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Notes on fractional Ornstein–Uhlenbeck random sheets

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In memory of Béla Brindza

Abstract. The paper explores, in a preliminary way, several models of fractional scalar random fields with two-dimensional parameter which extend the classical Ornstein–Uhlenbeck model. Issues of planar stochastic integration of deterministic scalar fields (all that is needed in our representation theorems) with respect to such fields are also addressed. The isotropic case is studied separately. The critical role of the extended Lamperti transformation providing connection between selfsimilarity and stationarity of random fields is emphasized. The motivation is provided by needs of physical models such as Burgers turbulence and interfacial growth.

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1. Introduction

Random surfaces encountered in the study of physical models such as Burgers turbulence or interface growth often display long-range dependence and fractional behavior caused by the underlying long-range dependence, see, e.g., [28] and [22]. Experimental data from observation of surface ocean waves, in particular in presence of surf beats, multivariate time series of the velocities of ocean drifters along their trajectories, and thin semiconductor film growth in chemical vapor deposition reactors produce planar spectra indicating presence of singularities, see, e.g., [10], [25] and [5], see, also, Figure 1 and Figure 2. Several recent papers, such as [3], [15], and [24], studied different aspects of time-dependent, linear and nonlinear evolution of such interfaces based on integro-differential equation. Statistical parametric estimation problems, essential for practical application of the above mentioned models, were addressed in [16]. Examples of singular planar spectra of interest in geophysical sciences can be found in [7]. In optics, singular spectra have been studied in the context of phase critical point densities in planar isotropic random waves, see [6]. The self-affine random surfaces characterized by power-law spectra have been also used in the study of surface structure, topography and roughness [30].

In the present paper we describe several models of fractional scalar random fields with two-dimensional parameter which extend the classical Ornstein–Uhlenbeck model. We will call such fields *fractional Ornstein–Uhlenbeck sheets* although the usage varies in the literature. The regular Ornstein–Uhlenbeck sheets have been extensively studied, in particular, in the context of stochastic partial differential equation, see e.g., [26], and other papers cited in the references section. Some statistical problems of random sheets and fields have been studied in [2] and [8], including specific problems of the quasilikelihood-based higher-order spectral estimation of random processes and fields with possible long-range dependence, see [1].

The present paper is preliminary in nature and contains mostly concepts and relatively straightforward calculations indicating opportunities for interesting work that needs to be done. The aim is to formulate some novel and flexible, but also analytically doable models of random fields for data displaying approximate spectral singularities at the origin or along the axes (and related long-range dependence) and probe what can be done

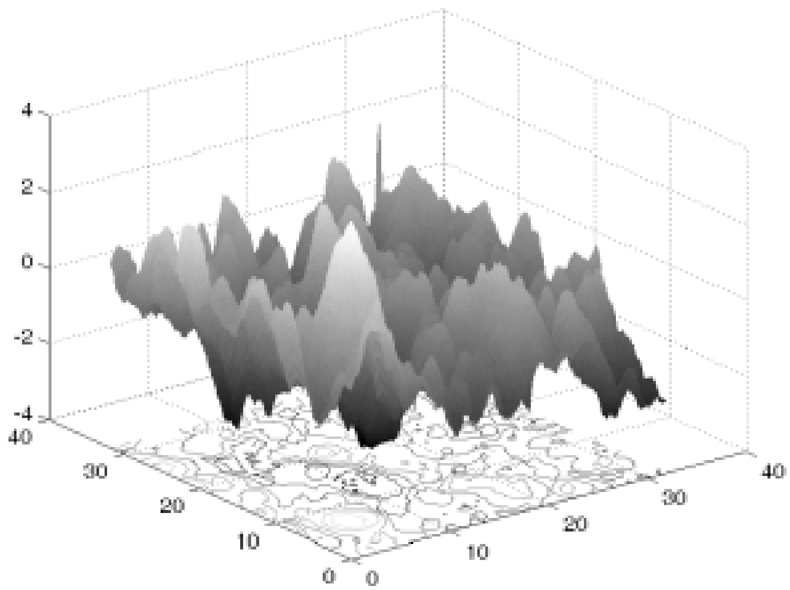

Figure 1. An atomic force microscopy (AFM) image of the surface of the diamond-like-carbon (DLC) thin semiconductor film deposited in the chemical vapor deposition (CVD) reactor in the Chemical Engineering Department at Case Western Reserve University (courtesy of J.A. Mann).

to explain their behavior. Of course, much work remains to be done, but we address some new fundamental issues arising in this context such as the concept of integration in the frequency domain for random sheets, including the special situation encountered in the fractional cases.

Two basic models of fractional Ornstein–Uhlenbeck sheets immediately suggest themselves. The first assumes the structure of the covariance to be of the product form

$$\text{const} \times \big(|x_1|^{2H_1}+|y_1|^{2H_1}-|x_1-y_1|^{2H_1}\big)\cdot\big(|x_2|^{2H_2}+|y_2|^{2H_2}-|x_2-y_2|^{2H_2}\big),$$

with anisotropic Hurst parameter (H_1, H_2), and the other insists on isotropy

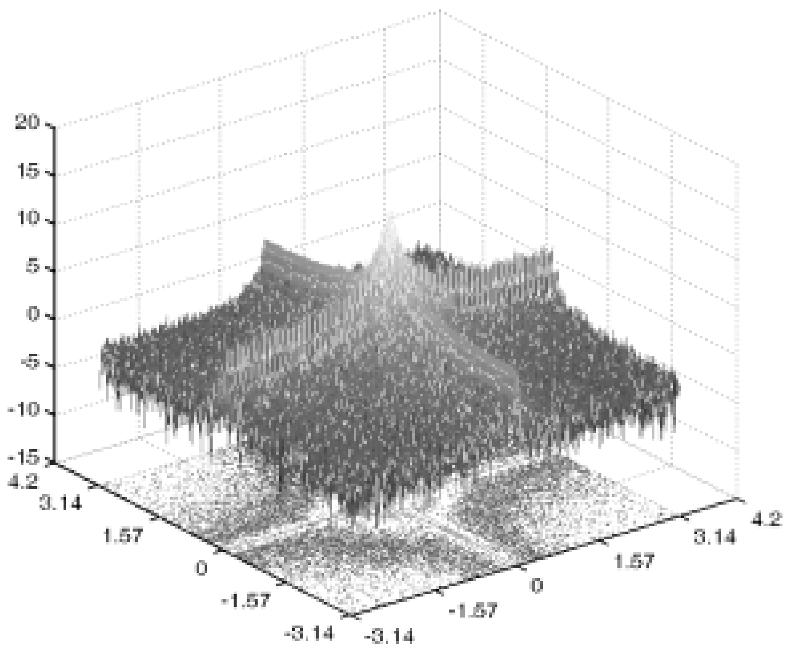

Figure 2. The two-dimensional log-spectrum of the random surface from Figure 1

and the structure of the covariance

$$\text{const} \times \left(\|\underline{x}\|^{2H} + \|\underline{y}\|^{2H} - \|\underline{x} - \underline{y}\|^{2H} \right),$$

with $\underline{x} = (x_1, x_2)$, $\underline{y} = (y_1, y_2)$. Both random sheets are generalizations of the fractional Brownian motion introduced by Mandelbrot and van Ness [21] and their definitions have an obvious extension to random fields with higher dimensional parameter.

Section 2 discusses the classical Brownian sheets from the perspective of planar stochastic integrals. Section 3 introduces the spectral approach to fractional Brownian sheets which then is used in Section 4 to study planar stochastic integration with respect to them. In Section 5 we turn out attention to the isotropic case. The paper is concluded by description of fractional Ornstein–Uhlenbeck sheets.

2D-Selfsimilarity and Stationarity. Like in the one-dimensional case, the issues of stationarity and selfsimilarity for random sheets are related via the Lamperti transformation. Walsh [26] noticed that if one considers the Brownian sheet B at exponential points $B\left(e^{x_1}, e^{x_2}\right)$ then the result is the Ornstein–Uhlenbeck sheet. It turns out that the Lamperti Transformation can be extended to provide a similar connection between selfsimilarity and stationarity in a more general situation. Let us observe that in view of the properties of the Brownian sheet it follows that

$$B\left(x_1, x_2\right) \stackrel{d}{=} a_1^{-1/2} a_2^{-1/2} B\left(a_1 x_1, a_2 x_2\right),$$

where the equality is in the distribution. The right-hand side is called renormalized dilatation of a stochastic process, see [18]. We use the same terminology for sheets.

Definition 1. Let $X\left(x_1, x_2\right)$, $\left(x_1, x_2\right) > 0$, be a random sheet. For $\underline{a} = (a_1, a_2) > 0$, and $\underline{H} = (H_1, H_2) > 0$, define the renormalized dilatation for $X(x_1, x_2)$ by the equation

$$\mathcal{D}_{\underline{H},\underline{a}} X(x_1, x_2) = a_1^{-H_1} a_2^{-H_2} X(a_1 x_1, a_2 x_2), \quad (x_1, x_2) > 0.$$

The sheet $X(x_1, x_2)$ will be called selfsimilar with selfsimilarity parameter (Hurst exponent) $\underline{H} = (H_1, H_2)$ if for each $\underline{a} > 0$, it is invariant, in distribution, under the renormalized dilatation $\mathcal{D}_{\underline{H},\underline{a}}$ i.e.

$$\mathcal{D}_{\underline{H},\underline{a}} X(x_1, x_2) \stackrel{d}{=} X(x_1, x_2), \quad (x_1, x_2) > 0,$$

where $\stackrel{d}{=}$ denotes the equality in the distribution, more precisely the equality of all finite dimensional distributions.

It is clear that $B^{(h_1,h_2)}$ is selfsimilar with selfsimilarity parameter $\underline{H} = (H_1, H_2) = \left(h_1 + \frac{1}{2}, h_2 + \frac{1}{2}\right)$, see (3.1) bellow.

The concept of the renormalized dilatation has been introduced by Lamperti [18] and widely used in the theory of stochastic processes; for recent applications see [9].

Definition 2. Let $X(x_1, x_2)$ be a stochastic sheet and $\underline{H} = (H_1, H_2) > 0$. The Lamperti Transformation $\mathcal{L}_{\underline{H}} X$ of X is defined by the following equation:

$$\mathcal{L}_{\underline{H}} X(x_1, x_2) = x_1^{H_1} x_2^{H_2} X(\log x_1, \log x_2), \quad (x_1, x_2) > 0.$$

Its inverse

$$\mathcal{L}_{\underline{H}}^{-1}X(x_1,x_2)=e^{-H_1x_1-H_2x_2}X\left(e^{x_1},e^{x_2}\right),\quad (x_1,x_2)\in\mathbb{R}^2.$$

Similarly to the scalar case we have

$$\mathcal{L}_{\underline{H}}^{-1}\mathcal{D}_{\underline{H},\underline{a}}\mathcal{L}_{\underline{H}}=S_{\log\underline{a}},$$

where $S_{\log\underline{a}}$ is the translation operator, i.e.,

$$S_{\log\underline{a}}X(x_1,x_2)=X(x_1+\log a_1,x_2+\log a_2).$$

The following Theorem shows that the Lamperti Transformation gives a one-to-one correspondence between selfsimilar sheets and stationary sheets.

Theorem 1. *If $X(x_1,x_2)$, $(x_1,x_2)>0$, is a selfsimilar sheet with selfsimilarity parameter $\underline{H}=(H_1,H_2)$ then its inverse Lamperti Transformation $Y=\mathcal{L}_{\underline{H}}^{-1}X$ is stationary, and if $Y(x_1,x_2)$, $(x_1,x_2)\in\mathbb{R}^2$, is a stationary sheet then its Lamperti Transformation $X=\mathcal{L}_{\underline{H}}Y$ is a selfsimilar sheet with selfsimilarity parameter $\underline{H}$.*

2. Planar integration with respect to Brownian sheet

Let $B=B(x_1,x_2)$, $(x_1,x_2)\in\mathbb{R}^2$, be a Brownian sheet with the spectral domain representation

$$B(x_1,x_2)=\int\limits_{\mathbb{R}^2}\frac{(e^{i2\pi x_1\omega_1}-1)}{i2\pi\omega_1}\cdot\frac{(e^{i2\pi x_2\omega_2}-1)}{i2\pi\omega_2}\,W(d\omega_1,d\omega_2),\tag{2.1}$$

where W is a complex, Gaussian, white noise spectral measure with $\mathsf{E}|W(d\omega_1,d\omega_2)|^2=d\omega_1 d\omega_2$. Denoting

$$e(x,\omega)=\frac{e^{i2\pi x\omega}-1}{i2\pi\omega}=\int_0^x e^{i2\pi\omega u}du,$$

we can write

$$B(x_1,x_2)=\int\limits_{\mathbb{R}^2}e(x_1,\omega_1)e(x_2,\omega_2)W(d\omega_1,d\omega_2).$$

We begin by summarizing some known results, see [4], [27], [26], concerning the problem of stochastic integration in the plane with respect to the Brownian sheet. We address this problem by utilizing the concept of the Skorohod integral in the frequency domain, a line of attack different from the usual approach.

At this point it is worthwhile to observe significant differences between the Brownian 1-D stochastic integral and the 2-D stochastic integral with respect to the Brownian sheet: For example,

$$\iint_{\underline{0}}^{\underline{T}} B(x_1,x_2)dB(x_1,x_2) \\ = \frac{B^2(T_1,T_2)-T_1T_2}{2} - \iint_{\underline{0}}^{\underline{T}} \partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2), \tag{2.2}$$

where our standard 2-D notation is $\underline{0}=(0,0), \underline{T}=(T_1,T_2)$, see [27], and (2.7) bellow. In what follows our basic tool is the *Diagram Formula* for multiple stochastic integrals in the frequency domain, see Major [20], which, for a Fourier transform $g \in L^2(\mathbb{R}^2)$ of a real function and a Fourier transform $f \in L^2(\mathbb{R}^{2k})$ of a real symmetric functions reads

$$\int_{\mathbb{R}^2} g(\underline{\omega})W(d\underline{\omega}) \int_{\mathbb{R}^{2k}} f\big(\underline{\omega}_{(1:k)}\big)W\big(d\underline{\omega}_{(1:k)}\big) \\ = \int_{\mathbb{R}^{2(k+1)}} f\big(\underline{\omega}_{(1:k)}\big)g(\underline{\omega}_{k+1})W\big(d\underline{\omega}_{(k+1)}\big) \\ + k\int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^2} \bar{g}(\underline{\omega}_k)f\big(\underline{\omega}_{(1:k)}\big)d\underline{\omega}_k W\big(d\underline{\omega}_{(k-1)}\big). \tag{2.3}$$

Here $\underline{\omega} \in \mathbb{R}^2$, $\underline{\omega}_{(1:k)} \in \mathbb{R}^{2k}$, and $W(d\underline{\omega})$ is a complex, Gaussian, white noise spectral measure on $\mathbb{R}^2$ with $\mathsf{E}\,|W(d\underline{\omega})|^2 = d\underline{\omega}$.

Furthermore, note that this multiple integral is invariant under the symmetrization operation *sym* on the integrand which, for $f\big(\underline{\omega}_{(1:k)}\big)$, where $\underline{\omega}_{(1:k)} = (\underline{\omega}_1, \underline{\omega}_2, \ldots, \underline{\omega}_k)$, $\underline{\omega}_j \in \mathbb{R}^2$, is defined by the formula

$$\operatorname*{sym}_{\underline{\omega}_{(1:k)}} f\big(\underline{\omega}_{(1:k)}\big) = \frac{1}{k!}\sum_{p\in\mathcal{P}_k} f\big(p\underline{\omega}_{(1:k)}\big), \tag{2.4}$$

where $\mathcal{P}_k$ is the set of all permutations of numbers $(1, 2, \ldots, k)$; if $p = (p_1, p_2, \ldots, p_k) \in \mathcal{P}_k$ then $p\underline{\omega}_{(1:k)} = (\underline{\omega}_{p_1}, \underline{\omega}_{p_2}, \ldots, \underline{\omega}_{p_k})$. Thus, for instance, the product $f(\underline{\omega}_{(1:k)})g(\underline{\omega}_{k+1})$ in (2.3) can be symmetrized without changing the value of the integral over $\mathbb{R}^{2k}$. Now, use the Diagram Formula (2.3) for the integral

$$\begin{aligned}
&\iint_{\underline{0}}^{\underline{T}} B(x_1, x_2) dB(x_1, x_2) \\
&= \iint_{\underline{0}}^{\underline{T}} \int_{\mathbb{R}^2} e(x_1, \omega_1) e(x_2, \omega_2) W(d\omega_1, d\omega_2) \\
&\quad \times \int_{\mathbb{R}^2} e^{i2\pi x_1 \lambda_1 + i2\pi x_2 \lambda_2} W(d\lambda_1, d\lambda_2) dx_1 dx_2 \qquad (2.5) \\
&= \int_{\mathbb{R}^4} \iint_{\underline{0}}^{\underline{T}} e(x_1, \omega_1) e(x_2, \omega_2) e^{i2\pi x_1 \lambda_1 + i2\pi x_2 \lambda_2} dx_1 dx_2 W(d\underline{\omega}, d\underline{\lambda}) + \mathrm{const}\,.
\end{aligned}$$

It is easy to see that

$$\int_0^T e(x, \omega) e^{i2\pi x \lambda} dx + \int_0^T e(x, \lambda) e^{i2\pi x \omega} dx = e(T, \omega) e(T, \lambda),$$

so that

$$\begin{aligned}
2 \operatorname*{sym}_{\underline{\omega}, \underline{\lambda}} &\iint_{\underline{0}}^{\underline{T}} e(x_1, \omega_1) e(x_2, \omega_2) e^{i2\pi(x_1\lambda_1 + x_2\lambda_2)} dx_1 dx_2 \\
&= e(T_1, \omega_1) e(T_2, \omega_2) e(T_1, \lambda_1) e(T_2, \lambda_2) \\
&\quad - \iint_{\underline{0}}^{\underline{T}} e(x_1, \lambda_1) e(x_2, \omega_2) e^{i2\pi(x_1\omega_1 + x_2\lambda_2)} \qquad (2.6) \\
&\quad + e(x_1, \omega_1) e(x_2, \lambda_2) e^{i2\pi(x_1\lambda_1 + x_2\omega_2)} dx_1 dx_2,
\end{aligned}$$

This function of $(\underline{\omega}, \underline{\lambda})$ is symmetric as a function of two vector variables, that is, its value will not change if $(\underline{\omega}, \underline{\lambda})$ is replaced by $(\underline{\lambda}, \underline{\omega})$, keeping the order inside the variables $\underline{\lambda} = (\lambda_1, \lambda_2)$ and $\underline{\omega} = (\omega_1, \omega_2)$ intact. The first term on the right-hand side of (2.6) satisfies equality

$$B^2(T_1, T_2) = \int_{\mathbb{R}^4} e(T_1, \omega_1) e(T_2, \omega_2) e(T_1, \lambda_1) e(T_2, \lambda_2) W(d\underline{\omega}, d\underline{\lambda}) + T_1 T_2,$$

while, for the first part of the second term, we have

$$\begin{aligned}
&\iint_{\underline{0}}^{\underline{T}} \partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2)\\
&= \iint_{\underline{0}}^{\underline{T}} \int_{\mathbb{R}^2} e^{i2\pi x_1\omega_1} e(x_2,\omega_2) W(d\omega_1,d\omega_2)\\
&\quad \times \int_{\mathbb{R}^2} e(x_1,\omega_1) e^{i2\pi x_2\omega_2} W(d\omega_1,d\omega_2)\,dx_1dx_2 \qquad (2.7)\\
&= \int_{\mathbb{R}^4} \iint_{\underline{0}}^{\underline{T}} e^{i2\pi x_1\omega_1} e(x_2,\omega_2) e(x_1,\lambda_1) e^{i2\pi x_2\lambda_2}\,dx_1dx_2 W(d\underline{\omega},d\underline{\lambda}) + \text{const}.,
\end{aligned}$$

with similar equality for the second part.

We shall show that the above constant zero which, in view of (2.5), (2.6) and (2.7), yields equation (2.2). Indeed, the typical term for both constants is the integral

$$\int_{\mathbb{R}} e(x,\omega)e^{-i2\pi x\omega}d\omega = \int_{\mathbb{R}} \frac{1-e^{-i2\pi x\omega}}{i2\pi\omega}d\omega, \qquad (2.8)$$

the principal value of which is zero.

Notice that the Abel and the Gauss mean of (2.8) is 1/2 but these facts do not affect the 2-D formula (2.2) the way they do the 1-D case, where one obtains the Itô version of BdB by choosing the principal value, and the Stratonovich version of BdB by choosing either the Abel or the Gauss mean. A general treatment of the Itô and Stratonovich integrals and their connections is given in the book [14], Chapter 3. We summarize the above discussion about the 2-D random measure generated by $\partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2)$ (see also Walsh [27]) in the following

Proposition 1. *The formula*

$$\begin{aligned}
&\iint_{\underline{0}}^{\underline{T}} \partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2)\\
&= \frac{B^2(T_1,T_2)-T_1T_2}{2} - \iint_{\underline{0}}^{\underline{T}} B(x_1,x_2)dB(x_1,x_2),
\end{aligned}$$

holds true for both Itô and Stratonovich integrals, but, for Itô integrals, we have

$$\iint_{\underline{0}}^{\underline{T}} \partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2) = \int_0^{T_1} B(x_1,T_2)\partial_1 B(x_1,T_2) - \iint_{\underline{0}}^{\underline{T}} B(x_1,x_2)dB(x_1,x_2),$$

and, for Stratonovich integrals, we have

$$\iint_{\underline{0}}^{\underline{T}} \partial_1 B(x_1,x_2)\partial_2 B(x_1,x_2) = \int_0^{T_1} B(x_1,T_2)\partial_1 B(x_1,T_2) - \iint_{\underline{0}}^{\underline{T}} B(x_1,x_2)dB(x_1,x_2) - \frac{T_1T_2}{2}.$$

Closing this section let us recall the spectral domain chaotic representation of a second-order isotropic and homogeneous sheet X subordinated to the Brownian sheet:

$$X(x_1,x_2) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^{2k}} e^{i2\pi\Sigma[\underline{x},\underline{\omega}_j]} f_k\big(\underline{\omega}_{(1:k)}\big) W\big(d\underline{\omega}_{(1:k)}\big), \tag{2.9}$$

where $[\underline{x},\underline{\omega}] = x_1\omega_1 + x_2\omega_2$ and the integrals are the multiple Wiener–Itô integrals (see, e.g., Major [20], and Kwapien and Woyczynski [17]) and

$$E|X(x_1,x_2)|^2 = \sum_{k=0}^{\infty} k! \int_{\mathbb{R}^{2k}} |f_k\big(\underline{\omega}_{(1:k)}\big)|^2 d\omega_{(1:k)} < \infty.$$

The stochastic integration of a homogeneous sheet X (subordinated to the Brownian sheet) according to the Brownian sheet, XdB, is carried out by the diagram formula in an obvious way, we follow this idea for the integration by the fractional Brownian sheet, as well.

3. Fractional Brownian sheet – frequency domain approach

The fractional Brownian sheet with parameter $(h_1,h_2) \in (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2})$, a generalization of the fractional Brownian motion introduced by

Mandelbrot and Ness [21] will be defined as follows: For $(x_1, x_2) \in \mathbb{R}_+^2$,

$$
\begin{aligned}
&B^{(h_1,h_2)}(x_1,x_2)\\
&\doteq \frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)}\left\{\int_0^{x_2}\int_0^{x_1}(x_1-y_1)^{h_1}(x_2-y_2)^{h_2}dB(y_1,y_2)_s\right.\\
&\quad -\int_{-\infty}^{0}\int_0^{x_1}(x_1-y_1)^{h_1}\left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right]dB(y_1,y_2)\\
&\quad -\int_0^{x_2}\int_{-\infty}^{0}\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right](x_2-y_2)^{h_2}dB(y_1,y_2)\\
&\quad \left.+\int_{-\infty}^{0}\int_{-\infty}^{0}\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right]\left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right]dB(y_1,y_2)\right\},
\end{aligned}
$$

where $B(y_1, y_2)$ is the standard Brownian sheet. Clearly, $B^{(0,0)}(x_1, x_2) = B(x_1, x_2)$.

The definition is closely related to the fractional integral operator

$$
\begin{aligned}
&I_{(x_1,x_2)}^{(h_1,h_2)}\left(f(y_1,y_2)\right)\\
&\quad = \frac{1}{\Gamma(h_1)\Gamma(h_2)}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}(x_1-y_1)^{1-h_1}(x_2-y_2)^{1-h_2}f(y_1,y_2)dy_1dy_2,
\end{aligned}
$$

since the fractional Brownian sheet happens to be the $(h_1, h_2)^{th}$ fractional integral of the Brownian sheet, but only after conditioning it to be zero at the origin. To avoid this difficulty we shall rewrite the definition of the fractional Brownian sheet $B^{(h_1,h_2)}(x_1, x_2)$ in the spectral domain, considering only the case $0 < h_k < \frac{1}{2}$, $k = 1, 2$. The proof of the spectral representation is similar in spirit to that for the fractional Brownian motion, see [13]. Another derivation of the spectral representation of $B^{(h_1,h_2)}$, based on its distributional properties, can be found in [12].

Theorem 2. *Let W be a complex Gaussian white noise spectral measure with $\mathsf{E}|W(d\omega_1, d\omega_2)|^2 = d\omega_1 d\omega_2$. Then we have the following spectral domain representation of the fractional Brownian sheet $B^{(h_1,h_2)}$:*

$$
\begin{aligned}
&B^{(h_1,h_2)}(x_1,x_2)\\
&\doteq \int_{\mathbb{R}^2}\frac{\left(e^{i2\pi x_1\omega_1}-1\right)\left(e^{i2\pi x_2\omega_2}-1\right)}{(i2\pi\omega_1)(i2\pi\omega_2)}(i\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}W(d\omega_1,d\omega_2).
\end{aligned}
\tag{3.1}
$$

Proof. In view of the definition of the fractional Brownian sheet we have

$$\begin{aligned}
&B^{(h_1,h_2)}(x_1,x_2) \doteq \frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)} \\
&\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big\{ \chi_{[(0,0),(x_1,x_2)]}(y_1,y_2)(x_1-y_1)^{h_1}(x_2-y_2)^{h_2} \\
&- \chi_{[(0,-\infty),(x_1,0)]}(y_1,y_2)(x_1-y_1)^{h_1}\left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right] \\
&- \chi_{[(-\infty,0),(0,x_2)]}(y_1,y_2)\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right](x_2-y_2)^{h_2} \\
&+ \chi_{[(-\infty,-\infty),(0,0)]}(y_1,y_2)\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right]\left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right]\Big\} \\
&\times dB(y_1,y_2) = \frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)} \int\limits_{\mathbb{R}^2} \Bigg[\int\limits_{\mathbb{R}^2} e^{i2\pi(y_1\omega_1+y_2\omega_2)} \\
&\times \Big\{ \chi_{[(0,0),(x_1,x_2)]}(y_1,y_2)(x_1-y_1)^{h_1}(x_2-y_2)^{h_2} \\
&- \chi_{[(0,-\infty),(x_1,0)]}(y_1,y_2)(x_1-y_1)^{h_1}\left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right] \\
&- \chi_{[(-\infty,0),(0,x_2)]}(y_1,y_2)\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right](x_2-y_2)^{h_2} \\
&+ \chi_{[(-\infty,-\infty),(0,0)]}(y_1,y_2)\left[(x_1-y_1)^{h_1}-(-y_1)^{h_1}\right] \\
&\times \left[(x_2-y_2)^{h_2}-(-y_2)^{h_2}\right]\Big\} dy_1 dy_2 \Bigg] W(d\omega_1,d\omega_2). \qquad (3.2)
\end{aligned}$$

Let us introduce function $\varphi((x_1,x_2),(\omega_1,\omega_2))$ such that

$$B^{(h_1,h_2)}(x_1,x_2) \doteq \int\limits_{\mathbb{R}^2} \varphi((x_1,x_2),(\omega_1,\omega_2))\, W(d\omega_1,d\omega_2).$$

For a fixed $(x_1,x_2)\in\mathbb{R}^2$, function $\varphi((x_1,x_2),(\omega_1,\omega_2))$ is an inverse Fourier-transform. We need to consider it only for $\omega_2>0$ because its value for $\omega_2<0$ is a complex conjugate. Unfortunately, the function being inverse Fourier-transformed is not in $L^1(\mathbb{R}^2)$ although it is in $L^2(\mathbb{R}^2)$. To circumvent this difficulty we shall calculate $\varphi((x_1,x_2),(\omega_1,\omega_2))$ for $z_k=\omega_k+i\lambda_k$, $\lambda_k<0$, and then take the limit as $\lambda_k\to 0$. Therefore,

$$\varphi((x_1,x_2),(z_1,z_2)) = \frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)}$$

$$\times \Bigg[\int\limits_{\mathbb{R}^2} e^{i2\pi(y_1 z_1 + y_2 z_2)} \Big\{ \chi_{[(0,0),(x_1,x_2)]}(y_1,y_2)(x_1-y_1)^{h_1}(x_2-y_2)^{h_2}$$
$$- \chi_{[(0,-\infty),(x_1,0)]}(y_1,y_2)(x_1-y_1)^{h_1}\left[(x_2-y_2)^{h_2} - (-y_2)^{h_2}\right]$$
$$- \chi_{[(-\infty,0),(0,x_2)]}(y_1,y_2)\left[(x_1-y_1)^{h_1} - (-y_1)^{h_1}\right](x_2-y_2)^{h_2}$$
$$+ \chi_{[(-\infty,-\infty),(0,0)]}(y_1,y_2)\left[(x_1-y_1)^{h_1} - (-y_1)^{h_1}\right]$$
$$\times \left[(x_2-y_2)^{h_2} - (-y_2)^{h_2}\right] \Big\} \, dy_1 dy_2 \Bigg]. \tag{3.3}$$

We separate these products as follows:

$$\varphi\left((x_1,x_2),(z_1,z_2)\right) = \frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)}$$
$$\times \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} e^{i2\pi(y_1z_1+y_2z_2)}(x_1-y_1)^{h_1}(x_2-y_2)^{h_2}dy_1dy_2$$
$$- \int_{-\infty}^{0}\int_{-\infty}^{x_1} e^{i2\pi(y_1z_1+y_2z_2)}(x_1-y_1)^{h_1}(-y_2)^{h_2}dy_1dy_2$$
$$- \int_{-\infty}^{x_2}\int_{-\infty}^{0} e^{i2\pi(y_1z_1+y_2z_2)}(-y_1)^{h_1}(x_2-y_2)^{h_2}dy_1dy_2$$
$$+ \int_{-\infty}^{0}\int_{-\infty}^{0} e^{i2\pi(y_1z_1+y_2z_2)}(-y_1)^{h_1}(-y_2)^{h_2}dy_1dy_2.$$

All of these integrals are splitting into one-variable integrals since, integrating by parts and transforming the interval of integration to $(0,\infty)$, we get

$$\frac{1}{\Gamma(h+1)}\int_{-\infty}^{x} e^{i2\pi zy}(x-y)^h dy = \frac{1}{\Gamma(h)}\frac{1}{i2\pi z}\int_{-\infty}^{x} e^{i2\pi zy}(x-y)^{h-1}dy$$
$$= \frac{1}{\Gamma(h)}\frac{e^{i2\pi zx}}{i2\pi z}\int_0^{\infty} e^{-i2\pi zu}u^{h-1}du.$$

Since (see, e.g., [11], formula 3.381.4.), for Re $z<0$, $\int_0^\infty e^{-i2\pi zu}u^{h-1}du = \Gamma(h)(i2\pi z)^{-h}$, we have

$$\varphi\left((x_1,x_2),(z_1,z_2)\right)$$
$$= \frac{(i2\pi z_1)^{-h_1}(i2\pi z_2)^{-h_2}}{(i2\pi z_1)(i2\pi z_2)}\left[e^{i2\pi(x_1z_1+x_2z_2)} - e^{i2\pi x_1z_1} - e^{i2\pi x_2z_2} + 1\right]$$

$$= \frac{\left(e^{i2\pi x_1 z_1} - 1\right)\left(e^{i2\pi x_2 z_2} - 1\right)}{(i2\pi z_1)(i2\pi z_2)}(i2\pi z_1)^{-h_1}(i2\pi z_2)^{-h_2}.$$

Taking the $(\lambda_1, \lambda_2) \to 0$ limit of this expression in the $L^2(\mathbb{R}^2)$-sense and then the same limit of (3.3), we can apply the Lebesgue Dominated Convergence theorem to obtain

$$\varphi\left((x_1, x_2), (\omega_1, \omega_2)\right) = \frac{\left(e^{i2\pi x_1 \omega_1} - 1\right)\left(e^{i2\pi x_2 \omega_2} - 1\right)}{(i2\pi \omega_1)(i2\pi \omega_2)}(i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2},$$

$$\omega_1\omega_2 \neq 0.$$

This concludes the proof of the theorem. □

The following proposition summarizes the most important and easily proved properties of the fractional Brownian sheet $B^{(h_1,h_2)}$. They should be compared with the well known similar properties of the fractional Brownian motion, see [21].

Proposition 2. *Let $B^{(h_1,h_2)}(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, be a fractional Brownian sheet. Then,*

(1) $\mathsf{E}B^{(h_1,h_2)}(x_1, x_2) = 0$;

(2) *$B^{(h_1,h_2)}(x_1, x_2)$ is mean-square continuous and continuous with probability 1;*

(3) *$B^{(h_1,h_2)}(x_1, x_2)$ has homogeneous increments, that is, the distribution of*

$$\Delta_{(u_1,u_2)}B^{(h_1,h_2)}(x_1,x_2) = B^{(h_1,h_2)}(x_1+u_1,x_2+u_2) - B^{(h_1,h_2)}(x_1+u_1,x_2) - B^{(h_1,h_2)}(x_1,x_2+u_2) + B^{(h_1,h_2)}(x_1,x_2)$$

does not depend on (x_1, x_2);

(4) *For any $(x_1, x_2) \in \mathbb{R}^2$, random sheet $B^{(h_1,h_2)}(x_1, x_2)$ is not differentiable at (x_1, x_2) with probability 1;*

(5) *The covariance structure of $B^{(h_1,h_2)}(x_1, x_2)$ is given by the formula*

$$\operatorname{cov}\left(B^{(h_1,h_2)}(x_1,x_2), B^{(h_1,h_2)}(y_1,y_2)\right) = \frac{\kappa_1(h_1)\kappa_1(h_2)}{4} \times \left(|x_1|^{2h_1+1} + |y_1|^{2h_1+1} - |x_1-y_1|^{2h_1+1}\right) \times \left(|x_2|^{2h_2+1} + |y_2|^{2h_2+1} - |x_2-y_2|^{2h_2+1}\right), \tag{3.4}$$

where

$$\kappa_1(h) \doteq \int_{\mathbb{R}} \left| \frac{e^{i2\pi\omega} - 1}{i2\pi\omega} \right|^2 |2\pi\omega|^{-2h} d\omega = \frac{\Gamma(1-2h)}{h(2h+1)\pi} \cos\left(\frac{\pi}{2}(1-2h)\right). \tag{3.5}$$

Proposition 3. *Fractional Brownian sheet is selfsimilar, that is, for each* $(a_1, a_2) > 0$,

$$B^{(h_1,h_2)}(a_1x_1, a_2x_2) = a_1^{H_1} a_2^{H_2} B^{(h_1,h_2)}(x_1, x_2),$$

where $H_k = h_k + 1/2$.

The last statement follows from the spectral representation (3.1) of the fractional Brownian sheet. Indeed, changing variables in (3.1) we observe that

$$W_{\underline{a}}(d\omega_1, d\omega_2) = a_1^{H_1+1/2} a_2^{H_2+1/2} W(d\omega_1/a_1, d\omega_2/a_2)$$

is a Gaussian stochastic spectral measure with $\mathsf{E}\left|W_{\underline{a}}(d\omega_1, d\omega_2)\right|^2 = a_1^{2H_1} a_2^{2H_2} d\omega_1 d\omega_2$.

This proposition and Theorem 1 imply that the inverse Lamperti transformation of the fractional Brownian sheet
$Y(x_1, x_2) = \mathcal{L}_{\underline{H}}^{-1} B^{(h_1,h_2)}(x_1, x_2)$ is a stationary sheet. Indeed the form of covariance of

$$Y(x_1, x_2) = e^{-H_1x_1 - H_2x_2} B^{(h_1,h_2)}\left(e^{x_1}, e^{x_2}\right),$$

follows from (3.4):

$$\begin{aligned}
&\operatorname{Cov}\left(Y(x_1, x_2), Y(y_1, y_2)\right) \\
&= \left(e^{H_1(x_1-y_1)} + e^{H_1(y_1-x_1)} - \left|e^{H_1(x_1-y_1)} - e^{H_1(y_1-x_1)}\right|^{2H_1}\right) \\
&\quad \times \left(e^{H_2(x_2-y_2)} + e^{H_2(y_2-x_2)} - \left|e^{H_2(x_2-y_2)} - e^{H_2(y_2-x_2)}\right|^{2H_2}\right) \\
&= \frac{\kappa_1(h_1)\kappa_1(h_2)}{4}\left(\cosh(|y_1 - x_1|) + 2^{2H_1-1}[\sinh(|y_1 - x_1|/2)]^{2H_1}\right) \\
&\quad \times \left(\cosh(|y_2 - x_2|) + 2^{2H_2-2}[\sinh(|y_2 - x_2|/2)]^{2H_2}\right).
\end{aligned} \tag{3.6}$$

If $(h_1,h_2)=(0,0)$, i.e. in the standard Brownian motion case, we shall have the covariance function corresponding to the spectrum (6.3) of an Ornstein–Uhlenbeck sheet; this particular case has been noticed by Walsh [26].

Remark 1. It can be seen, for instance, that the Lamperti Transformation of the fractional Brownian sheet is *not long-range dependent*.

The covariance (3.6) corresponds to a spectrum which can be obtained from the original covariances of the fractional Brownian sheet using the Mellin transform. These connections open up a new field which we plan to study in more detail in another paper.

4. Planar integration with respect to the fractional Brownian sheet

Formula (3.1) gives the following formal representation of the "derivative process" [21]:

$$dB^{(h_1,h_2)}(x_1,x_2)$$
$$=\int_{\mathbb{R}^2} e^{i2\pi(x_1\omega_1+x_2\omega_2)}dx_1dx_2(i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}W(d\omega_1,d\omega_2).$$

Rigorously, the "derivative" exists only in the generalized, distributional sense. We will establish the invertibility of $B^{(h_1,h_2)}(x_1,x_2)$. For that purpose we define formally the integral of a non–random function with respect to $B^{(h_1,h_2)}(x_1,x_2)$ via the formula

$$\int_{\mathbb{R}^2} f(x_1,x_2)B^{(h_1,h_2)}(dx_1,dx_2)$$
$$=\int_{\mathbb{R}^2} f(x_1,x_2)\int_{\mathbb{R}^2} e^{i2\pi(x_1\omega_1+x_2\omega_2)}(i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}W(d\omega_1,d\omega_2)dx_1dx_2$$
$$=\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} e^{i2\pi(x_1\omega_1+x_2\omega_2)}f(x_1,x_2)dx_1dx_2\right](i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}W(d\omega_1,d\omega_2).$$

This formal definition can be made rigorous under an additional assumption.

Definition 3. Let $f:\mathbb{R}^2\to\mathbb{R}^2$, $f\in L^2(\mathbb{R}^{2k})$, and

$$\int_{\mathbb{R}^2} \left|\check{f}(\omega_1,\omega_2)\right|^2 |\omega_1|^{-2h_1}|\omega_2|^{-2h_2} d\omega_1 d\omega_2<\infty,$$

where $\check{f}$ is the inverse Fourier transformation of f, i.e. $\check{f}(\omega_1,\omega_2)=\int_{\mathbb{R}^2} e^{i2\pi(x_1\omega_1+x_2\omega_2)}f(x_1,x_2)\ dx_1dx_2$. Then,

$$\int_{\mathbb{R}^2} f(x_1,x_2)B^{(h_1,h_2)}(dx_1,dx_2)$$
$$\doteq \int_{\mathbb{R}^2} \check{f}(\omega_1,\omega_2)(i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}W(d\omega_1,d\omega_2).$$

As an application of the above integral we provide the following inversion formula for the fractional Brownian sheet:

$$\begin{aligned}
&B(x_1,x_2)\\
&\quad=\frac{1}{\Gamma(1+h_1)\Gamma(1+h_2)}\int_{\mathbb{R}^2}\Big\{\chi_{[(0,0),(x_1,x_2)]}(y_1,y_2)(x_1-y_1)^{-h_1}(x_2-y_2)^{-h_2}\\
&\qquad-\chi_{[(0,-\infty),(x_1,0)]}(y_1,y_2)(x_1-y_1)^{-h_1}\big[(x_2-y_2)^{-h_2}-(-y_2)^{-h_2}\big]\\
&\qquad-\chi_{[(-\infty,0),(0,x_2)]}(y_1,y_2)\big[(x_1-y_1)^{-h_1}-(-y_1)^{-h_1}\big](x_2-y_2)^{-h_2}\\
&\qquad\times\big[(x_2-y_2)^{-h_2}-(-y_2)^{-h_2}\big]\Big\}B^{(h_1,h_2)}(dy_1,dy_2).
\end{aligned}\tag{4.1}$$

Let us denote by $L_2(B^{(h_1,h_2)})$ the space of all homogeneous sheets $X(x_1,x_2)$ which are measurable with respect to the σ-algebra $\mathcal{F}(B^{(h_1,h_2)})\doteq\sigma\{B^{(h_1,h_2)}(x_1,x_2),\ \underline{x}\in\mathbb{R}^2\}$, i.e.,

$$\begin{aligned}
&L_2(B^{(h_1,h_2)})\\
&\quad\doteq\big\{X(x_1,x_2)\in L^2\big(\mathcal{F}(B^{(h_1,h_2)})\big),\ X \text{ is second order homogeneous}\big\}.
\end{aligned}$$

Any random sheet $X(x_1,x_2) \in L_2\left(B^{(h_1,h_2)}\right)$ has a spectral chaotic expansion

$$X(x_1,x_2) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^{2k}} e^{i2\pi\Sigma[\underline{x},\underline{\omega}_j]} f_k\left(\underline{\omega}_{(1:k)}\right) \times \prod_1^k (i2\pi\underline{\omega}_j)^{-(h_1,h_2)} W\left(d\underline{\omega}_{(1:k)}\right), \tag{4.2}$$

where $(i2\pi\underline{\omega})^{-(h_1,h_2)} = (i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}$, and

$$\mathsf{E}\,|X(x_1,x_2)|^2 = \sum_{k=0}^{\infty} \frac{k!}{(2\pi)^{2k}} \int_{\mathbb{R}^{2k}} \left| f_k\left(\underline{\omega}_{(1:k)}\right)\right|^2 \prod_1^k |\underline{\omega}_j|^{-(h_1,h_2)} d\omega_{(1:k)} < \infty.$$

Vice versa, every random sheet $X(x_1,x_2)$ of the form (4.2) is in $L_2\left(B^{(h_1,h_2)}\right)$. Functions f_k will be called here *transfer functions*; they are unique, up to a permutation of their variables $\underline{\omega}_j \in \mathbb{R}^2$. The proofs for both formulae (4.1) and (4.2) is similar to the proof of the analogous properties for the fractional Brownian motion, see [13]. We provide here the stochastic integral $B^{(h_1,h_2)}dB^{(h_1,h_2)}$, as an example of the stochastic integration with respect to the fractional Brownian sheet.

Proposition 4. *The formulae*

$$\iint_{\underline{0}}^{\underline{T}} B^{(h_1,h_2)}(x_1,x_2)dB^{(h_1,h_2)}(x_1,x_2) = \frac{\left[B^{(h_1,h_2)}(T_1,T_2)\right]^2}{2} - \iint_{\underline{0}}^{\underline{T}} \partial_1 B^{(h_1,h_2)}(x_1,x_2)\partial_2 B^{(h_1,h_2)}(x_1,x_2),$$

and

$$\begin{aligned}\iint_{\underline{0}}^{\underline{T}} &\partial_1 B^{(h_1,h_2)}(x_1,x_2)\partial_2 B^{(h_1,h_2)}(x_1,x_2) \\ &= \int_0^{T_1} B^{(h_1,h_2)}(x_1,T_2)\partial_1 B^{(h_1,h_2)}(x_1,T_2) \\ &\quad - \iint_{\underline{0}}^{\underline{T}} B^{(h_1,h_2)}(x_1,x_2)dB^{(h_1,h_2)}(x_1,x_2) \\ &\quad - \frac{\kappa_1(h_1)\kappa_1(h_2)}{4} T_1^{1+2h_1}T_2^{1+2h_2},\end{aligned}$$

are valid for both Itô and Stratonovich integrals.

Proof. First consider

$$\left[B^{(h_1,h_2)}(T_1,T_2)\right]^2 = \int\limits_{\mathbb{R}^4} e(T_1,\omega_1)e(T_2,\omega_2)e(T_1,\lambda_1)e(T_2,\lambda_2)$$
$$\times (i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}(i2\pi\lambda_1)^{-h_1}(i2\pi\lambda_2)^{-h_2}W(d\underline{\omega},d\underline{\lambda})$$
$$+T_1^{1+2h_1}T_2^{1+2h_2}\kappa_1(h_1)\kappa_1(h_2),$$

where $\kappa_1(h)$ is given by (3.5). Now, use the Diagram Formula (2.3) again for the integral

$$\iint_{\underline{0}}^{\underline{T}} B^{(h_1,h_2)}(x_1,x_2)dB^{(h_1,h_2)}(x_1,x_2)$$
$$= \int\limits_{\mathbb{R}^4} \iint_{\underline{0}}^{\underline{T}} e(x_1,\omega_1)e(x_2,\omega_2)e^{i2\pi x_1\lambda_1+i2\pi x_2\lambda_2}$$
$$\times (i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}(i2\pi\lambda_1)^{-h_1}(i2\pi\lambda_2)^{-h_2}dx_1dx_2W(d\underline{\omega},d\underline{\lambda}) + c_1.$$

The constant term c_1 is calculated by the integral

$$\int\limits_{\mathbb{R}} e(x,\omega)e^{-i2\pi x\omega}|2\pi\omega|^{-2h_1}d\omega = \int\limits_{\mathbb{R}} \frac{1-e^{-i2\pi x\omega}}{i2\pi\omega}|2\pi\omega|^{-2h_1}d\omega,$$

which contrary to the (2.8) has a unique value, namely $\frac{\kappa_1(h_1)}{2}(1+2h_1)x^{2h_1}$, hence

$$c_1 = \frac{\kappa_1(h_1)\kappa_1(h_2)}{4}T_1^{1+2h_1}T_2^{1+2h_2}.$$

Now, (2.6) and

$$\iint_{\underline{0}}^{\underline{T}} \partial_1 B^{(h_1,h_2)}(x_1,x_2)\partial_2 B^{(h_1,h_2)}(x_1,x_2)$$
$$= \int\limits_{\mathbb{R}^4} \iint_{\underline{0}}^{\underline{T}} e(x_2,\omega_2)e(x_1,\lambda_1)e^{i2\pi(x_1\omega_1+x_2\lambda_2)}$$
$$\times (i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}(i2\pi\lambda_1)^{-h_1}(i2\pi\lambda_2)^{-h_2}dx_1dx_2W(d\underline{\omega},d\underline{\lambda}) + c_1,$$

implies

$$\iint_{\underline{0}}^{\underline{T}} B^{(h_1,h_2)}(x_1,x_2)dB^{(h_1,h_2)}(x_1,x_2)$$

$$= \frac{\left[B^{(h_1,h_2)}(T_1,T_2)\right]^2}{2} - \iint_{\underline{0}}^{\underline{T}} \partial_1 B^{(h_1,h_2)}(x_1,x_2)\partial_2 B^{(h_1,h_2)}(x_1,x_2).$$

Similar calculation leads to the second equation of the proposition. □

5. Isotropic fractional sheets

An important example of a fractional random sheet is the *fractional Lévy's Brownian sheet* $B_L^{(h)}(\underline{x})$ with isotropic increments (see, e.g., LINDSTRØM [19]) which, for parameter $h = H - 1/2$, $H \in (0,1)$, is defined by the stochastic integral

$$B_L^{(h)}(\underline{x}) \doteq c_h \int_{\mathbb{R}^2} \left[\|\underline{x}-\underline{y}\|^{h-1/2} - \|\underline{y}\|^{h-1/2}\right] dB(\underline{y}),$$

where $\underline{x} = (x_1,x_2)$, $\underline{y} = (y_1,y_2)$, and $\|\,.\,\|$ denotes the Euclidean norm. Distribution-wise, fractional Lévy's Brownian sheet has the spectral representation

$$B_L^{(h)}(\underline{x}) \doteq \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]}-1}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} W(d\underline{\omega}),$$

where $\underline{\omega} = (\omega_1,\omega_2)$, and W is a complex Gaussian white noise spectral measure and $\mathsf{E}|W(d\omega_1,d\omega_2)|^2 = d\omega_1 d\omega_2$. Basic properties of $B_L^{(h)}(\underline{x})$ are listed in the following

Proposition 5. *Let $B_L^{(h)}(\underline{x})$ be a fractional Lévy's Brownian sheet. Then,*

(1) $B_L^{(h)}(\underline{x})$ *is centered;*

(2) $B_L^{(h)}(\underline{x})$ *is mean square continuous and continuous with probability 1;*

(3) *The covariance structure of* $B_L^{(h)}(\underline{x})$ *is given by formula*

$$\begin{aligned}&\operatorname{Cov}\left(B_L^{(h)}(\underline{x}), B_L^{(h)}(\underline{y})\right)\\ &= \kappa_2(h)\big(\|\underline{x}\|^{2(h+1/2)} + \|\underline{y}\|^{2(h+1/2)} - \|\underline{x}-\underline{y}\|^{2(h+1/2)}\big),\end{aligned} \tag{5.1}$$

where

$$\kappa_2(h) = \kappa_1(h) 2^{2h+1}\mathbf{B}(h+1, h+1),$$

$\kappa_1(h)$ *is the constant featuring in (3.5) and* $\mathbf{B}(\cdot,\cdot)$ *is the Beta Function.*

(4) $B_L^{(h)}(\underline{x})$ *has isotropic increments, in particular along each direction*

$$B_L^{(h)}(\underline{x}) - B_L^{(h)}(\underline{y}) \stackrel{d}{=} B_L^{(h)}(\underline{x}-\underline{y}) - B_L^{(h)}(\underline{0}),$$

as a consequence the "total" increments $\Delta B_{L2}^{(h)}$ *are homogeneous.*

(5) $B_L^{(h)}(\underline{x})$ *is selfsimilar, that is, for each* $a > 0$,

$$B_L^{(h)}(a\underline{x}) \stackrel{d}{=} a^{h+1} B_L^{(h)}(\underline{x}).$$

Proof. The continuity follows easily from the Kolmogorov Theorem since $B_L^{(h)}$ is Gaussian with the particular covariance structure (5.1). The isotropy can be demonstrated as follows: Observe that

$$B_L^{(h)}(\underline{x}) - B_L^{(h)}(\underline{y}) = \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]} - e^{i2\pi[\underline{y},\underline{\omega}]}}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} W(d\underline{\omega}).$$

Then change the variable to $\|\underline{x}-\underline{y}\|\underline{\omega}$ to obtain

$$\begin{aligned}B_L^{(h)}(\underline{x}) - B_L^{(h)}(\underline{y}) &= \|\underline{x}-\underline{y}\|^{h+1/2} \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]/\|\underline{x}-\underline{y}\|} - e^{i2\pi[\underline{y},\underline{\omega}]/\|\underline{x}-\underline{y}\|}}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} W(d\underline{\omega})\\ &= \|\underline{x}-\underline{y}\|^{h+1/2} \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{y},\underline{\omega}]/\|\underline{x}-\underline{y}\|}(e^{i2\pi[\underline{u}_\alpha,\underline{\omega}]} - 1)}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} W(d\underline{\omega}),\end{aligned}$$

where $\underline{u}_\alpha$ is a unit vector depending on the angle α between $\underline{x}-\underline{y}$ and one of the axis. It is clear now that the variance of $B_L^{(h)}(\underline{x}) - B_L^{(h)}(\underline{y})$ depends only on $\|\underline{x}-\underline{y}\|$ and it is equal to the variance of $B_L^{(h)}(\underline{x}-\underline{y})$. Evaluation of the value of constant $\kappa_2(h)$ follows from the following variance calculation:

$$\int\limits_{\mathbb{R}^2} \left| \frac{e^{i2\pi[\underline{x},\underline{\omega}]}-1}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} \right|^2 d\underline{\omega} = \|\underline{x}\|^{2(h+1/2)} \int\limits_{\mathbb{R}^2} \left| \frac{e^{i2\pi[\underline{u}_\alpha,\underline{\omega}]}-1}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} \right|^2 d\underline{\omega}$$

$$= \|\underline{x}\|^{2(h+1/2)} \int_0^{2\pi} \int_0^{\infty} \left| \frac{e^{i2\pi\rho\cos\beta}-1}{(i2\pi\rho)^{3/2+h}} \right|^2 \rho d\rho d\beta$$

$$= \|\underline{x}\|^{2(h+1/2)} \int_0^{2\pi} |\cos\beta|^{1+2h} \int_0^{\infty} \frac{|e^{i2\pi\rho}-1|^2}{(2\pi\rho)^{2(1+h)}} d\rho d\beta$$

$$= \|\underline{x}\|^{2(h+1/2)} \kappa_1(h) 2^{2(h+1)} \mathbf{B}(h+1,h+1),$$

where $\mathbf{B}(\cdot,\cdot)$ is the Beta Function. □

Notice that the above argument also shows that the increments of the fractional Lévy's Brownian sheet are stationary in the strong sense, that is, they are invariant under rigid body motions, see [23] for the time domain approach.

As in (3.1) we can write a formal representation for the "derivative process" of the fractional Lévy's Brownian motion:

$$dB_L^{(h)}(\underline{x}) = \int\limits_{\mathbb{R}^2} e^{i2\pi[\underline{x},\underline{\omega}]} d\underline{x} \frac{(i2\pi\omega_1)(i2\pi\omega_2)}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} W(d\underline{\omega}).$$

Again, rigorously, it exists only in the sense of Schwartzian distributions. For any $h \in (-1/2, 1/2)$, its spectrum

$$\left| \frac{(i2\pi\omega_1)(i2\pi\omega_2)}{(i2\pi\|\underline{\omega}\|)^{3/2+h}} \right|^2 \tag{5.2}$$

vanishes at the origin, and its memory parameter is 0. Since in many physical applications fractional model fields with long-range dependence, i.e. fields with singular spectrum, are important (see, e.g., [28]), we will now introduce another fractional Brownian sheet as a model enjoying these properties. It is markedly different from the, usual in this situation, model of the Gaussian h-fractional noise.

Definition 4. The h-fractional Brownian sheet $B_F^{(h)}$ is defined by the formula

$$B_F^{(h)}(\underline{x}) \doteq \int\limits_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]} - 1 - i2\pi[\underline{x},\underline{\omega}]}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} W(d\underline{\omega}). \tag{5.3}$$

Note that the h-fractional Brownian sheet $B_F^{(h)}$ is well defined since its transfer function is square integrable. The basic properties of the h-fractional Brownian sheet $B_F^{(h)}$ are listed in the following

Proposition 6. *Let $B_F^{(h)}$ be the h-fractional Brownian sheet defined in* (5.3). *Then,*

(1) $B_F^{(h)}(\underline{x})$ *is centered;*

(2) $B_F^{(h)}(\underline{x})$ *is mean-square continuous and continuous with probability 1;*

(3) *The "total" increments $\Delta B_F^{(h)}$ of $B_F^{(h)}$ are homogeneous, that is, $\Delta_{\underline{u}} B_F^{(h)}(\underline{x})$ is independent of $\underline{x}$; Actually, the variance*

$$\begin{aligned}\operatorname{Var}\left(\Delta_{\underline{u}} B_F^{(h)}(\underline{x})\right) &= \int\limits_{\mathbb{R}^2} \left| \frac{(e^{i2\pi\omega_1 u_1} - 1)(e^{i2\pi\omega_2 u_2} - 1)}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} \right|^2 d\underline{\omega} \\ &= 2 \iint\limits_{\underline{0}}^{\underline{u}} x_1 x_2 C_2(x_1, x_2) dx_1 dx_2,\end{aligned}$$

where the covariance structure $C_2(x_1, x_2)$ is given bellow (5.5).

(4) *The singular spectrum of the derivative sheet*

$$dB_F^{(h)}(\underline{x}) = \int\limits_{\mathbb{R}^2} e^{i2\pi[\underline{x},\underline{\omega}]} d\underline{x} \frac{1}{(i2\pi\|\underline{\omega}\|)^{1/2+h}} W(d\underline{\omega}),$$

of $B_F^{(h)}$ is of the form

$$\left| \frac{1}{2\pi\|\underline{\omega}\|^{1/2+h}} \right|^2 = \frac{1}{(2\pi)^2\|\underline{\omega}\|^{1+2h}}, \tag{5.4}$$

and the derivative sheet is an isotropic Gaussian sheet with the covariance structure

$$C_2(x_1, x_2) = \frac{2^{2h}\|\underline{x}\|^{2h-1}}{(2\pi)^{3-2h}} \mathbf{B}(h,h)\Gamma(1-2h)\cos(\pi(1-2h)). \tag{5.5}$$

Proof. The stationarity immediately follows from the following calculation (the rest of the proposition is evident):

$$\begin{aligned}\Delta_{\underline{u}} B_F^{(h)}(\underline{x}) &= \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x}+\underline{u},\underline{\omega}]} - e^{i2\pi[\underline{x}+u_1\underline{e}_1,\underline{\omega}]} - e^{i2\pi[\underline{x}+u_2\underline{e}_2,\underline{\omega}]} + e^{i2\pi[\underline{x},\underline{\omega}]}}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} W(d\underline{\omega}) \\ &= \int_{\mathbb{R}^2} \frac{(e^{i2\pi\omega_1(x_1+u_1)} - e^{i2\pi\omega_1 x_1})(e^{i2\pi\omega_2(x_2+u_2)} - e^{i2\pi\omega_2 x_2})}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} W(d\underline{\omega}) \\ &= \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]}(e^{i2\pi\omega_1 u_1} - 1)(e^{i2\pi\omega_2 u_2} - 1)}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} W(d\underline{\omega}),\end{aligned}$$

where $\underline{e}_j$ is the j^{th} basis vector. The variance

$$\begin{aligned}\operatorname{Var}\left(\Delta_{\underline{u}} B_F^{(h)}\right) &= \int_{\mathbb{R}}^{2} \left| \frac{e^{i2\pi[\underline{x},\underline{\omega}]}(e^{i2\pi\omega_1 u_1} - 1)(e^{i2\pi\omega_2 u_2} - 1)}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} \right|^2 d\underline{\omega} \\ &= \int_{\mathbb{R}}^{2} \left| \frac{(e^{i2\pi\omega_1 u_1} - 1)(e^{i2\pi\omega_2 u_2} - 1)}{(i2\pi\omega_1)(i2\pi\omega_2)(i2\pi\|\underline{\omega}\|)^{1/2+h}} \right|^2 d\underline{\omega} \\ &= \int_{\mathbb{R}}^{2} \left| \iint_{\underline{0}}^{\underline{u}} e^{i2\pi[\underline{x},\underline{\omega}]} d\underline{x} \right|^2 \frac{1}{(i2\pi\|\underline{\omega}\|)^{1+2h}} d\underline{\omega} \\ &= 2 \iint_{\underline{0}}^{\underline{u}} x_1 x_2 C_2(x_1, x_2) dx_1 dx_2,\end{aligned}$$

where $C_2(x_1, x_2)$ is given bellow, see (5.6). The calculation of the covariance structure for the derivative sheet is similarly straightforward:

$$\begin{aligned}C_2(x_1, x_2) &= \int_{\mathbb{R}^2} \frac{e^{i2\pi[\underline{x},\underline{\omega}]}}{(2\pi)^2\|\underline{\omega}\|^{1+2h}} d\underline{\omega} \\ &= \int_0^{2\pi} \int_0^{\infty} \frac{e^{i2\pi\rho\|\underline{x}\|\cos\vartheta}}{(2\pi)^2\rho^{2h}} d\rho d\vartheta \\ &= \frac{2\|\underline{x}\|^{2h-1}}{(2\pi)^2} \int_0^{\pi/2} (\cos\vartheta)^{2h-1} d\vartheta \left[2\operatorname{Re} \int_0^{\infty} \frac{e^{i2\pi\rho}}{\rho^{2h}} d\rho \right]\end{aligned}$$

$$= \frac{4\|\underline{x}\|^{2h-1}}{(2\pi)^2} 2^{2h-2} B(h,h)\Gamma(2h+1) \operatorname{Re}\left[(2\pi)^{2h-1} e^{-i\pi(2h-1)}\right]$$

$$= \frac{4\|\underline{x}\|^{2h-1}}{(2\pi)^{3-2h}} 2^{2h-2} \mathbf{B}(h,h)\Gamma(1-2h)\cos(\pi(1-2h)), \tag{5.6}$$

where we applied the well known formula $\int_0^\infty e^{iy\rho}\rho^{1-u}d\rho = \Gamma(u)y^{-u}e^{-i\pi u}$ (see, e.g., [11]). □

The selfsimilarity of both fractional Lévy's sheet and h-fractional Brownian sheet introduced above follows from their spectral representations and changing variables in a Gaussian stochastic spectral measure since $\mathsf{E}\left|W(d\omega_1/a_1, d\omega_2/a_2)\right|^2 = (a_1a_2)^{-1/2}d\omega_1 d\omega_2$. Thus we have

$$B_L^{(h)}(a\underline{x}) = a^H B_L^{(h)}(\underline{x}),$$

where $H = h + 1/2$, and

$$B_F^{(h)}(a\underline{x}) = a^H B_F^{(h)}(\underline{x}),$$

where $H = h + 3/2$.

6. Fractional Ornstein–Uhlenbeck sheet

A Gaussian homogeneous sheet $X(x_1, x_2)$ on the plane is called an *Ornstein–Uhlenbeck sheet* if, $\mathsf{E}X(x_1, x_2) = 0$, and if its covariance function

$$\operatorname{Cov}\big(X(x_1,x_2), X(0,0)\big) = \sigma_X^2 e^{-(\alpha_1|x_1| + \alpha_2|x_2|)},$$

where $\alpha_1, \alpha_2 > 0$. The later condition for α_1, α_2 will be kept for the rest of the section as well.

The Ornstein–Uhlenbeck sheet is a unique homogeneous solution of the stochastic differential equation

$$\begin{aligned} dX(x_1,x_2) = {} & \alpha_1\partial_2 X(x_1,x_2)dx_1 + \alpha_2\partial_1 X(x_1,x_2)dx_2 \\ & - \alpha_1\alpha_2 X(x_1,x_2)dx_1dx_2 + \sigma dB(x_1,x_2), \end{aligned} \tag{6.1}$$

where

$$\iint_{\underline{0}}^{\underline{T}} dX(x_1,x_2) = X(T_1,T_2) - X(T_1,0) - X(T_1,0) + X(0,0),$$

and

$$\int_0^{T_1} \partial_1 X(x_1, x_2) = X(T_1, x_2) - X(0, x_2).$$

In the spectral domain it has the representation

$$X(x_1, x_2) = \int_{\mathbb{R}^2} \frac{e^{i2\pi(x_1\omega_1 + x_2\omega_2)}}{(i2\pi\omega_1 - \alpha_1)(i2\pi\omega_2 - \alpha_2)} \sigma W(d\omega_1, d\omega_2), \tag{6.2}$$

and its spectral density is

$$S_2(\omega_1, \omega_2) = \frac{\sigma^2}{\left(\alpha_1^2 + (2\pi\omega_1)^2\right)\left(\alpha_2^2 + (2\pi\omega_2)^2\right)}. \tag{6.3}$$

It is easy to see that the chaotic representation form (2.9) of an Ornstein–Uhlenbeck sheet contains only a single, Gaussian, term.

If the driving term $B(x_1, x_2)$ in (6.1) is replaced by the fractional Brownian sheet $B^{(h_1,h_2)}$ and we consider stochastic differential equation

$$\begin{aligned} dX(x_1, x_2) = \alpha_1 \partial_2 X(x_1, x_2) dx_1 + \alpha_2 \partial_1 X(x_1, x_2) dx_2 \\ - \alpha_1 \alpha_2 X(x_1, x_2) dx_1 dx_2 + \sigma dB^{(h_1,h_2)}(x_1, x_2), \end{aligned}$$

with the same boundary conditions as (6.1), then the resulting homogeneous solution random sheet will be called a *fractional Ornstein–Uhlenbeck sheet*. Its spectral representation is

$$\begin{aligned} X(x_1, x_2) = \int_{\mathbb{R}^2} & e^{i2\pi(x_1\omega_1 + x_2\omega_2)} \\ & \times \frac{(i2\pi\omega_1)^{-h_1}(i2\pi\omega_2)^{-h_2}}{(i2\pi\omega_1 - \alpha_1)(i2\pi\omega_2 - \alpha_2)} \sigma W(d\omega_1, d\omega_2), \end{aligned} \tag{6.4}$$

and its spectral density

$$S_{X,2}(\omega_1, \omega_2) = \frac{\sigma^2 |2\pi\omega_1|^{-2h_1} |2\pi\omega_2|^{-2h_2}}{\left(\alpha_1^2 + (2\pi\omega_1)^2\right)\left(\alpha_2^2 + (2\pi\omega_2)^2\right)}. \tag{6.5}$$

In the chaotic representation (4.2) the fractional Ornstein–Uhlenbeck sheet contains only one, Gaussian, term.

Although it is well known that the Ornstein–Uhlenbeck field can not be isotropic, see [29], there are important applications for Ornstein–Uhlenbeck sheets with h-fractional Brownian sheet $B_F^{(h)}$ input, see (5.3). They have the spectral representation in the form

$$X(x_1, x_2) = \int_{\mathbb{R}^2} e^{i2\pi(x_1\omega_1 + x_2\omega_2)} \frac{(i2\pi\|\underline{\omega}\|)^{-(1/2+h)}}{(i2\pi\omega_1 - \alpha_1)(i2\pi\omega_2 - \alpha_2)} \sigma W(d\omega_1, d\omega_2),$$

and the spectral density

$$S_{X,2}(\omega_1, \omega_2) = \frac{\sigma^2 (2\pi\|\underline{\omega}\|)^{-(1+2h)}}{\big(\alpha_1^2 + (2\pi\omega_1)^2\big)\big(\alpha_2^2 + (2\pi\omega_2)^2\big)}.$$

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