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Automorphisms of Hilbert space effect algebras equipped with Jordan triple product, the two-dimensional case

By JANKO MAROVT (Maribor) and TATJANA PETEK (Maribor)

Abstract. It is known that the automorphisms of the Hilbert space effect algebras equipped with Jordan triple product, where Hilbert space is of dimension not less than three, are implemented by unitary or antiunitary operators. The aim of this paper is to show that the same asertion holds true in the two-dimensional case.

Let $\mathcal{H}$ be a real or complex Hilbert space. Operator interval $[0, I]$ of all positive operators on the Hilbert space $\mathcal{H}$, which are bounded by the identity I, is called the Hilbert space effect algebra. Elements of $[0, I]$ are called effects. In general $A, B \in [0, I]$ does not imply that $AB \in [0, I]$. However, the so-called Jordan triple product $A * B = ABA$ is always an effect for any $A, B \in [0, I]$.

Molnár proved in [8] that the automorphisms of $[0, I]$, where $\dim \mathcal{H} \geq 3$ and effect algebra is equipped with the Jordan triple product, are implemented by unitary or antiunitary operators. We will show that this is also true for $\dim \mathcal{H} = 2$.

Let us mention that the concept of effects is important in studies of quantum mechanics and plays a fundamental role in the mathematical description of quantum measurement (see for example the first chapter in [6] or [1], [2], [7]). In some modern subfields of quantum mechanics like

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quantum computation or quantum information, the two-dimensional case is of particular importance.

Theorem 1. *Let $\mathcal{H}$ be a two-dimensional real or complex Hilbert space and let $\varphi : [0, I] \to [0, I]$ be a bijective function satisfying*

$$\varphi(ABA) = \varphi(A)\varphi(B)\varphi(A), \qquad A, B \in [0, I]\,.$$

Then φ is of the form

$$\varphi(A) = UAU^*, \qquad A \in [0, I]\,,$$

where U is either a unitary or an antiunitary operator on $\mathcal{H}$.

Proof. As in [8] we prove that φ sends projections to projections and preserves the partial ordering $\leq$ among the projections in both directions. In particular, we obtain that

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(I) = I.$$

This also means that φ preserves rank of the projections.

As usually we denote

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The projection $\varphi(E_{11})$ is of rank-one, hence E_{11} and $\varphi(E_{11})$ are similar. They are, of course, also Hermitian and therefore (see for example [5]), there exists a unitary matrix V such that

$$\varphi\left(E_{11}\right) = VE_{11}V^*.$$

As in [8] we observe that $\varphi(I - P) = I - \varphi(P)$, where P is a projection and therefore

$$\varphi\left(E_{22}\right) = VE_{22}V^*.$$

Let $\psi(A) = V^*\varphi(A)V$, $A \in [0, I]$. It is easy to see that the function ψ has the same properties as φ, so without a loss of generality we can assume that $\varphi(E_{11}) = E_{11}$ and $\varphi(E_{22}) = E_{22}$.

Let $A = \left[\begin{smallmatrix} a & v \\ \overline{v} & d \end{smallmatrix}\right]$ be an arbitrary effect. We will prove that

$$\varphi(A) = \begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix}, \quad \text{where } |v| = |u|.$$

In the same way as in [8] we see that $\varphi(\lambda P) = \lambda\varphi(P)$ for every $\lambda \in [0,1]$ and every rank-one projection P. Hence,

$$\begin{aligned} aE_{11} &= a\varphi(E_{11}) = \varphi(aE_{11}) = \varphi(E_{11}AE_{11}) \\ &= \varphi(E_{11})\varphi(A)\varphi(E_{11}) = E_{11}\varphi(A)E_{11} \end{aligned}$$

and similarly

$$dE_{22} = E_{22}\varphi(A)E_{22}.$$

This yields that

$$\varphi(A) = \begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix}.$$

If $A = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ then $\varphi(A^2) = \left[\begin{smallmatrix} a^2 & w \\ \overline{w} & d^2 \end{smallmatrix}\right]$. We know that φ sends identity to identity, which gives us $\varphi(A^2) = \varphi(AIA) = \varphi(A)^2$ and hence

$$\begin{bmatrix} a^2 & w \\ \overline{w} & d^2 \end{bmatrix} = \begin{bmatrix} a^2 + |u|^2 & (a+d)u \\ (a+d)\overline{u} & d^2 + |u|^2 \end{bmatrix}.$$

It follows that $u = 0$ and therefore

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Let $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right]$ be a nonzero effect. Then, on one hand we compute

$$\varphi\left(\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\right) = \begin{bmatrix} \alpha a^2 + |v|^2\,\delta & l \\ \overline{l} & |v|^2\alpha + \delta d^2 \end{bmatrix}$$

and on the other hand, we get

$$\begin{aligned} \varphi\left(\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\right)\varphi\left(\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\right)\varphi\left(\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\right) &= \begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix} \\ &= \begin{bmatrix} \alpha a^2 + |u|^2\delta & \alpha a u + \delta d u \\ \alpha a\overline{u} + \delta d\overline{u} & |u|^2\alpha + \delta d^2 \end{bmatrix}. \end{aligned}$$

So $|v| = |u|$.

Next, we take a projection $\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$. Then $\varphi\left(\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\right) = \begin{bmatrix} \frac12 & \frac12\epsilon \\ \frac12\overline{\epsilon} & \frac12 \end{bmatrix}$, where $|\epsilon| = 1$. Let $U = \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}$. Because U is unitary, $UE_{11}U^* = E_{11}$, and $UE_{22}U^* = E_{22}$, we may assume without a loss of generality that

$$\varphi\left(\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\right) = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}.$$

Let us show that $\operatorname{Re} v = \operatorname{Re} u$. On one hand, we get

$$\varphi\left(\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\right)\varphi\left(\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\right) = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix}\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$$
$$= \frac14 \begin{bmatrix} a+d+\overline{u}+u & a+d+\overline{u}+u \\ a+d+\overline{u}+u & a+d+\overline{u}+u \end{bmatrix}.$$

On the other hand, this equals to

$$\varphi\left(\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\right) = \frac14 \begin{bmatrix} a+d+\overline{v}+v & k \\ \overline{k} & a+d+\overline{v}+v \end{bmatrix}.$$

Therefore $\operatorname{Re} u = \operatorname{Re} v$. Equation $|u| = |v|$ now implies $u = v$ or $u = \overline{v}$.

We have proved for an effect A that $\varphi(A) = A$ or $\varphi(A) = \overline{A}$, where $\overline{A}$ is the complex conjugate of A. Next, we will show that $\varphi(A) = A$ for every $A \in [0, I]$ or $\varphi(A) = \overline{A}$ for every $A \in [0, I]$.

Let us take a projection $\begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}$ and assume

$$\varphi\left(\begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}\right) = \begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}.$$

Then

$$\varphi\left(\begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} a & v \\ \overline{v} & d \end{bmatrix}\right)\varphi\left(\begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}\right) = \begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}\begin{bmatrix} a & u \\ \overline{u} & d \end{bmatrix}\begin{bmatrix} \frac12 & \frac{\mathrm{i}}2 \\ -\frac{\mathrm{i}}2 & \frac12 \end{bmatrix}$$
$$= \frac14 \begin{bmatrix} a+d+\mathrm{i}(\overline{u}-u) & \mathrm{i}(a+d)+u-\overline{u} \\ -\mathrm{i}(a+d)+\overline{u}-u & a+d+\mathrm{i}(\overline{u}-u) \end{bmatrix},$$

which equals to

$$\frac{1}{4}\begin{bmatrix} a+d+\mathrm{i}(\overline{v}-v) & m \\ \overline{m} & a+d+\mathrm{i}(\overline{v}-v) \end{bmatrix}.$$

Hence $\operatorname{Im} u = \operatorname{Im} v$ and therefore $u = v$.

Similarly, if we assume that

$$\varphi\left(\begin{bmatrix} \frac{1}{2} & \frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{2} & \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & \frac{1}{2} \end{bmatrix},$$

we obtain $\operatorname{Im} v = -\operatorname{Im} u$ and therefore

$$u = \overline{v},$$

which completes the proof. □

Jordan algebra $B_S(\mathcal{H})$ of all bounded linear self-adjoint operators on the Hilbert space $\mathcal{H}$ plays a significant role in quantum mechanics, so it is important to determine automorphisms of $B_S(\mathcal{H})$ equipped with the Jordan triple product. Molnár described such automorphisms in [8] for $\dim \mathcal{H} \geq 3$. The next theorem clarifies the situation for the two-dimensional case. We prove the following theorem in the same way as the previous one, using [8].

Theorem 2. *Let $\mathcal{H}$ be a two-dimensional real or complex Hilbert space and let $\varphi : B_S(\mathcal{H}) \to B_S(\mathcal{H})$ be a bijective function satisfying*

$$\varphi(ABA) = \varphi(A)\varphi(B)\varphi(A), \qquad A, B \in B_S(\mathcal{H}).$$

Then there is an either unitary or antiunitary operator U on $\mathcal{H}$ such that either

$$\varphi(A) = UAU^*, \qquad A \in B_S(\mathcal{H}),$$

or

$$\varphi(A) = -UAU^*, \qquad A \in B_S(\mathcal{H}).$$

Remark 3. Gudder and Nagy have recently introduced in [4]the concept of the so-called sequential product $A \circ B = \sqrt{A}\,B\sqrt{A}$, $A, B \in [0, I]$, which also has a serious physical meaning. As a consequence of Molnár's result [8, Theorem 3], Gudder and Greechie proved in [3] that the automorphisms of $[0, I]$, where $\dim \mathcal{H} \geq 3$ and effect algebra is equipped with the sequential product, are also implemented by unitary or antiunitary operators. MOLNÁR has recently proved in [9] that the same is also true in the two-dimensional case. This assertion can also be derived directly from Theorem 1. For details see [3, the proof of Theorem 2.7].

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JANKO MAROVT
FACULTY OF ECONOMICS AND BUSINESS
UNIVERSITY OF MARIBOR
RAZLAGOVA 14, 2000 MARIBOR
SLOVENIA

E-mail: janko.marovt@uni-mb.si

TATJANA PETEK
FACULTY OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE
UNIVERSITY OF MARIBOR
SMETANOVA 17, 2000 MARIBOR
SLOVENIA

E-mail: tatjana.petek@uni-mb.si

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