \\
&\quad + \sum_{k=1}^{\infty} P\left\{ \sum_{i=1}^{2^{k+1}} |\langle X_i, \theta\rangle| I(|\langle X_i, \theta\rangle| \leq r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)}) > \varepsilon r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)} \right\} \\
&= I_1 + I_2.
\end{aligned}$$

In view of (1.4) and (2.3), we conclude that

$$I_1 \leq C_2 \sup_{k\geq 1} 2^{k+1} P\left\{ |\langle X, \theta\rangle| > r_{2^{k+1}}^{-1} \right\} \sum_{k=1}^{\infty} f(2^k)^{-(1-\varepsilon_0)} < \infty.$$

Moreover, by Markov's inequality together with (1.4), (2.1) and (2.3) we have, using $\alpha(\theta) < 1$, that for an individual summand of I_2 we have

$$
\begin{aligned}
&P\Big\{\sum_{i=1}^{2^{k+1}} |\langle X_i,\theta\rangle| I(|\langle X_i,\theta\rangle| \le r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)}) > \varepsilon r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)}\Big\}\\
&\le \varepsilon^{-1} r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} 2^{k+1} E|\langle X,\theta\rangle| I(|\langle X,\theta\rangle| \le r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)})\\
&= \varepsilon^{-1} r_{2^{k+1}} f(2^k)^{-1/\alpha(\theta)} 2^{k+1} U_1\big(r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)},\theta\big)\\
&= \varepsilon^{-1} \frac{U_1\big(r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)},\theta\big)}{r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)} V_0\big(r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)},\theta\big)}\\
&\quad\cdot \frac{V_0\big(r_{2^{k+1}}^{-1} f(2^k)^{1/\alpha(\theta)},\theta\big)}{V_0\big(r_{2^{k+1}}^{-1},\theta\big)} 2^{k+1} V_0\big(r_{2^{k+1}}^{-1},\theta\big)\\
&\le C f(2^k)^{-(1-\varepsilon_0)}
\end{aligned}
$$

for all large k and some constant $C > 0$. Hence, by our assumption on f we conclude $I_2 < \infty$, so (2.6) holds true. This completes the proof. □

Proof of Theorem 1.5. By analyzing the proof of Theorem 1.1 carefully, it is easy to see that the result of Theorem 1.1 and hence of Corollary 1.2 also hold if $h(k/n)$ is replaced by $a_{n,k}$ where the real triangular array $\{a_{n,k} : 1 \le k \le n,\ n \ge 1\}$ fulfills the following two conditions:

(a) $\sup_{n\ge1}\big(\sum_{k=1}^{n-1} |a_{n,k} - a_{n,k-1}| + |a_{n,n}|\big) < \infty$.

(b) There exist increasing sequences $(n(k))_k$ and $(m(k))_k$ such that $\sup_{k\ge1}(n(k+1) - n(k)) < \infty$ and $\liminf_{k\to\infty} |a_{n(k),m(k)}| > 0$.

Now, under the assumptions of Theorem 1.5, let $a_{n,k} = \frac{1}{n}\sum_{i=k}^{n} h(i/n)$. Then it is easy to see that $\{a_{n,k}\}$ fulfills (a) and (b) above. An application of the above mentioned variant of Theorem 1.1 to the present case gives, after a change of the order of summation, the desired result. □

Before we give a proof of Theorem 1.7 and its corollary, we first prove a special case sufficient for our purpose. Recall from [9] that a (c^E, c) operator semistable law is called spectrally simple, if every eigenvalue of E has the same real part.

Proposition 2.3. *Let the distribution of* Y *be a full* (c^E, c) *operator semistable, spectrally simple, nonnormal law on a finite dimensional vector space* V *and let* X *belong to the strict generalized domain of semistable attraction of* Y *i.e.* (1.3) *holds. Let* $f : [1, \infty] \to (0, \infty)$ *nondecreasing with* $\lim_{x\to\infty} f(x) = \infty$. *Then*

(a) *If there exists a* $\varepsilon_0 > 0$ *such that* $\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty$, *then*

$$\limsup_{n\to\infty} \frac{\|B_n S_n\|}{f(n)^{1/\alpha}} = 0 \quad a.s.$$

(b) *If there exists a* $\varepsilon_0 > 0$ *such that* $\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty$, *then*

$$\limsup_{n\to\infty} \frac{\|B_n S_n\|}{f(n)^{1/\alpha}} = \infty \quad a.s.$$

and especially

$$\limsup_{n\to\infty} \|B_n S_n\|^{\frac{1}{\log\log n}} = e^{1/\alpha} \quad a.s.$$

where $B_n \in RV(-E)$ *is the embedding sequence and* $1/\alpha$ *is the real part of the eigenvalues of* E.

Proof. (a) Let $n_k = 2^k$. Since

$$\max_{n_k\le n<n_{k+1}} \frac{\|B_n S_n\|}{f(n)^{1/\alpha}} \le \sup_{k\ge 1} \sup_{n_k\le n<n_{k+1}} \|B_n B_{n_{k+1}}^{-1}\| \max_{n_k\le n<n_{k+1}} \frac{\|B_{n_{k+1}} S_n\|}{f(n_k)^{1/\alpha}}$$

and by (4.7) of [10] we have $\sup_{k\ge1} \sup_{n_k\le n<n_{k+1}} \|B_n B_{n_{k+1}}^{-1}\| < \infty$, it is enough to prove that

$$\max_{n_k\le n<n_{k+1}} \frac{\|B_{n_{k+1}} S_n\|}{f(n_k)^{1/\alpha}} \to 0 \quad \text{a.s. as } k \to \infty$$

holds true. Let $\{\theta^{(1)}, \dots, \theta^{(m)}\}$ be an orthonormal basis of V. Since

$$\|B_{n_{k+1}} S_n\|^2 = |\langle B_{n_{k+1}} S_n, \theta^{(1)}\rangle|^2 + \cdots + |\langle B_{n_{k+1}} S_n, \theta^{(m)}\rangle|^2$$

it suffices to show that for any $1 \le j \le m$ we have

$$\max_{n_k\le n<n_{k+1}} \frac{|\langle B_{n_{k+1}} S_n, \theta^{(j)}\rangle|}{f(n_k)^{1/\alpha}} \to 0 \quad \text{a.s. as } k \to \infty. \tag{2.7}$$

Fix any $1 \leq j \leq m$. Then (2.7) follows, if we can show that

$$\max_{n_k \leq n < n_{k+1}} \frac{\left|\langle B_{n_{k+1}} S_n, \theta^{(j)}\rangle - nE\left[\langle B_{n_{k+1}} X, \theta^{(j)}\rangle I(|\langle B_{n_{k+1}} X, \theta^{(j)}\rangle| \leq f(n_k)^{1/\alpha})\right]\right|}{f(n_k)^{1/\alpha}} \to 0 \qquad (2.8)$$

a.s. as $k \to \infty$ and

$$\frac{n_k E\left[\langle B_{n_{k+1}} X, \theta^{(j)}\rangle I(|\langle B_{n_{k+1}} X, \theta^{(j)}\rangle| \leq f(n_k)^{1/\alpha})\right]}{f(n_k)^{1/\alpha}} \to 0 \quad \text{as } k \to \infty. \qquad (2.9)$$

Write $B^*_{n_{k+1}} \theta^{(j)} = r_k \theta_k$ for some $r_k > 0$ and $\|\theta_k\| = 1$. For $k \geq 1$ and $\varepsilon > 0$ let

$$A_k = \Big\{ \max_{n_k \leq n <_{k+1}} \big| r_k \langle S_n, \theta_k\rangle - nE\big[r_k \langle X, \theta_k\rangle I(|r_k \langle X, \theta_k\rangle| \leq f(n_k)^{1/\alpha})\big]\big| \\ > \varepsilon f(n_k)^{1/\alpha} \Big\}.$$

For the proof of (2.8), by Borel–Cantelli's Lemma, we have to show that $\sum_{k=1}^{\infty} P(A_k) < \infty$. Now let

$$E_k = \big\{ |r_k \langle X_i, \theta_k\rangle| > f(n_k)^{1/\alpha} \text{ for at least one } 1 \leq i \leq n_{k+1} \big\}$$

and

$$G_k = \Big\{ \max_{n_k \leq n < n_{k+1}} \Big| r_k \sum_{i=1}^{n} \langle X_i, \theta_k\rangle I(r_k |\langle X_i, \theta_k\rangle| \leq f(n_k)^{1/\alpha}) \\ - nE\big[r_k \langle X_i, \theta_k\rangle I(r_k |\langle X_i, \theta_k\rangle| \leq f(n_k)^{1/\alpha})\big]\Big| > \varepsilon f(n_k)^{1/\alpha} \Big\}.$$

Then $A_k \subset E_k \cup G_k$. Moreover, using that $\sup nP\{|\langle B_n X, \theta^{(j)}\rangle| > 1\} < \infty$, together with Corollary 4.15 of [12] we obtain, by setting $\delta = \alpha\varepsilon_0$, that

$$\begin{aligned} P(E_k) &\leq n_{k+1} V_0(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k) \\ &= \frac{V_0(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k)}{V_0(r_k^{-1}, \theta_k)} n_{k+1} V_0(r_k^{-1}, \theta_k) \\ &\leq C f(n_k)^{-1+\varepsilon_0} \end{aligned}$$

for all large k and some constant $C > 0$. Moreover, in view of Theorem 4.20 and Corollary 4.15 of [12], we get using Kolmogoroff's inequality, that

$$\begin{aligned}
P(G_k) &\leq \varepsilon^{-2} f(n_k)^{-2/\alpha} n_{k+1} E\big[r_k^2 \langle X, \theta_k\rangle^2 I(|\langle X, \theta_k\rangle| \leq r_k^{-1} f(n_k)^{1/\alpha})\big] \\
&= \varepsilon^{-2} n_{k+1} \big(f(n_k)^{-1} r_k\big)^2 U_2\big(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k\big) \\
&= \varepsilon^{-2} \frac{U_2\big(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k\big)}{\big(r_k^{-1} f(n_k)^{1/\alpha}\big)^2 V_0\big(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k\big)} \\
&\quad \cdot \frac{V_0\big(r_k^{-1} f(n_k)^{1/\alpha}, \theta_k\big)}{V_0(r_k^{-1}, \theta_k)} n_{k+1} V_0(r_k^{-1}, \theta_k) \\
&\leq C \varepsilon^{-2} f(n_k)^{-1+\varepsilon_0}
\end{aligned}$$

for some constant $C > 0$ and all large k. Hence $P(A_k) \leq C f(n_k)^{-1+\varepsilon_0}$ for all large k, so by our assumption on f, (2.8) holds true.

For the proof of (2.9) note that since by [11] the sequence $(B_n S_n)$ is stochastically compact, we have $B_{n_k} S_{n_k} / f(n_k)^{1/\alpha} \to 0$ in probability. Using that (B_n) is regularly varying that implies that for any $1 \leq j \leq m$

$$\frac{\langle B_{n_{k+1}} S_{n_k}, \theta^{(j)}\rangle}{f(n_k)^{1/\alpha}} \to 0 \quad \text{in probability.} \tag{2.10}$$

Now (2.8) implies that

$$\frac{\big|\langle B_{n_{k+1}} S_{n_k}, \theta^{(j)}\rangle - n_k E\big[\langle B_{n_{k+1}} X, \theta^{(j)}\rangle I(|\langle B_{n_{k+1}} X, \theta^{(j)}\rangle| \leq f(n_k)^{1/\alpha})\big]\big|}{f(n_k)^{1/\alpha}} \to 0$$

in probability, as $k \to \infty$, so by (2.10) it follows that (2.9) holds true.

The proof of (b) is similar to the proof of (4.16) in [10] and therefore omitted. $\square$

Proof of Theorem 1.7. Using Proposition 2.3, the result of Theorem 1.7 follows along the lines of the proof of Theorem 2.6 in [10]. $\square$

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CHEN PINGYAN
DEPARTMENT OF MATHEMATICS
JINAN UNIVERSITY
GUANGZHOU, 510630
P.R. CHINA

E-mail: chenpingyan@263.net

HANS-PETER SCHEFFLER
FACHBEREICH MATHEMATIK
UNIVERSITY OF DORTMUND
44221 DORTMUND
GERMANY

E-mail: hps@math.uni-dortmund.de

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